# MASTERCLASS EXERCISES

Mikala Ørsnes Jansen, Lecture 2

**Exit path categories.**

*Exercise 1.* Match the stratified spaces with the exit path 1-categories.

For our purposes, a stratified space $X = \bigcup_{i \in I} X_i$ is a topological space $X$ with a specified partition $\{X_i\}_{i \in I}$ indexed by a poset $I$ such that $i \leq j$ if and only if $X_i \subset \overline{X_j}$. We additionally require that the partitions are sufficiently well-behaved (conically stratified), but we will not go into the details of that here.

In the examples below, the partitions are illustrated by using different colours and the poset structure is exactly given by the closure relations (*blue* < *green* < *yellow*). All the given examples are conically stratified and have exit path $\infty$-categories which are equivalent to 1-categories.

The exit path 1-category of a stratified space $X$ with partition $\{X_i\}_{i \in I}$ is (ignoring technicalities) given as follows: the objects are the points of $X$, the morphisms are (equivalence classes of) paths of the following form

$$\gamma \colon [0,1] \to X, \qquad \gamma(0) \in X_i \quad \text{and} \quad \gamma((0,1]) \subset X_j \quad \text{for some } i \leq j,$$

where equivalence is given by a stratified homotopy equivalence. In other words, a path either stays within one stratum (the case $i = j$) or starts in one stratum $X_i$ at $t = 0$ and then immediately moves into an adjacent stratum $X_j$ where is remains for all $t > 0$ (the case $i < j$). Composition is given by concatenation of paths (and identifying the representative of the form specified above).

Identify the exit path 1-category of the given stratified spaces from the list of categories. The categories are represented by a skeletal subcategory (path connected points within a stratum are equivalent); identity morphisms (constant paths) are omitted; unlabelled arrows are unique morphisms; $\mathbb{Z}\langle a \rangle$ means the infinite cyclic group generated by $a$; $\{c_i\}_{i \in \mathbb{Z}}$ is a set of morphisms in bijection with $\mathbb{Z}$; composition rules are specified when necessary.

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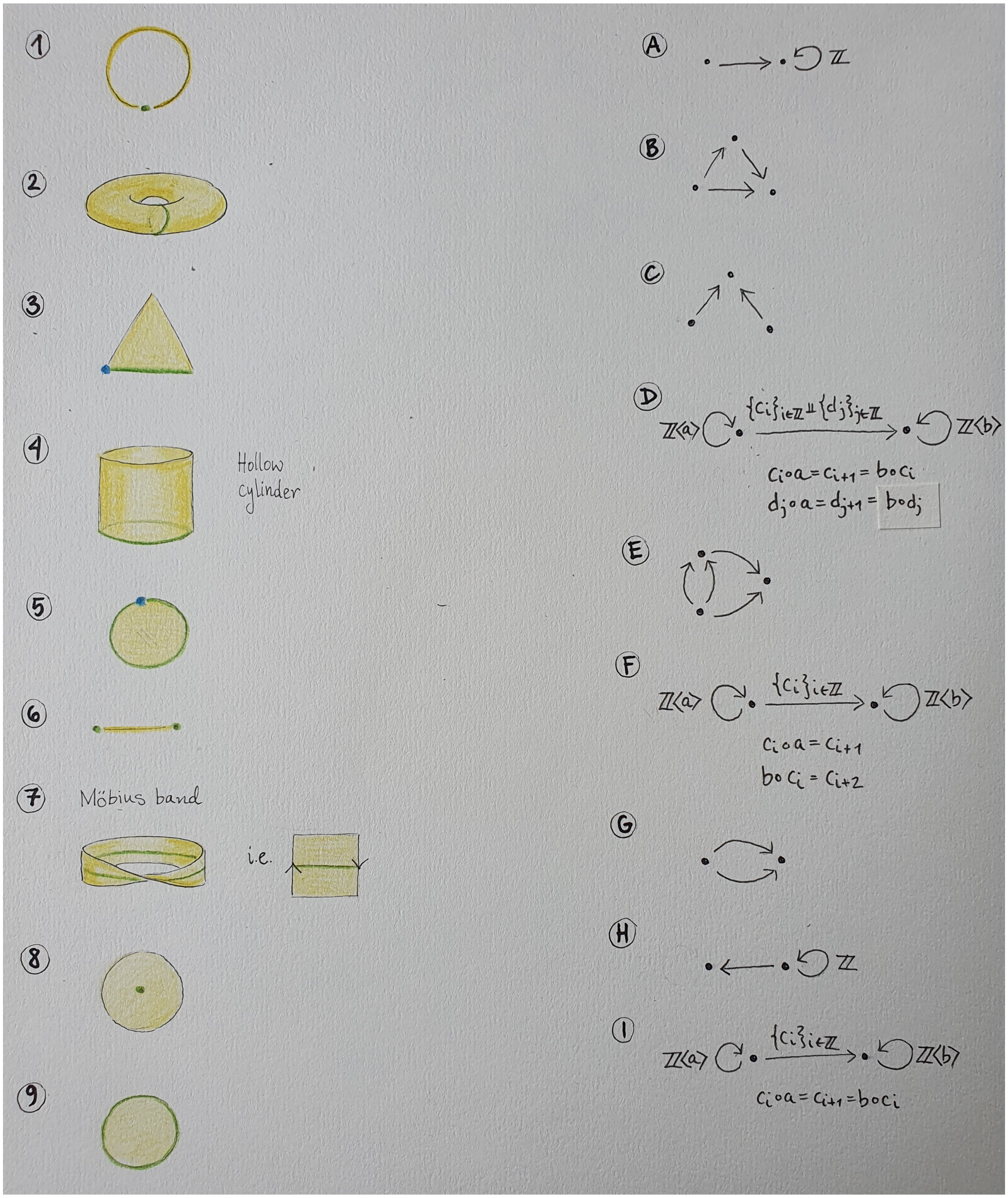
1
2
3
4
Hollow cylinder
5
6
7
Möbius band
i.e.
8
9
A
$\bullet \longrightarrow \bullet \circlearrowleft \mathbb{Z}$
B
C
D
$\mathbb{Z}\langle a\rangle \circlearrowright \bullet \xrightarrow{\{c_i\}_{i\in\mathbb{Z}} \amalg \{d_j\}_{j\in\mathbb{Z}}} \bullet \circlearrowleft \mathbb{Z}\langle b\rangle$
$c_i \circ a = c_{i+1} = b \circ c_i$
$d_j \circ a = d_{j+1} = b \circ d_j$
E
F
$\mathbb{Z}\langle a\rangle \circlearrowright \bullet \xrightarrow{\{c_i\}_{i\in\mathbb{Z}}} \bullet \circlearrowleft \mathbb{Z}\langle b\rangle$
$c_i \circ a = c_{i+1}$
$b \circ c_i = c_{i+2}$
G
H
$\bullet \longleftarrow \bullet \circlearrowleft \mathbb{Z}$
I
$\mathbb{Z}\langle a\rangle \circlearrowright \bullet \xrightarrow{\{c_i\}_{i\in\mathbb{Z}}} \bullet \circlearrowleft \mathbb{Z}\langle b\rangle$
$c_i \circ a = c_{i+1} = b \circ c_i$

## Tara Brendle, Lecture 2

Let $C_*$ denote the usual chain complex for cellular homology. Let $\mathcal{A}$ denote the arc complex $\mathcal{A}(S_g^1)$, and let $\mathcal{A}_\infty$ denote the arc complex at infinity $\mathcal{A}_\infty(S_g^1)$.

(1) Show that $\mathcal{A}_\infty \simeq C(S_g^1)$.
(2) Use $C(S_g^1) \simeq \bigvee S^{2g-2}$ to show that the shifted chain complex (with the same boundary maps as $C_*$)
$$F_k := C_{2g-1+k}(\mathcal{A} \backslash \mathcal{A}_\infty; \mathbb{Z}).$$
gives a finite resolution of the Steinberg module $\mathrm{St}(S_g^1)$ of length $4g-3$.

## Jenny Wilson, Lecture 2

### Simplicial methods and a lemma of Quillen

Recall the definition of the link of a simplex in a simplicial complex.

**Definition (Link).** Let $X$ be a simplicial complex, and let $\sigma = [s_0, \ldots, s_p]$ be a simplex in $X$. The *link* $\mathrm{Link}_X(\sigma)$ of $\sigma$ is the subcomplex of $X$ of simplices
$$\{[t_0, \ldots, t_q] \mid [s_0, \ldots, s_p, t_0, \ldots, t_q] \text{ is a simplex of } X\}.$$

(3) **(Bonus).** What can you say about links of simplices in the Tits buildings $\mathcal{T}_n(\mathbb{F})$?

The following is Quillen's definition of a Cohen–Macaulay simplicial complex. We caution that this definition is stronger than other definitions of Cohen–Macaulay appearing in the literature.

**Definition (CM complex).** A $d$-dimensional simplicial complex $X$ is *Cohen–Macaulay (CM)* if
**(i)** $X$ is $(d-1)$-connected,
**(ii)** $\mathrm{Link}_X(\sigma)$ is $(d-2-\dim(\sigma))$-connected for every simplex $\sigma$ in $X$.

(We may simplify the statement of the definition if we consider the empty set a $(-1)$-simplex, and $X$ its link).

(4) Verify that the following simplicial complexes are CM.
   (a) an $n$-simplex, and its barycentric subdivision
   (b) the boundary of an $n$-simplex, and its barycentric subdivision
   (c) **(Bonus)** the join of CM simplicial complexes

In Church–Ellenberg–Farb's proof of Ash–Rudolph's theorem, we used the following lemma. The lemma follows from Quillen's paper "Homotopy properties of the poset of nontrivial $p$-subgroups of a group," Theorem 9.1 and Corollary 9.7.

For a poset $X$, let $|X|$ denote its geometric realization. A map $f : X \to Y$ of posets is *strictly increasing* if $f(x_1) > f(x_2)$ for all $x_1 > x_2$.

**Lemma (Quillen)** Let $f : X \to Y$ be a strictly increasing map of posets. Assume that $|Y|$ is a $d$-dimensional CM complex. For $y \in Y$, let $f_y$ denote the subposet
$$f_y = \{x \in X \mid f(x) \leq y\} \subseteq X$$

and further assume that $|f_y|$ is CM for all $y$. Then $|X|$ is CM, and $f_* : \widetilde{H}_d(|X|) \to \widetilde{H}_d(|Y|)$ surjects.

(5) **(Bonus).** Prove Quillen's lemma. *Hint:* See Quillen's paper .

## Maazen's theorem for $n = 2$

**Definition (Complex of partial bases).** Fix a PID $R$. Let $B_n(R)$ denote the *complex of partial bases* in $R^n$. Its vertices are primitive elements $v_0$ of $R^n$, and vertices $\{v_0, \ldots, v_p\}$ span a $p$-simplex precisely when they are a subset of a basis for $R^n$ (possibly equal to a basis of $R^n$).

(6) Fix a PID $R$. Let $P_n(R)$ denote the poset of partial bases of $R^n$ under inclusion. Show that the geometric realization $|P_n(R)|$ is equal to the barycentric subdivision of $B_n(R)$.

The goal of this section is to prove the following result of Maazen in the case $n = 2$.

**Theorem (Maazen).** Let $R$ be a Euclidean domain. Let $P_n(R)$ denote the poset of partial bases of $R^n$ under inclusion. Then $|P_n(R)|$ is Cohen–Macaulay.

(7) Fix a Euclidean ring $R$ with norm $|\cdot|$, and let $n = 2$.
   (a) Explain why, to prove Maazen's theorem in the case $n = 2$, it suffices to show that the graph $|P_2(R)|$ is connected, equivalently, that $B_2(R)$ is connected. Thus given a vertex indexed by a primitive vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in $R^2$, it suffices to find a path to the vertex $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
   (b) Prove the following claim: Given a primitive vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in $R^2$ with $|b| > 0$, there is a basis
$$\left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right\}$$
of $R^2$ with $|d| < |b|$.
*Hint:* First choose any partial basis $\left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c' \\ d' \end{bmatrix} \right\}$, and consider elements $\begin{bmatrix} c \\ d \end{bmatrix}$ of the form
$$\begin{bmatrix} c' \\ d' \end{bmatrix} - q \begin{bmatrix} a \\ b \end{bmatrix}, \qquad q \in R.$$
   (c) Explain why the claim completes the proof.

(8) **(Bonus)**. Can you generalize this proof strategy to $n \geq 2$?

## A proof of Ash–Rudolph

The objective of the second lecture was a simplified proof of Ash–Rudolph's theorem following Church–Farb–Putman.

**Theorem (Ash–Rudolph).** Let $R$ be a Euclidean ring and $\mathbb{F}$ its field of fractions. Then the top homology of the associated Tits building $\widetilde{H}_{n-2}(\mathcal{T}_n(\mathbb{F}))$ is generated by *integral* apartment classes.

(9) Review the main argument of the lecture: use Quillen's lemma and Maazen's theorem to prove Ash–Rudolph's theorem.

## Kai-Uwe Bux, Lecture 2

The spine $Y_n$ of outer space can be described as the geometric realization of the category $\mathcal{C}_n$ with marked graphs as objects and forest collapses as morphisms. The local structure of $Y_n$ is therefore understood by looking at the *down link* of a graph $\Gamma$, i.e. the graphs obtained from $\Gamma$ by collapsing a forest, and the *up link* of $\Gamma$, i.e. the graphs that can be collapsed down to $\Gamma$.

Let $\Gamma$ be a finite graph without terminal vertices. The complex $\mathcal{F}$ of forests in $\Gamma$ is the simplicial complex whose vertices are edges in $\Gamma$ and a a collection of edges forms a simplex if they form a sub-forest of $\Gamma$. Show that $\mathcal{F}$ is homotopy equivalent to a wedge of spheres. Determine the dimension of the spheres in terms of $\Gamma$. (The complex of forests of $\Gamma$ is a description of the down link.)

Let $M$ be a finite set. A *partition* of $M$ is a way of writing $M$ as a disjoint union $M = S \sqcup T$ of two non-empty proper subsets $S, T \subset M$. Two partitions $(S_0, T_0)$ and $(S_1, T_1)$ are *nested* if one of the four possible inclusions

$$S_0 \subset S_1, \qquad T_0 \subset S_1, \qquad S_0 \subset T_1, \text{or} \quad T_0 \subset T_1$$

holds. The partition complex $\mathcal{P}$ of $M$ is the simplicial simplex where a $k$-simplex consists of $k+1$ pairwise nested partitions. Show that the partition complex is homotopy equivalent to a wedge of spheres. Determine the dimension in terms of $M$. (The partition complex models the space of blow-ups at a node $v \in \Gamma$ – the set $M$ consists of the half-edges issuing from $v$.)