

SPRAN GALATIS

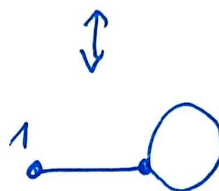
HIGH DIMENSIONAL COTHOLOGY OF MODULI SPACES II

LAST TIME:

$$M_g \rightarrow \hat{M}_g = \text{HARVEY BORDIFICATION}$$

CORNER STRATA

$\updownarrow$
ISOMORPHISM CLASS
OF $(\Gamma, w) \in \mathcal{J}_g$



TROPICAL MODULI SPACE M_g^{trop}

$$M_g^{\text{trop}} = \text{codim } \mathbb{R}_{\geq 0}^{\text{EF}} \ni l \text{ "length"}$$

$$(\Gamma, w) \in \mathcal{J}_g \quad e \in \text{EF} \mapsto l(e) \geq 0$$

$$= \coprod_{(\Gamma, w)} \mathbb{R}_{\geq 0}^{\text{EF}} / \sim$$

$\mathcal{J}_g^{\text{op}} \xrightarrow{E} \text{Set}$
 $\uparrow$
LENGTH = 0
 $\leftrightarrow e$ collapsed
in the limit

$M_g^{\text{trop}} = \text{isomorphism class of } (\Gamma, w, l)$

OPEN SUBSPACE $w \equiv 0$

$\leadsto \text{OUTER SPACE } \text{out}(F_g) \subset M_g^{\text{trop}}$

IN M_g^{trop} , CAN CONTRACT EVERYTHING TO g .

$$H_c^*(M_g^{\text{trop}}) = \text{INTERESTING}$$

NOTE:

E Finite set.

$$H_c^i(R_{>0}^E) \cong \begin{cases} \mathbb{Q} & i = |E| \\ 0 & \text{else.} \end{cases}$$

$$H_c^n(R_{>0}^E) = \mathbb{Q}$$

$\uparrow \quad \quad \uparrow$
 $s_n \quad \quad \text{sign}$

ANONICAL IDENTIFICATION ONLY UP TO SIGN.

$\exists \mathbb{Q}$ -CHAIN Cplx $C^{(g)}$ WITH GENERATORS $[\Gamma, \omega, <]$

$(\Gamma, \omega) \in \mathcal{I}_g$, $<$ TOTAL ORDER ON $E\Gamma$

RELATION: $[\Gamma, \omega, <] = \pm [\Gamma', \omega', <']$ IF

$\exists \varphi: (\Gamma, \omega) \rightarrow (\Gamma', \omega')$, $\pm = \text{sign}(E\Gamma \xrightarrow[\varphi]{\cong} E\Gamma')$

EX:  HAS A SIGN REVERSING AUTOMORPHISM

$\rightarrow [\Gamma, \omega, <] = 0$ IF $\exists \varphi \in \text{Aut}(\Gamma, \omega)$ s.t. $\text{sign}(\varphi_{E\Gamma}) = -1$.

BOUNDARY MAP: $\partial: C^{(g)} \rightarrow C^{(g)}$

$$[\Gamma, \omega, <] \mapsto \sum_{i=0}^n (-1)^i [\Gamma / e_i, \omega, < /_{E\Gamma \setminus \{e_i\}}]$$

$E\Gamma = \{e_0, \dots, e_n\}$

CLAIM: THIS IS COMPATIBLE WITH THE RELATIONS.

Thm: $H_c^*(M_g^{\text{trop}}) \cong H^*(C^{(g)})^\vee$

$$H_c^i(M_g^{\text{trop}}) \longleftrightarrow \text{GRAPHS WITH } |E\Gamma| = i.$$

PF: FILTER M_g^{trop} BY CARDINALITY OF EDGE SET.

$\leadsto$ ASSOCIATED GRADED $\coprod_{>0} (R_{>0}^{\text{EF}}) / \text{Aut}(\Gamma, w)$

(SAME ARGUMENT AS $H^*(\text{SIMPLICIAL CPX})$ OUT OF $H^*(\Delta^p, \partial\Delta^p)$ NOW USING $H_c^*(R_{>0}^{\text{EF}})$ INSTEAD + WORRYING ABOUT AUT.
("SYMMETRIC Δ -COMPLEX") $\left| (H_c^*(R_{>0}^{\text{EF}}) \cong H_c^*(R_{>0}^{\text{EF}}) \right|$

THIS GRAPH CPX APPEARED MANY PLACES ~~EARLIER~~ EARLIER, STARTING WITH KONTSEVICH IN JO'S, IN CERTAIN VARIANTS.

$$\begin{array}{ccccccc} 0 & \longrightarrow & B^{(g)} & \longrightarrow & C^{(g)} & \longrightarrow & A^{(g)} \longrightarrow 0 \\ & & \uparrow & & & & \uparrow \\ & & \exists v \in \Gamma & & & & \text{version} \\ & & \text{s.t. } w(v) \neq 0 & & & & \text{WITHOUT} \\ & & & & & & \text{WEIGHTS.} \end{array}$$

lemma: $\mathcal{B}^{(g)}$ ACYCLIC.

$$H_c^* \left(\frac{\text{Outer space}}{\text{Out}(F_n)} \right) = H^*(A^{(g)})^\vee \cong H_c^*(M_g^{\text{trop}})$$

Tomorrow

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$M_g \rightarrow M_g^{\text{trop}}$ & WHY IT INDUCES INJECTION IN $H_c^*(-)$.

COMPUTER CALCULATIONS BY
BAR-NATAN - MCKAY 2001:

$\begin{cases} \text{IMPLIES} \end{cases}$

$$H_c^{15}(M_6^{\text{trop}}) \neq 0$$

$$H_c^{19}(M_8^{\text{trop}}) \neq 0, \quad H_c^{27}(M_{10}^{\text{trop}}) \neq 0.$$

$\leadsto$ ODD DEGREE CLASSES IN $H_c^*(M_g)$.

MORE SYSTEMATIC APPROACH:

EASY: $W_g = \text{wheel with } g \text{ spokes} \in A^{(g)}$

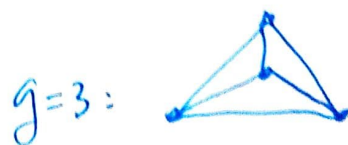
WHEEL WITH g SPOKES
(2deg EDGES, $g+1$ VERTICES)
 $W=0$.

($g \geq 3$.)

$[W_g, <]$ CYCLE:  $= 0$

$\neq 0$ IFF g ODD.  $= 0$

$[W_g, <] \neq 0$ in H_* ?

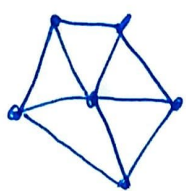


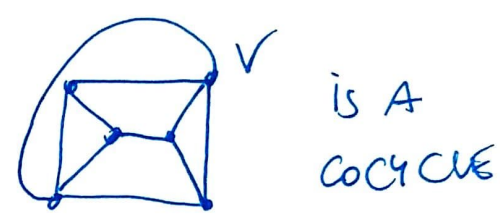
CANNOT BE THE
BOUNDARY OF ANYTHING.

Thm (Wilwagner)

$$[w_g, <] \neq 0 \text{ in Homology.}$$

BEST PROOF: FIND A COCYCLE $\sigma_g \in (A^{(g)})^V$ s.t.
 $\sigma_g(w_g, <) \neq 0.$

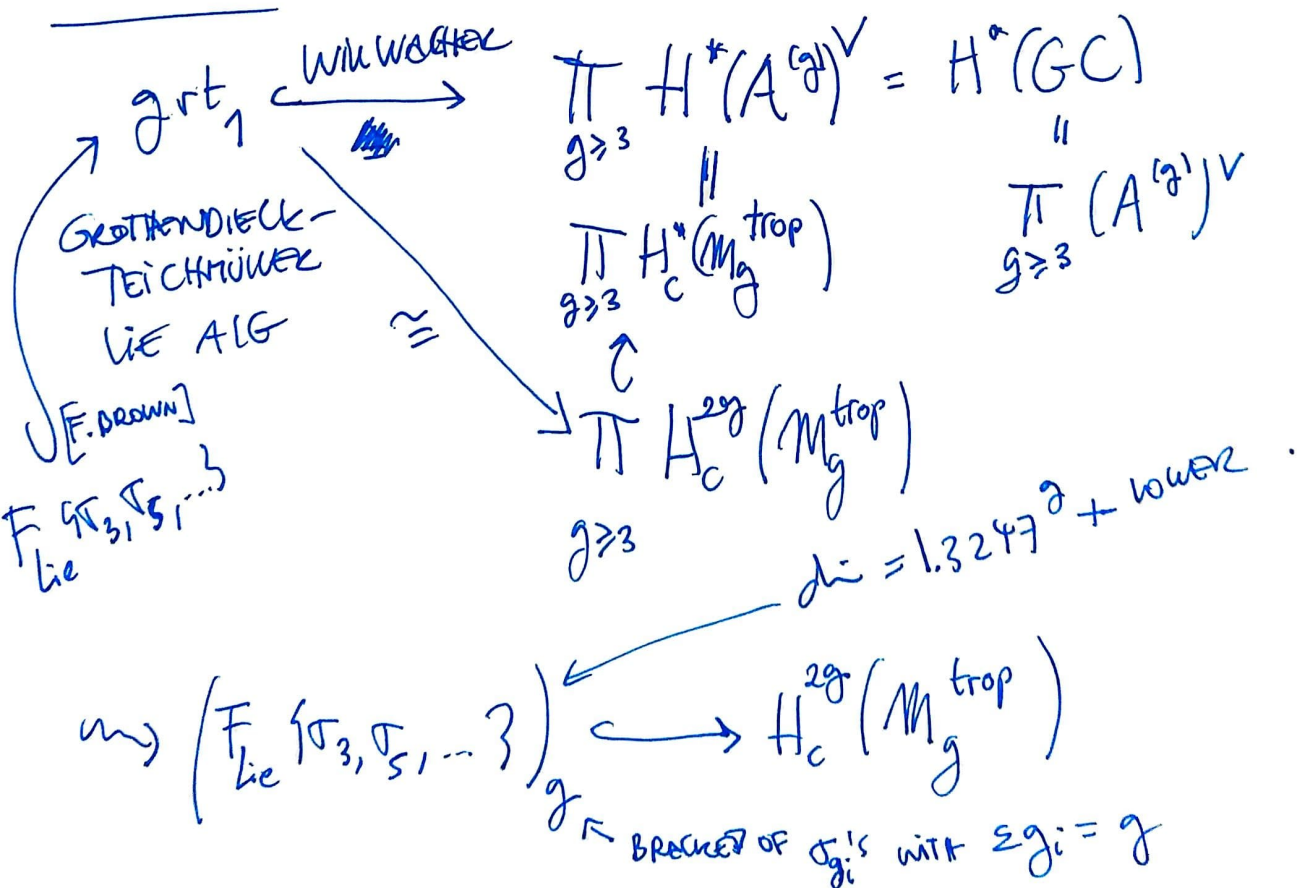
eg $g=5$:  $\pm \frac{5}{2}$



$$\Rightarrow [w_5, <] \neq 0 \in H_1(A^{(5)})$$

AS THE ^{ABOVE} COCYCLE CLASS
 WILL EVALUATE
 NON-TRIVIAL ON IT.

+ GET A PREFERRED
 ELEMENT OF $H_c^{10}(M_5^{trop})$
 FROM THE CYCLE.



"PREFORMED" $\sigma_g \in \text{Hom}(A^g, \mathbb{R})$

1.3247... is THE REAL ROOT OF SOME CUBIC POLYNOMIAL.

g odd. $\sigma_g \in H_c^{2g}(M_g^{\text{trop}})$

$\downarrow$
 $H_c^{2g}(M_g) \cong H_{4g-6}(\text{Mod}_g)$

CONSTRUCTION OF THE DUAL COCYCLE (MUMFORD)

NEED

$\sigma_g : (\Gamma, \angle) \mapsto \sigma_g(\Gamma, \angle) \in \mathbb{R}$

$|\Gamma| = 2g$

$|\Gamma| = g+1$

s.t. $\sum \pm \sigma_g(\Gamma_e, \angle|_{\Gamma_e}) = 0$
 $|\Gamma| = 2g+1$

AND $\sigma_g(W_g, \angle) \neq 0$
 $(\in \mathcal{Z}(g) \cdot \mathbb{Q}^x)$

idea: $M_{g, g+2} \cong I_{nj}(V\Gamma, \mathbb{C}) / \mathbb{C}^x \cdot X(\mathbb{C})$ $\dim_{\mathbb{R}} = 2g-2$

$(\Gamma, \angle) \xrightarrow{\text{Mum}} \omega_{(\Gamma, \angle)} \in \Omega^{2g-2} \left(\downarrow \right), \sigma_g(\Gamma, \angle) = \int \omega_{(\Gamma, \angle)}$