SØREN GALATIUS

HIGH DIMENSIONAL COH. OF MODULI SPACES III

RECALL:

$$\mathbb{F}_{Lie}\{\sigma_3, \sigma_5, \ldots\} \hookrightarrow \prod_{g \geq 3} H_c^{2g}(\mathcal{M}_g^{trop})$$

(BROWN) $\mathbb{F}_{Lie}\{\sigma_3,\sigma_5,\ldots\} \hookrightarrow \mathfrak{grt}_1 \xrightarrow{\cong} \prod_{g\geq 3} H_c^{2g}(\mathcal{M}_g^{trop})$ (WILLWACHER)

$\mathcal{M}_g \longrightarrow \mathcal{M}_g^{trop}$ PROPER

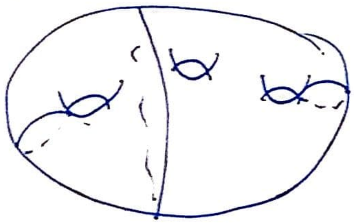
GOING TO DO $\longleftrightarrow$ LONG EDGE IN Γ

$\updownarrow$

SHORT GEODESIC IN S_g

$$\mathcal{M}_g = \left\{ \begin{array}{l} \text{GENUS } g \text{ 2-MFD} \\ \text{w/ HYP METRIC} \end{array} \right\} \Big/ \text{ISOM.}$$

$\exists\, \varepsilon$ s.t. GEODESICS OF LENGTH $\leq \varepsilon$ ARE EITHER EQUAL OR DISJOINT. eg. $\varepsilon = \operatorname{arcosh}\left(\frac{5}{4}\right)$

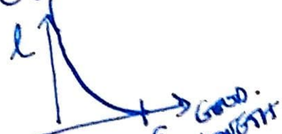
 $\longmapsto$ 

$\mathcal{M}_g \longrightarrow \mathcal{M}_g^{trop}$

$(S, h) \longmapsto (\Gamma, \omega, \ell)$ WHERE Γ HAS ONE VERTEX FOR EACH COMP. OF $S \setminus$ SHORT GEODESICS, WITH WEIGHT THEIR GENUS, AND ONE EDGE FOR EACH GEODESIC.

$\uparrow$ HYP. METRIC

ℓ: LENGTH OF GEODESICS "RENORMALIZED":



$$H_c^*(\mathcal{M}_g^{trop}) \hookrightarrow H_c^*(\mathcal{M}_g)$$

IMAGE $= W_0 H_c^*(\mathcal{M}_g) \subset H_c^*(\mathcal{M}_g)$

"WEIGHT ZERO" — TO BE EXPLAINED.

$$C^{(g)} \xrightarrow{\sim} A^{(g)}$$

$\cup$

W_g WHEEL CLASS $\qquad \exists\, \sigma_g \in (C^{(g)})^\vee$ COCYCLE s.t. $\sigma_g(W_g) \neq 0$

$$\sigma_g \in H_c^{2g}(\mathcal{M}_g^{trop}) \longrightarrow H_c^{2g}(\mathcal{M}_g)$$

$\exists$ SPECTRAL SEQUENCE $\Rightarrow H_c^*(\mathcal{M}_g) = H^*(\hat{\mathcal{M}}_g, \partial \hat{\mathcal{M}}_g)$

HARVEY COMPACTIFICATION

"RESOLVE" $\partial \hat{\mathcal{M}}_g$

GIVEN $X \xrightarrow{f} Y$ ~~[illegible]~~ CONTINUOUS MAP

$\rightsquigarrow$ SIMPLICIAL SPACE $\quad X \leftrightarrows X \times_Y X \leftrightarrows X \times_Y X \times_Y X \cdots$

$$X = X_0, \qquad X \times_Y X = X_1$$

$|X_\bullet| \longrightarrow Y \qquad$ MILD CONDITION $\Rightarrow |X_\bullet| \xrightarrow{\sim} f(X) \subset Y$

SS $\quad E_1^{s,t} = H^t(X_s^{\text{FAT DIAG.}}) \Rightarrow H^{s+t}(|X_\bullet|) = H^{s+t}(f(X))$

FROM SIMPLICIAL FILTRATION.

SIMILARLY, AUGMENTED VERSION GIVES SS

$$E_1^{s,t} = H^t(\underbrace{X \times_Y \cdots \times_Y X}_{s \text{ MANY}}, \text{FAT DIAG}) \Rightarrow H^{s+t}(Y, f(X))$$

$\Sigma_s \circlearrowright$

SYM. GROUP

SIMILAR IDEA USING THE Σ_s ACTION NOW GIVES SS

$$E_1^{s,t} = H^t(X \times_Y \cdots \times_Y X, \text{DIAG})^{\Sigma_s} \Rightarrow H^{s+t}(Y, f(X))$$

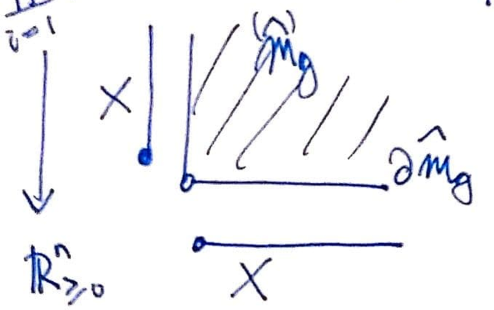
($\leftrightarrow$ "RESOLUTION BY UNORDERED SIMPLICES") ~ $\Delta^{op} \xrightarrow{Z} Top$, $\Delta^{op} \to (\text{FIN. SETS})^{op} \to Top$; $\coprod Z_p \times \Delta^p / \sim$ ...

APPLY TO $Y = \hat{M}_g$ $\qquad f(X) = \partial \hat{M}_g$

$f \uparrow$

$\coprod_{i=1}^{n} \{x \mid x_i = 0\}$

$X = \{(S, h, x) \mid x$ CHOSEN, COLLAPSED GEODESIC $\}$



$X \times_Y X \cdots \times_Y X \leftrightarrow$ MANY CHOSEN GEODESICS

$\rightsquigarrow E_1^{s,t} = H^t$(MODULI SPACE OF GENUS g HYP. WITH s CHOSEN DISTINCT, ORDERED COLLAPSED GEODESICS $)^{\Sigma_s} \Rightarrow H^*(\hat{M}_g, \partial \hat{M}_g) \cong H^*_c(M_g)$

$(s \geq 0)$ $\Updownarrow$ (AUGMENTED VERSION)

<u>$k=0$</u>

$$E_1^{s,0} = H^0(\dots)^{\Sigma_s}$$

$$\updownarrow$$

$$\pi_0 \cong \{(\Gamma, \omega, <)\}$$
$$|E\Gamma| = s$$

AND $d_1 \leftrightarrow$ STRATUM INCLUSION

$$\updownarrow$$

EDGE COLLAPSE

$$\rightsquigarrow (E_1^{s,0}, d_1) \cong C^{(g)}_\bullet .$$

$$E_2^{*,0} \longrightarrow H^*(\hat{M}_g, \partial\hat{M}_g) = H^*_c(M_g).$$ BY EDGE HOMOMORPHISM.

WANT TO SHOW THIS MAP IS INJECTIVE.

$\Longleftrightarrow$ NO DIFFERENTIALS GO INTO THAT LINE.

<u>MODIFY IDEA:</u> $\hat{M}_g \longrightarrow \overline{M}_g$ = DELIGNE-MUMFORD COMP.

DEFINED LIKE $\hat{M}_g$ BUT FORGET THE TWIST PARAMETER WHEN LENGTH PARAM $\to 0$.

<u>LOCAL PICTURE:</u>

$$\begin{tikzcd}
\mathbb{R}^{3g-3}_{>0} \times \mathbb{R}^{3g-3} \arrow[r] \arrow[d] & T_g \arrow[d] \arrow[dr] & \\
\mathbb{R}^{3g-3}_{>0} \times (\mathbb{R}/2\pi\mathbb{Z})^{3g-3} \arrow[r] \arrow[d, equal] & T_g/\mathbb{Z}^{3g-3} & T_g/Mod_g \\
(\mathbb{R}_{>0} \times S^1)^{3g-3} & &
\end{tikzcd}$$

DEHN TWIST

$$\mathbb{Z}^{3g} \hookrightarrow Mod_g$$

COMPACTIFIES TO $\mathbb{R}^2$

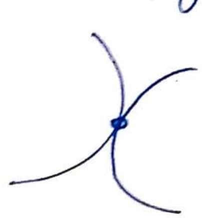
$\rightsquigarrow$ COMPACT SMOOTH ORBIFOLD, REAL. DIM $6g-6$.

($\rightsquigarrow$ SMOOTH PROPER STACK$/\mathrm{Spec}(\mathbb{Z})$)

$$H^*_c(\mathcal{M}_g) = H^*(\overline{\mathcal{M}}_g, \text{"}\partial \overline{\mathcal{M}}_g\text{"})$$

$\partial \overline{\mathcal{M}}_g$ = DEGENERATE LOCUS (codim 2) = NODAL "CURVES" (ALG GEN. ℂ-CURVES/

LOCALLY $\{(z,w) \in \mathbb{C}^2 / zw = 0\}$



(SAME IDEA AS BEFORE) $\nearrow$

$E_1^{s,t}$ (MODULI SPACE OF NODAL CURVES C, S CHOSEN NODES $z_1, \dots, z_s \in C$)$^{\Sigma_s}$

$\leftarrow$ CPX ALG. VARIETIES

$\hat{\mathcal{M}}_g \longrightarrow \overline{\mathcal{M}}_g$

INDUCES A MAP OF SS THAT IS AN ISO ON

π_0(STRATA) $\Rightarrow$ $E^{*,0}$ = AS FOR CURVES.

$\cong C^{(g)}$

$$\begin{tikzcd}
\text{SMOOTH PROJECTIVE VAR} \arrow[d] \arrow[rr] & & \text{PURE HODGE STRUCT} \arrow[dl] \arrow[ddd] \\
\mathbb{C}\text{-VARIETY} \arrow[r] \arrow[d, "\text{FORGET}"'] & \text{MIXED HODGE STR} \arrow[d] & \\
\text{Top} \arrow[r, "H^t"'] & \mathbb{Q}\text{-VECT SPACES} &
\end{tikzcd}$$

(= SPLITTING OF H^n IN $H^{p,q}$)

PURE HODGE STRUCT.: ON $H^i(X, \mathbb{R}) = \left\{ \omega \in \Omega^i(X) \,\middle|\, \begin{array}{l} d\omega = 0 \\ d^*\omega = 0 \end{array} \right\}$ $\circlearrowleft \mathbb{C}^*$

X SMOOTH PROJ. VARIETY

$$H^i(X, \mathbb{C}) \cong \bigoplus_{a+b=i} H^{a,b}$$

$$(z \cdot \omega)(v_1, \dots, v_i) = \omega(z v_1, \dots, z v_i)$$

MIXED HODGE STRUCT: SIMILAR...

VIA $G_{MHS} \twoheadrightarrow \mathbb{C}^*$ (DELIGNE HODGE II)

ALG GROUP

$H^i(X;\mathbb{R})$ X $\mathbb{C}$-VARIETY

COMPATIBLE WITH SS $\rightsquigarrow$ SS OF MIXED HODGE STRUCTURES

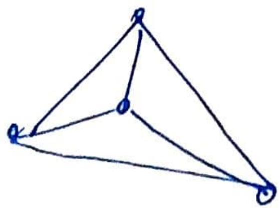
(BECAUSE THESE FORM AN ABELIAN CAT)

$\rightsquigarrow$ NO DIFFERENTIALS GOING BETWEEN DIFFERENT LINES

$\Rightarrow$ SS COLLAPSES AT E_2

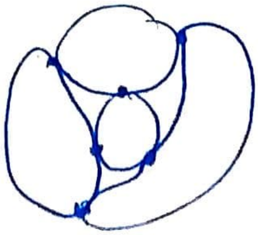
$\Rightarrow$ EDGE HOMOMORPHISM INJECTIVE.

E^2-PAGE: GRAPHS WITH VERTICES DECORATED BY COHOMOLOGY CLASSES



$+$ ERROR TERMS

$\rightarrow$



$\rightsquigarrow (S^1)^6 / \mathrm{Aut}(\triangle) \longrightarrow H_6(\mathrm{Mod}_3)$

MOVING A LITTLE AWAY FROM ∂