Moduli spaces

A *moduli space* is a space (topological space, algebraic variety, or similar) used to classify certain objects or bundles.

**Example 1.** 1. *The Grassmannian* $\text{Gr}(k, \mathbb{R}^n)$ *is the set of all* $k$-*dimensional subspaces of* $\mathbb{R}^n$. As a space it can be constructed as follows. Start with $\mathbb{R}^n$ and take the subset of tuples $(v_1, \ldots, v_k)$ with linear independent vectors. This is called a Stiefel manifold $V(k, \mathbb{R}^n)$. The Grassmanian $\text{Gr}(k, \mathbb{R}^n)$ is the quotient space of $V(k, \mathbb{R}^n)$ identifying tuples $(v_1, \ldots, v_k)$ that span the same subspace of $\mathbb{R}^n$. The set $[X, \text{Gr}(k, \mathbb{R}^n)]$ of maps up to homotopy classifies $k$-dimensional vector bundles over $X$.

2. *The moduli space* $\mathcal{M}_g$ *of algebraic curves of genus* $g$ *classifies surface bundles in the sense that the set* $[X, \mathcal{M}_g]$ *of maps up to homotopy classifies surface bundles with fiber of genus* $g$ *over* $X$.

For all examples in this masterclass, we will consider a group action of a group $G$ on a contractible space $X$. The moduli space is then the homotopy quotient of this group action, that is the classifying space $BG$. The cohomology of $BG$ defines the group cohomology of $G$. Depending on the action (all of them are properly discontinuous but some have finite stabilizers), this space is closely related to the proper quotient $X/G$. In all of our examples, it has the same rational cohomology.

**Example 2.** 1. *The Teichmüller space* $X = \mathcal{T}_g$ *of a surface of genus* $g$ *is a space that parametrizes complex structures on* $S$ *up to the action of diffeomorphisms that are isotopic to the identity diffeomorphism. The mapping class group $G = \text{Mod}_g$ *of diffeomorphism up to isotopy of a surface of genus* $g$ *acts on* $\mathcal{T}_g$ *and the moduli space* $\mathcal{M}_g$ *is the homotopy quotient.*

2. $G = \text{GL}_k(\mathbb{R})$ *acts on the Stiefel manifold* $X = V(k, \mathbb{R}^\infty)$ *and the Grassmanian* $\text{Gr}(k, \mathbb{R}^\infty)$ *is* $X/G = V(k, \mathbb{R}^\infty)/\text{GL}_k(\mathbb{R})$. *This is the only example where the group action is free and thus* $X/G \cong BG$.

3. *Culler–Vogtmann’s Outer Space* $X = \mathcal{C}V_n$ *is the space of marked metric connected graphs with fundamental group the free group* $F_n$ *and volume 1. The outer automorphism group* $G = \text{Out}(F_n)$ *acts on* $\mathcal{C}V_n$ *and we are interested in the cohomology of* $\mathcal{C}V_n/\text{Out}(F_n)$.

4. *The special linear group* $G = \text{SL}_n(\mathbb{Z})$ *acts on the symmetric space* $X = \text{Sym}_n$ *of all positive definite quadratic forms on* $\mathbb{R}^n$ *(or equivalently on the space of positive definite symmetric* $n \times n$ *matrices). We are interested in the locally symmetric space* $X/G = \text{Sym}_n/\text{SL}_n(\mathbb{Z})$. 
5. The symplectic group $G = \text{Sp}_{2g}(\mathbb{Z})$ acts on the Siegel upper half space $X = \mathfrak{H}_g$ of complex symmetric $g \times g$ matrices whose imaginary part is positive definite. The moduli space $A_g$ of abelian varieties of genus $g$ is $X/G = \mathfrak{H}_g/\text{Sp}_{2g}(\mathbb{Z})$.

Cohomological stability

During the masterclass, we will focus on the sequences of groups $\text{Mod}_g$, $\text{Out}(F_n)$, $\text{SL}_n(\mathbb{Z})$, and $\text{Sp}_{2g}(\mathbb{Z})$ and their group cohomology. For these four sequences of groups (and many more), we have cohomological stability:

**Theorem 3** (Harer, Hatcher–Vogtmann, Dwyer, Charney, ...). For the sequences $(G_n)_{n \in \mathbb{N}_0} = (\text{Mod}_n)_{n \in \mathbb{N}_0}, (\text{Out}(F_n))_{n \in \mathbb{N}_0}, (\text{SL}_n(\mathbb{Z}))_{n \in \mathbb{N}_0}, (\text{Sp}_{2n}(\mathbb{Z}))_{n \in \mathbb{N}_0}$, the group cohomology

$$H^i(G_n; \mathbb{Z})$$

is independent of $n$ for large $n$ in comparison to $i$.

We even know what those values are in the stable range:

**Theorem 4.**

$$H^*(\text{Mod}_g; \mathbb{Q}) \cong \mathbb{Q}[x_2, x_4, x_6, \ldots] \quad \text{(Madsen–Weiss)}$$

$$H^*(\text{Out}(F_n); \mathbb{Q}) \cong \mathbb{Q} \quad \text{(Galatius)}$$

$$H^*(\text{SL}_n(\mathbb{Z}); \mathbb{Q}) \cong \mathbb{Q}[x_5, x_9, x_{13}, \ldots] \quad \text{(Borel)}$$

$$H^*(\text{Sp}_{2n}(\mathbb{Z}); \mathbb{Q}) \cong \mathbb{Q}[x_2, x_6, x_{10}, \ldots] \quad \text{(Borel)}$$

High dimensional cohomology

But what about $H^i(G_n)$ for large $i$ in comparison to $n$? It turns out that rationally, it vanishes.

**Definition 5.** The virtual cohomological dimension $\text{vcd}(G)$ of a group $G$ is the largest number $i$ such that there is a torsion-free finite-index subgroup $H$ of $G$ and a $\mathbb{Z}H$-module $M$ with $H^i(H; M) \neq 0$.

In our cases, that implies in particular that

$$H^i(G; \mathbb{Q}) = 0$$

for $i > \text{vcd}(G)$. And the virtual cohomological dimensions of the groups we are considering is finite:

**Theorem 6.**

$$\text{vcd}(\text{Mod}_g) = 4g - 5 \quad \text{(Harer)}$$

$$\text{vcd}(\text{Out}(F_n)) = 2n - 2 \quad \text{(Culler–Vogtmann)}$$

$$\text{vcd}(\text{SL}_n(\mathbb{Z})) = \binom{g}{2} \quad \text{(Borel–Serre)}$$

$$\text{vcd}(\text{Sp}_{2g}(\mathbb{Z})) = g^2 \quad \text{(Borel–Serre)}$$

It turns out that even more is true. These groups are virtual duality groups which means they satisfy a property similar to Poincaré duality:
Theorem 7 (Harer, Bestvina–Feighn, Borel–Serre). For $G = \text{Mod}_g, \text{Out}(F_n), \text{SL}_n(\mathbb{Z}), \text{Sp}_{2g}(\mathbb{Z})$, there is a (not one-dimensional) duality module $D$, such that

$$H^{\text{vcd}}(G)^{−i}(G; \mathbb{Q}) \cong H_i(G; D).$$

For $\text{SL}_n(\mathbb{Z})$, Borel–Serre determine the dualizing module be the rationalized Steinberg module $D = \mathbb{Q} \otimes \text{St}_n \mathbb{Q}$.

The Steinberg module of a field $K$ is the top homology

$$\text{St}_n K := \tilde{H}_{n−2}(\mathcal{T}_n(K); \mathbb{Z})$$

of the Tits building $\mathcal{T}_n(K)$, the simplicial complex whose $p$-simplices correspond to flags of subspaces

$$0 \subseteq V_0 \subseteq \cdots \subseteq V_p \subseteq K^n.$$ 

Solomon–Tits proved that $\mathcal{T}_n(K)$ is $(n−2)$-spherical. In particular, the Steinberg module describes all of the reduced homology of $\mathcal{T}_n(K)$. The figure below depicts a typical homology class, a so-called apartment class associated to a basis $v_1, v_2, v_3 \in K^3$.

![Diagram of a typical homology class](image)

For $\text{Sp}_{2g}(\mathbb{Z})$, Borel–Serre prove that dualizing module is the symplectic Steinberg module.

For $\text{Mod}_g$, Harer found a spherical simplicial complex whose reduced rational homology is the dualizing module.

For $\text{Out}(F_n)$, less is known about the dualizing module.

Conjecture 8 (Church–Farb–Putman). For $(G_n)_{n \in \mathbb{N}_0} = (\text{Mod}_n)_{n \in \mathbb{N}_0}, (\text{SL}_n(\mathbb{Z}))_{n \in \mathbb{N}_0},$ the group cohomology

$$H^{\text{vcd}}(G_n)^{−i}(G_n; \mathbb{Q}) \cong H_i(G_n; D) = 0$$

is independent of $n$ for large $n$ in comparison to $i$.

In summary, we get the following picture for the cohomology $H^i(G_n; \mathbb{Q})$ for the sequences of groups $(G_n)_{n \in \mathbb{N}_0} = (\text{Mod}_n)_{n \in \mathbb{N}_0}, (\text{Out}(F_n))_{n \in \mathbb{N}_0}, (\text{SL}_n(\mathbb{Z}))_{n \in \mathbb{N}_0}, (\text{Sp}_{2n}(\mathbb{Z}))_{n \in \mathbb{N}_0}$. 

3
Progress on the conjecture:

Theorem 9.

\[ H^{(2)}(\text{SL}_n(\mathbb{Z}); \mathbb{Q}) = 0 \quad \text{for } n \geq 2 \quad (\text{Lee–Szczarba}) \]
\[ H^{(2) - 1}(\text{SL}_n(\mathbb{Z}); \mathbb{Q}) = 0 \quad \text{for } n \geq 3 \quad (\text{Church–Farb}) \]
\[ H^{4g - 5}(\text{Mod}_g; \mathbb{Q}) = 0 \quad \text{for } g \geq 2 \quad (\text{Church–Farb–Putman}) \]
\[ H^{4g - 6}(\text{Mod}_g; \mathbb{Q}) \neq 0 \quad \text{for } g \geq 7 \quad (\text{Chan–Galatius–Payne}) \]

Goals of the masterclass

- Understand how to prove that a group is a duality group and compute its dualizing module.
- Understand how to use a description of the dualizing module to compute cohomology groups.
- Understand why the Church–Farb–Putman conjecture is wrong for mapping class groups.
- Understand the cohomology of subgroups.