Masterclass: High dimensional cohomology of moduli spaces – Introduction

Peter Patzt

June 28, 2021

Moduli spaces

A *moduli space* is a space (topological space, algebraic variety, or similar) is used to classify certain objects or bundles.

Example 1. *1. The Grassmannian* $\mathrm{Gr}(k, \mathbb{R}^n)$ *is the set of all* k*-dimensional subspaces of* $\mathbb{R}^n$*. As a space it can be constructed as follows. Start with* $(\mathbb{R}^n)^k$ *and take the subset of tuples* $(v_1, \dots, v_k)$ *with linear independent vectors. This is called a Stiefel manifold* $V(k, \mathbb{R}^n)$*. The Grassmanian* $\mathrm{Gr}(k, \mathbb{R}^n)$ *is the quotient space of* $V(k, \mathbb{R}^n)$ *identifying tuples* $(v_1, \dots, v_k)$ *that span the same subspace of* $\mathbb{R}^n$*. The set* $[X, \mathrm{Gr}(k, \mathbb{R}^\infty)]$ *of maps up to homotopy classifies* k*-dimensional vector bundles over* X*.*

2. The moduli space $\mathcal{M}_g$ *of algebraic curves of genus* g *classifies surface bundles in the sense that the set* $[X, \mathcal{M}_g]$ *of maps up to homotopy classifies surface bundles with fiber of genus* g *over* X*.*

For all examples in this masterclass, we will consider a group action of a group G on a contractible space X. The moduli space is then the homotopy quotient of this group action, that is the classifying space BG. The cohomology of BG defines the group cohomology of G. Depending on the action (all of them are properly discontinuous but some have finite stabilizers), this space is closely related to the proper quotient X/G. In all of our examples, it has the same rational cohomology.

Example 2. *1. The Teichmüller space* $X = \mathcal{T}_g$ *of a surface of genus* g *is a space that parametrizes complex structures on* S *up to the action of diffeomorphisms that are isotopic to the identity diffeomorphism. The mapping class group* $G = \mathrm{Mod}_g$ *of diffeomorphism up to isotopy of a surface of genus* g *acts on* $\mathcal{T}_g$ *and the moduli space* $\mathcal{M}_g$ *is the homotopy quotient.*

2. $G = \mathrm{GL}_k(\mathbb{R})$ *acts on the Stiefel manifold* $X = V(k, \mathbb{R}^\infty)$ *and the Grassmanian* $\mathrm{Gr}(k, \mathbb{R}^\infty)$ *is* $X/G = V(k, \mathbb{R}^\infty)/\mathrm{GL}_k(\mathbb{R})$*. This is the only example where the group action is free and thus* $X/G \simeq BG$*.*

3. Culler–Vogtmann's Outer Space $X = CV_n$ *is the space of marked metric connected graphs with fundamental group the free group* F_n *and volume* 1*. The outer automorphism group* $G = \mathrm{Out}(F_n)$ *acts on* CV_n *and we are interested in the cohomology of* $CV_n/\mathrm{Out}(F_n)$*.*

4. The special linear group $G = \mathrm{SL}_n(\mathbb{Z})$ *acts on the symmetric space* $X = \mathrm{Sym}_n$ *of all positive definite quadratic forms on* $\mathbb{R}^n$ *(or equivalently on the space of positive definite symmetric* $n \times n$ *matrices). We are interested in the locally symmetric space* $X/G = \mathrm{Sym}_n/\mathrm{SL}_n(\mathbb{Z})$*.*

5. *The symplectic group* $G = \mathrm{Sp}_{2g}(\mathbb{Z})$ *acts on the Siegel upper half space* $X = \mathfrak{H}_g$ *of complex symmetric* $g \times g$ *matrices whose imaginary part is positive definite. The moduli space* $\mathcal{A}_g$ *of abelian varieties of genus* g *is* $X/G = \mathfrak{H}_g/\mathrm{Sp}_{2g}(\mathbb{Z})$*.*

Cohomological stability

During the masterclass, we will focus on the sequences of groups Mod_g, $\mathrm{Out}(F_n)$, $\mathrm{SL}_n(\mathbb{Z})$, and $\mathrm{Sp}_{2g}(\mathbb{Z})$ and their group cohomology. For these four sequences of groups (and many more), we have cohomological stability:

Theorem 3 (Harer, Hatcher–Vogtmann, Dwyer, Charney, ...)**.** *For the sequences* $(G_n)_{n\in\mathbb{N}_0} = (\mathrm{Mod}_n)_{n\in\mathbb{N}_0}, (\mathrm{Out}(F_n))_{n\in\mathbb{N}_0}, (\mathrm{SL}_n(\mathbb{Z}))_{n\in\mathbb{N}_0}, (\mathrm{Sp}_{2n}(\mathbb{Z}))_{n\in\mathbb{N}_0}$*, the group cohomology*

$$H^i(G_n;\mathbb{Z})$$

is independent of n *for large* n *in comparison to* i*.*

We even know what those values are in the stable range:

Theorem 4.

$$\begin{aligned} H^*(\mathrm{Mod}_\infty;\mathbb{Q}) &\cong \mathbb{Q}[x_2, x_4, x_6, \dots] && \textit{(Madsen–Weiss)} \\ H^*(\mathrm{Out}(F_\infty);\mathbb{Q}) &\cong \mathbb{Q} && \textit{(Galatius)} \\ H^*(\mathrm{SL}_\infty(\mathbb{Z});\mathbb{Q}) &\cong \mathbb{Q}[x_5, x_9, x_{13}, \dots] && \textit{(Borel)} \\ H^*(\mathrm{Sp}_\infty(\mathbb{Z});\mathbb{Q}) &\cong \mathbb{Q}[x_2, x_6, x_{10}, \dots] && \textit{(Borel)} \end{aligned}$$

High dimensional cohomology

But what about $H^i(G_n)$ for large i in comparison to n? It turns out that rationally, it vanishes.

Definition 5. *The* virtual cohomological dimension $\mathrm{vcd}(G)$ *of a group* G *is the largest number* i *such that there is a torsion-free finite-index subgroup* H *of* G *and a* $\mathbb{Z}H$*-module* M *with* $H^i(H;M) \neq 0$*.*

In our cases, that implies in particular that

$$H^i(G;\mathbb{Q}) = 0$$

for $i > \mathrm{vcd}(G)$. And the virtual cohomological dimensions of the groups we are considering is finite:

Theorem 6.

$$\begin{aligned} \mathrm{vcd}(\mathrm{Mod}_g) &= 4g-5 && \textit{(Harer)} \\ \mathrm{vcd}(\mathrm{Out}(F_n)) &= 2n-2 && \textit{(Culler–Vogtmann)} \\ \mathrm{vcd}(\mathrm{SL}_n(\mathbb{Z})) &= \binom{g}{2} && \textit{(Borel–Serre)} \\ \mathrm{vcd}(\mathrm{Sp}_{2g}(\mathbb{Z})) &= g^2 && \textit{(Borel–Serre)} \end{aligned}$$

It turns out that even more is true. These groups are virtual duality groups which means they satisfy a property similar to Poincaré duality:

Theorem 7 (Harer, Bestvina–Feighn, Borel–Serre)**.** *For $G = \mathrm{Mod}_g, \mathrm{Out}(F_n), \mathrm{SL}_n(\mathbb{Z}), \mathrm{Sp}_{2g}(\mathbb{Z})$, there is a (not one-dimensional) duality module D, such that*

$$H^{\mathrm{vcd}(G)-i}(G;\mathbb{Q}) \cong H_i(G;D).$$

For $\mathrm{SL}_n \mathbb{Z}$, Borel–Serre determine the dualizing module be the rationalized Steinberg module $D = \mathbb{Q} \otimes \mathrm{St}_n \mathbb{Q}$.

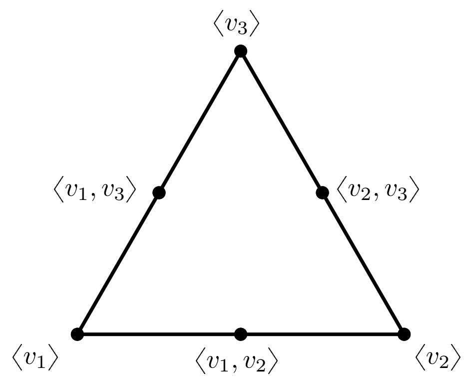
The *Steinberg module* of a field K is the top homology

$$\mathrm{St}_n K := \tilde{H}_{n-2}(\mathcal{T}_n(K);\mathbb{Z})$$

of the *Tits building* $\mathcal{T}_n(K)$, the simplicial complex whose p-simplices correspond to flags of subspaces

$$0 \subsetneq V_0 \subsetneq \cdots \subsetneq V_p \subsetneq K^n.$$

Solomon–Tits proved that $\mathcal{T}_n(K)$ is $(n-2)$-spherical. In particular, the Steinberg module describes all of the reduced homology of $\mathcal{T}_n(K)$. The figure below depicts a typical homology class, a socalled apartment class associated to a basis $v_1, v_2, v_3 \in K^3$.



For $\mathrm{Sp}_{2g}(\mathbb{Z})$, Borel–Serre prove that dualizing module is the symplectic Steinberg module.

For Mod_g, Harer found a spherical simplicial complex whose reduced rational homology is the dualizing module.

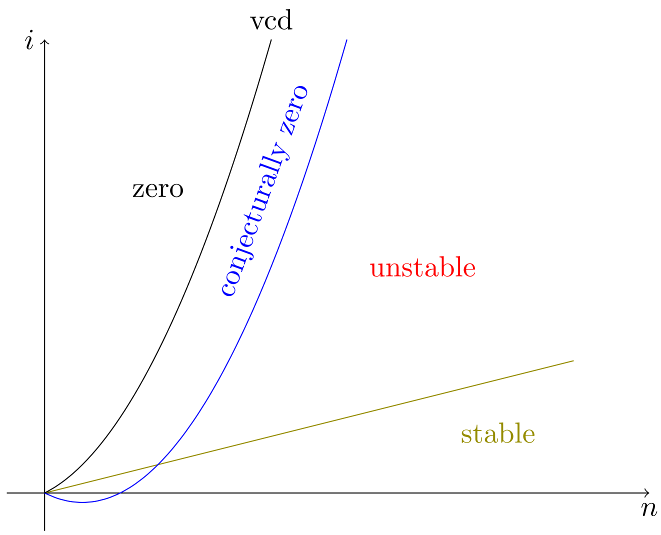
For $\mathrm{Out}(F_n)$, less is known about the dualizing module.

Conjecture 8 (Church–Farb–Putman)**.** *For $(G_n)_{n\in\mathbb{N}_0} = (\mathrm{Mod}_n)_{n\in\mathbb{N}_0}, (\mathrm{SL}_n(\mathbb{Z}))_{n\in\mathbb{N}_0}$, the group cohomology*

$$H^{\mathrm{vcd}(G_n)-i}(G_n;\mathbb{Q}) \cong H_i(G_n;D) = 0$$

is independent of n for large n in comparison to i.

In summary, we get the following picture for the cohomology $H^i(G_n;\mathbb{Q})$ for the sequences of groups $(G_n)_{n\in\mathbb{N}_0} = (\mathrm{Mod}_n)_{n\in\mathbb{N}_0}, (\mathrm{Out}(F_n))_{n\in\mathbb{N}_0}, (\mathrm{SL}_n(\mathbb{Z}))_{n\in\mathbb{N}_0}, (\mathrm{Sp}_{2n}(\mathbb{Z}))_{n\in\mathbb{N}_0}$.



Progress on the conjecture:

Theorem 9.

$$\begin{aligned}
H^{\binom{n}{2}}(\mathrm{SL}_n(\mathbb{Z});\mathbb{Q}) &= 0 \quad \textit{for } n \geqslant 2 \qquad \textit{(Lee–Szczarba)}\\
H^{\binom{n}{2}-1}(\mathrm{SL}_n(\mathbb{Z});\mathbb{Q}) &= 0 \quad \textit{for } n \geqslant 3 \qquad \textit{(Church–Farb)}\\
H^{4g-5}(\mathrm{Mod}_g;\mathbb{Q}) &= 0 \quad \textit{for } g \geqslant 2 \qquad \textit{(Church–Farb–Putman)}\\
H^{4g-6}(\mathrm{Mod}_g;\mathbb{Q}) &\neq 0 \quad \textit{for } g \geqslant 7 \qquad \textit{(Chan–Galatius–Payne)}
\end{aligned}$$

Goals of the masterclass

- Understand how to prove that a group is a duality group and compute its dualizing module.
- Understand how to use a description of the dualzing module to compute cohomology groups.
- Understand why the Church–Farb–Putman conjecture is wrong for mapping class groups.
- Understand the cohomology of subgroups.