Problem Session

Masterclass on "High dimensional cohomology of moduli spaces"

1 Benjamin Brück: Homotopy type of the poset of free factor systems

Let F_n denote the free group on n letters. A free factor of F_n is a subgroup $H \leqslant F_n$ such that there is another subgroup C with $H * C = F_n$. Here $*$ denotes internal free product. A free factor system is a collection of conjugacy classes of nontrivial proper free factors $\{[A_1], \ldots, [A_k]\}$ with $A_1 * \ldots * A_k$ a free factor. Say $\{[A_1], \ldots, [A_k]\} \leqslant \{[B_1], \ldots, [B_j]\}$ if for all i, there exists j with $A_i \leqslant B_j$ (for some representatives of the conjugacy classes). Let FFS_n denote the poset of free factor systems.

Question 1.1. *Is FFS_n homotopy equivalent to a wedge of spheres?*

2 Jeremy Miller: High dimensional coomology of $\mathrm{Aut}(F_n)$

Recall that $2n-2$ is the virtual cohomological dimension of $\mathrm{Aut}(F_n)$ (Culler–Vogtmann [CV86]).

Question 2.1. *Is $H^{2n-2}(\mathrm{Aut}(F_n); \mathbb{Q}) = 0$ for n sufficiently large?*

This is consistent with all computer evidence and similar to what happens for mapping class groups (Farb–Putman [CFP12] and Morita–Sakasai–Suzuki [MSS13]). Chan–Galatius–Payne [CGP] proved that the cohomology of the mapping class group one degree below the virtual cohomological dimension is nonzero in sufficiently large genus. One might guess that something similar happens for $\mathrm{Aut}(F_n)$.

3 Vignesh Subramanian: Steinberg modules in representation theory

Steinberg modules play an important role in the representation theory of $\mathrm{GL}_n(F)$ for F a finite field.

Question 3.1. *Are there applications of Steinberg modules of infinite fields to representation theory?*

4 Jeremy Miller: Generators for Steinberg modules of PIDs

Let R be a ring and $\mathbf{k}$ its field of fractions. Let $St_n(\mathbf{k})$ denote the Steinberg module of $\mathrm{SL}_n(\mathbf{k})$. Miller–Nagpal–Patzt [MNP20] (also see Galatius–Kupers–Randal-Williams [GKRW]) describe $\mathrm{GL}_n(\mathbf{k}) \times \mathrm{GL}_m(\mathbf{k})$-equivariant maps

$$\mathrm{St}_n(\mathbf{k}) \otimes \mathrm{St}_m(\mathbf{k}) \longrightarrow \mathrm{St}_{n+m}(\mathbf{k}).$$

The Ash-Rudolph theorem is equivalent to the statement that if R is Euclidean, then

$$\operatorname{Ind}_{\mathrm{GL}_1(R)\times\ldots\times\mathrm{GL}_1(R)}^{\mathrm{GL}_n(R)} \mathrm{St}_1(\mathbf{k})\otimes\ldots\otimes\mathrm{St}_1(\mathbf{k}) \longrightarrow \mathrm{St}_n(\mathbf{k})$$

is surjective.

Question 4.1. *If R is a PID, is*

$$\operatorname{Ind}_{\mathrm{GL}_2(R)\times\ldots\times\mathrm{GL}_2(R)}^{\mathrm{GL}_{2n}(R)} \mathrm{St}_2(\mathbf{k})\otimes\ldots\otimes\mathrm{St}_2(\mathbf{k}) \longrightarrow \mathrm{St}_{2n}(\mathbf{k})$$

and

$$\operatorname{Ind}_{\mathrm{GL}_2(R)\times\ldots\times\mathrm{GL}_2(R)\times\mathrm{GL}_1(R)}^{\mathrm{GL}_{2n+1}(R)} \mathrm{St}_2(\mathbf{k})\otimes\ldots\otimes\mathrm{St}_2(\mathbf{k})\otimes\mathrm{St}_1(\mathbf{k}) \longrightarrow \mathrm{St}_{2n+1}(\mathbf{k})$$

surjective?

This would give a generating set for Steinberg modules for all n in terms of generators for $\mathrm{St}_2(\mathbf{k})$.

5 Tara Brendle: Generators and relations for Steinberg modules of mapping class groups

Let $S_{g,b}^r$ be a surface of genus g with b boundary components and r marked points. Let $\mathrm{Mod}(S_{g,b}^r)$ be the associated mapping class group and let $\mathrm{St}(S_{g,b}^r)$ be the dualizing module of $\mathrm{Mod}(S_{g,b}^r)$. Broaddus [Bro12] proved that $\mathrm{St}(S_{g,b}^r)$ is a cyclic $\mathrm{Mod}(S_{g,b}^r)$-module for $b = 0$ and $r = 0$ or 1.

Question 5.1. *Is $\mathrm{St}(S_{g,b}^r)$ is a cyclic $\mathrm{Mod}(S_{g,b}^r)$-module for all values of g, b and r?*

The answer to this question would likely have implications for the following question.

Question 5.2. *Does the rational cohomology of $\mathrm{Mod}(S_{g,b}^r)$ vanish in its virtual cohomological dimension for g sufficiently large compared to b and r?*

This question has an affirmative answer for small values of b and r (Farb–Putman [CFP12] and Morita–Sakasai–Suzuki [MSS13]).

In cases where we know generating sets, it is natural to ask for a description of relations.

Question 5.3. *Is there a presentation of $\mathrm{St}(S_{g,b}^r)$ in the spirit of Bykovskiĭ [Byk03] (also see Church–Putman [CP17])?*

See [Bro12, Section 4.2].

6 Kai-Uwe Bux: Presentation of Torelli groups

The following question is classical.

Question 6.1. *Are the Torelli subgroups of mapping class groups and automorphism groups of free groups finitely presented for genus/rank sufficiently large?*

7 Peter Patzt: Steinberg modules of orthogonal groups

Let R be a ring and $\mathbf{k}$ its field of fractions. Gunnels [Gun00] proved that if R is Euclidean, then $\mathrm{St}(\mathrm{Sp}_n(\mathbf{k}))$ is a cyclic $\mathbb{Z}[\mathrm{Sp}_n(\mathbf{k})]$-module with vanishing coinvariants for $n \geqslant 2$.

Question 7.1. *Is $\mathrm{St}(O_{n,n}(\mathbf{k}))$ is a cyclic $\mathbb{Z}[O_{n,n}(\mathbf{k})]$-module with vanishing coinvariants for $n \geqslant 2$?*

One can ask a similar question for other families of groups.

8 Nathalie Wahl: Duality for automorphism groups of free products

Bestvina–Feighn [BF00] proved that $\mathrm{Aut}(F_n) = \mathrm{Aut}(\mathbb{Z} * \ldots * \mathbb{Z})$ is a virtual duality groups.

Question 8.1. *Are there other examples of groups G such that $\mathrm{Aut}(G * \ldots * G)$ is (or isn't) a virtual duality group?*

9 Alexander Kupers: Vanishing with twisted coefficients

Question 9.1. *Let M be a nontrivial algebraic representation of $\mathrm{SL}_n(\mathbb{Q})$. Are there conditions on M (maybe it has regular highest weight as in [LS04]) that imply that $H_i(\mathrm{SL}_n(\mathbb{Z}); M \otimes \mathrm{St}_n(\mathbb{Q}))$ vanishes?*

Possibly the theory of automorphic forms could be useful here and make it easier than the case where M is the trivial representation (conjecture of Church–Farb–Putman [CFP14]).

10 Peter Patzt: Compute Hecke eigenvalues in the vcd of congruence subgroups

Let $\Gamma_n(p)$ denote the kernel of $\mathrm{SL}_n(\mathbb{Z}) \longrightarrow \mathrm{SL}_n(\mathbb{Z}/p)$.

Question 10.1. *What are the Hecke eigenvalues of the action of the corresponding Hecke algebra on $H^{\binom{n}{2}}(\Gamma_n(p))$?*

Note that $H^{\binom{n}{2}}(\Gamma_n(p))$ has been computed for $p = 2, 3, 5$ for all n [LS76, MPP21]. See for example Ash–Stevens [AS86] for information on the Hecke action and its applications to number theory.

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