On (not) the dualizing module of Aut(F_n)

`` Much of the work on mapping class groups and automorphisms of free groups is motivated by the idea that these sequences of groups are strongly analogous, and should have many properties in common. This program is occasionally derailed by uncooperative facts but has in general proved to be a successful strategy, leading to fundamental discoveries about the structure of these groups."
-Martin Bridson and Karen Vogtmann

SL_n(R)

Def
A submodule N ⊆ M is called a summand if there is C with $M = N \oplus C$.
Let T(M) be the realization of the poset of nontrivial, proper summands of M.

Thm (Solomon-Tits)
Let R be a Dedekind domain. Then T(R^n) is (n-3)-connected.

Def
St(R^n)=$\widetilde{H}_{n-2}$ (T(R^n)).

Aut(F_n)

Def
A group H ⊆ G is called a free factor if there is C with G=H*C.
Let T(G) be the realization of the poset of nontrivial, proper free factors of G.

Thm (Hatcher-Vogtmann)
T(F_n) is (n-3)-connected.

Def
St(F_n)=$\widetilde{H}_{n-2}$ (T(F_n)).

Def
A group G is called a rational duality group if there is some number v and G-rep D such that:

H^{v-i}(G;D)=H_i(G;M ⊗ D).

(everything over char 0)

v is called the virtual cohomological dimension of G.
M is called the dualizing module.

Thm (Borel-Serre)
Let R be a number ring. Then SL_n(R) is a virtual duality group.

vcd(SL_n(Z))= $\binom{n}{2}$

Thm (Borel-Serre)
Let R be a number ring. St(R^n) is the dualizing module of SL_n(R).

Thm (Bestvina-Feighn)
Aut(F_n) is a virtual duality group.

vcd(Aut(F_n))= $2n-2$

Conjecture (Hatcher-Vogtmann)
St(F_n) is the dualizing module of Aut(F_n).

"This program is occasionally derailed by uncooperative facts…''

Thm (Himes-M.-Nariman)
St(F_5) is not the dualizing module of Aut(F_5).

Proof overview

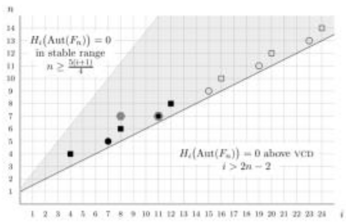


Table copied from:
Assembling homology classes in automorphism groups of free groups

[James Conant](), [Allen Hatcher](), [Martin Kassabov](), [Karen Vogtmann]()
https://arxiv.org/abs/1501.02351v4

Thm (Hatcher-Vogtman)
$H^{v_5-1}(\mathrm{Aut}(F_5))=\mathbb{Q}$

Thm (Himes-M.-Nariman)
$H_1(\mathrm{Aut}(F_n);\mathrm{St}(F_n))=0$ for $n \geq 2$.

Remark
It is unknown if $H^{v_n}(\mathrm{Aut}(F_n))=0$ for large n.

Proof strategy:
Construct a flat presentation:

$\mathrm{St}(F_n) \leftarrow G_n \leftarrow R_n$

With $(G_n)_{\mathrm{Aut}(F_n)}=0$
and $(R_n)_{\mathrm{Aut}(F_n)}=0$
for $n \geq 2$.

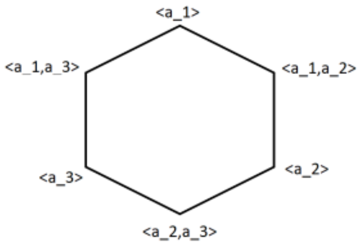
Review:
If M is a G-representation, then $M_G=H_0(G;M)$ is M mod the submodule generated by m-gm.

For simplicity, I will focus on generators.

To prove $H_0(\mathrm{Aut}(F_n);\mathrm{St}(F_n))=0$ for $n \geq 2$, it suffices to find a generating set of $\mathrm{St}(F_n)$ that maps to 0 in $H_0(\mathrm{Aut}(F_n);\mathrm{St}(F_n))=\mathrm{St}(F_n)_{\mathrm{Aut}(F_n)}$.

Let $a_1,\ldots,a_n$ be a basis of F_n.
Let $[[a_1,\ldots,a_n]]$ be the subcomplex of $T(F_n)$ generated by proper nonempty subsets of $\{a_1,a_2,\ldots,a_n\}$.

Ex
n=3



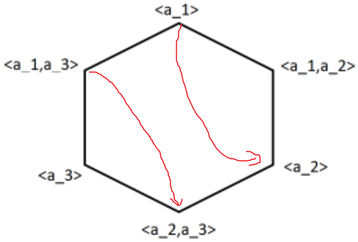
Prop
$[[a_1,\ldots,a_n]] \cong S^{n-2}$

Let $[[a_1,\ldots,a_n]]$ also denote the image of $[S^{n-2}]$ in $\mathrm{St}(F_n)=\tilde H_{n-2}(T(F_n))$

Thm (Hatcher-Vogtmann,Costa)

$St(F_n)$ is generated by classes of the form $[[a_1,\dots,a_n]]$.

Let $f:F_3 \to F_3$ be $f(a_1)=f(a_2)$, $f(a_2)=a_1$, and $f(a_3)=a_3$



For $n \geq 2$, there is f in $Aut(F_n)$ with $f([[a_1,\dots,a_n]]) =-[[a_1,\dots,a_n]]$.

So $[[a_1,\dots,a_n]] =-[[a_1,\dots,a_n]]$ in $St(F_n)_{Aut(F_n)}$

So $[[a_1,\dots,a_n]] =0$ in $St(F_n)_{Aut(F_n)}$

So $H_0(Aut(F_n);St(F_n)) = St(F_n)_{Aut(F_n)}=0$.

Thm (HMN)

The relations in $St(F_n)$ are generated by:

i) Order of the basis in $[[a_1,\dots,a_n]]$ only matters up to sign.

ii) $[[a_1,a_2\dots,a_n]]= [[a_1^{-1},a_2\dots,a_n]]$

iii) $[[a_1,a_2,\dots,a_n]] - [[a_1a_2,a_2,\dots,a_n]] + [[a_1a_2,a_1,\dots,a_n]]=0$

Relations vanish in coinvariance so $H_1(Aut(F_n);St(F_n))=0$

Example of relation 3:

$[[x,y]]=x-y$ in $\tilde H_0(T(F_2))$

$[[a_1,a_2]] - [[a_1a_2,a_2]] + [[a_1a_2,a_1]]=$

$a_1-a_2 -(a_1a_2-a_2) + (a_1a_2-a_1) = 0$

Simplicial complexes and 3-manifolds

Obvious thm
Let S be a set. Then $\{a_i-b_i\}$ generate $\tilde H_0(S)$ if and only if the graph with vertices S and with edges $[a_i,b_i]$ is connected.

Def
An element of F_n is called primitive if it is part of a basis.

Ex

Let $F_2=\langle a,b\rangle$
Then $a^2 b^2$ is not part of a basis since $\begin{bmatrix}2\\2\end{bmatrix}$

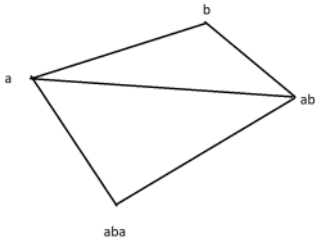
is not part of a basis of Z^2.

Def
Let $B(F_2)$ be the graph with vertices rank 1 free factors of F_2 and edges connecting free factors A and B if $F_2=A*B$.

Note that a rank 1 free factor is the same data as a primitive vector up to taking inverses.

Alt Def
Let $B(F_2)$ be the graph with edges $\{x^\pm\}$ with x primitive in F_2. $[x^\pm,y^\pm]$ forms an edge if x,y is a basis of F_2.

Observation
Zero skeleton of $B(F_2)$ is $T(F_2)$.
Edges come from our proposed generating set.
Thus:
$B(F_2)$ is connected if and only if $St(F_2)$ is generated by classes of the form $[[a_1,a_2]]$.



b,aba does not form a basis.
$b-aba$ in $St(F_2)=\tilde H_0(T(F_2))$

$b-aba=(aba-a)+(a-b)$

(too lazy to write $^\pm$)

Def
Let $B(F_n)$ be the graph with edges $\{x^\pm\}$ with x primitive in F_n. $[x_0^\pm,x_1^\pm,\ldots,x_p^\pm]$ forms a p-simplex if $x_0,\ldots,x_p$ is a subset of a basis.

Thm (Costa)
$B(F_n)$ is $(n-2)$-connected.

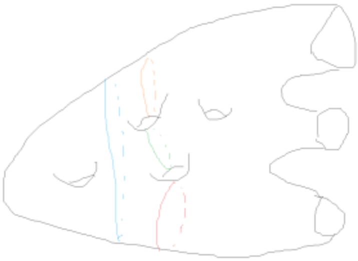
Cor
$St(F_n)$ is generated by classes of the form $[[a_1,\ldots,a_n]]$.

3-manifolds

Let $M_{n,m} = \#_n S^2 \times S^1 - m$ balls

$M_{4,3}=$

Def
Let $Y(M_{n,m})$ be the simplicial complex with vertices isotopy classes of embedded non-separating 2-spheres in $M_{n,m}$. $S_0,\ldots,S_p$ form a p-simplex if they can be isotoped to be disjoint and non-separating.



Blue is not a vertex.
Red, green, and orange are vertices.
[red,green,orange] is not a 2-simplex.
[green,orange] is a 1-simplex.

Prop
$Y(M_{2,1}) \cong B(F_2)$

Pick a basepoint in $\partial M_{2,1}$.
Send S to $\pi_1(M_{2,1} - S)$ which is rank 1 free factor of $\pi_1(M_{2,1})=F_2$.

Def
Let $S(M_{n,m})$ be the simplicial complex with vertices isotopy classes of embedded 2-spheres in $M_{n,m}$.
$S_0,\ldots,S_p$ form a p-simplex if they can be isotoped to be disjoint.

Thm (Hatcher)
$S(M_{n,m})$ is contractible.

To prove $Y(M_{2,1})$ is connected, first find a path in $S(M_{n,m})$. Then try to push the path to be in $Y(M_{2,1})$.

Similar ideas work for bigger n (work of Hatcher-Vogtmann).