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ROBIN SROKA ON THE HIGH-DIM RATIONAL COHOMOLOGY OF SYMPLECTIC GPS

BASED ON jt WORK WITH BRÜCK - PATZT

$$0 \to \text{Torelli} \to MCG_g \twoheadrightarrow Sp_{2g}(\mathbb{Z}) \to 0$$

$$\mathcal{A}_g = [Sp_{2g}(\mathbb{Z}) \backslash \mathfrak{h}_g]$$

NOTATION $\mathbb{Q}^{2n} \geq \mathbb{Z}^{2n} \ni \{e_1, \dots, e_n, \bar{e}_n, \dots, \bar{e}_1\}$

SYMPLECTIC GROUPS

DEF: SYMPLECTIC FORM $\omega : \mathbb{Q}^{2n} \times \mathbb{Q}^{2n} \to \mathbb{Q}$

$$(*) \begin{cases} (1) & \omega(e_i, e_j) = 0 = \omega(\bar{e}_i, \bar{e}_j) \\ (2) & \omega(e_i, \bar{e}_j) = \delta_{ij} = -\omega(\bar{e}_j, e_i) \end{cases}$$

$(e_i, \bar{e}_i)$ SYMPLECTIC PAIRS.

DEF: $M \in Sp_{2n}(\mathbb{Q}) \leq Sl_{2n}(\mathbb{Q})$

$\Leftrightarrow M = (M_1, \dots, M_n, M_{\bar{n}}, \dots, M_{\bar{1}})$ SYMPLECTIC BASIS (I.E. SATISFYING (*))

COLUMN

$$Sp_{2n}(\mathbb{Z}) = Sp_{2n}(\mathbb{Q}) \cap Sl_{2n}(\mathbb{Z}).$$

THM (BOREL–SERRE). $\mathrm{vcd}(Sp_{2n}(\mathbb{Z})) = n^2$

- $Sp_{2n}(\mathbb{Z})$ is a VIRTUAL BIERI–ECKMANN DUALITY GROUP, WITH DUALITY MODULE St_n^{ω}:

$$H^{n^2-i}(Sp_{2n}(\mathbb{Z}); \mathbb{Q}) \cong H_i(Sp_{2n}\mathbb{Z}; \mathbb{Q} \otimes_{\mathbb{Z}} St_n^{\omega})$$

(JUST LIKE FOR SL_n!)

QUESTION: $H^{n^2-i}(Sp_{2n}(\mathbb{Z}); \mathbb{Q}) \overset{?}{=} 0 \quad n \gg i$?

FOCUS TODAY: $i=0$ AND $i=1$.

(codim 0) IN LITERATURE:

THM (GUNNELLS): St_n^{ω} IS GENERATED BY INTEGRAL APARTMENT CLASSES, eg.

$$\mathbb{Z}[Sp_{2n}\mathbb{Z}] \xrightarrow{[-]} St_n^{\omega}$$

IN EXERCISE

$$H^{n^2}(Sp_{2n}(\mathbb{Z}); \mathbb{Q}) \cong (\mathbb{Q} \otimes St_n^{\omega})_{Sp_{2n}(\mathbb{Z})} = 0$$

GOAL TODAY: ALTERNATIVE PROOF OF GUNNELLS' THM BASED ON AN IDEA OF PUTMAN.

(Codim 1) — WORK IN PROGRESS [BRÜCK–PATZT–S]

$$H^{n^2-1}(Sp_{2n}\mathbb{Z}; \mathbb{Q}) \overset{?}{=} 0.$$

SYMPLECTIC STEINBERG MODULE St_n^ω

DEF: A SUBSPACE V OF $\mathbb{Q}^{2n}$ IS **ISOTROPIC** IF

$\omega|_V \equiv 0$.

EX: • $L \leq \mathbb{Q}^{2n}$ LINE ALWAYS ISOTROPIC.

• $Sp_{2n}(\mathbb{Z}) \ni M = (M_1, \dots, M_n, \bar{M}_n, \dots, \bar{M}_1)$

$\cup\!\!\shortmid$

SUBSET $\mathcal{S}$ NOT CONTAINING A SYMPLECTIC PAIR

$\rightsquigarrow \mathrm{Span}_{\mathbb{Q}}\langle \mathcal{S} \rangle$ ISOTROPIC IN $\mathbb{Q}^{2n}$.

FACT: A MAXIMAL ISOTROPIC SUBSPACE HAS dim n.

DEF: THE SYMPLECTIC TITS BUILDING IS THE POSET

$T_n^\omega = (\{0 \neq V \lneq \mathbb{Q}^{2n} \mid V \text{ ISOTROPIC}\}, \subseteq)$

($\uparrow$ V CANNOT BE $\mathbb{Q}^{2n}$ AS ISOTROPIC)

AS SIMPLICIAL CPX,

k-SIMPLICES $\longleftrightarrow$ $0 \neq V_0 \subsetneq \dots \subsetneq V_k$

$\dim T_n^\omega = n-1$. $\quad Sp_{2n}(\mathbb{Q}) \curvearrowright T_n^\omega$.

<u>THM</u> (SOLOMON–TITS) $\Pi_n^\omega \simeq \bigvee S^{n-1}$

<u>THM</u> (BOREL – SERRE) $St_n^\omega \cong \tilde{H}_{n-1}(\Pi_n^\omega; \mathbb{Z})$.

MAP IN GUNNELLS' THM

$$[\ ]: \mathbb{Z}[Sp_{2n}\mathbb{Z}] \longrightarrow St_n^\omega \cong \tilde{H}_{n-1}(\Pi_n^\omega; \mathbb{Z})$$

$$M \longmapsto [M] = \text{"APARTMENT CLASS" ASSOCIATED WITH } M.$$

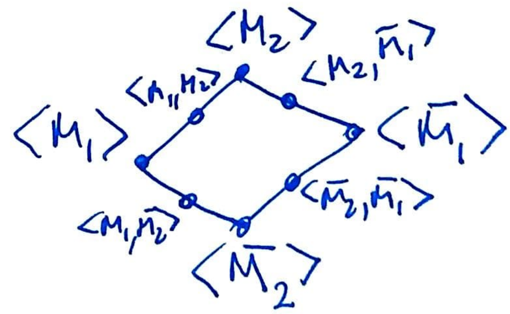
$$S^{n-1} \xhookrightarrow{M} \Pi_n^\omega .$$

<u>n=1</u> $M = (M_1, \overline{M}_1)$

$$S^0 = \cdot \quad \cdot \xhookrightarrow{M} \langle M_1 \rangle \cdot \quad \cdot \langle \overline{M}_1 \rangle \subseteq \Pi_1^\omega$$

<u>n=2</u> $M = (M_1, M_2, \overline{M}_2, \overline{M}_1)$

$$\Sigma S^0 = S^1 \xhookrightarrow{M}$$



$\subseteq \Pi_2^\omega$

<u>n=3</u> $\Sigma^2 S^0 = S^2 \xhookrightarrow{M} \cdots$

<u>REM</u>: CAN LIKEWISE DEFINE A MAP

$$[\ ]: \mathbb{Z}[Sp_{2n}\mathbb{Q}] \twoheadrightarrow St_n^{\omega}$$

SOLOMON-TITS IMPLIES THAT THIS MAP IS SURJECTIVE (SEE THE EXERCISES).

<u>PROOF THAT</u>: $[\ ]: \mathbb{Z}[Sp_{2n}\mathbb{Z}] \twoheadrightarrow St_n^{\omega}$

(IN THE SPIRIT OF CHURCH-FARB-PUTMAN FOR SL_n)

<u>STRATEGY</u>:

THERE ARE SIMPLICIAL CPX'S X AND Y AND $X \subseteq Y$

$$\begin{tikzcd}
\mathbb{Z}[Sp_{2n}(\mathbb{Z})] \arrow[d, two heads, "\alpha_n"] \arrow[dd, bend left, "{[\ ]}", start anchor=east, end anchor=north east, to path={-- (\tikztotarget)}] \\
H_n(Y,X;\mathbb{Z}) \arrow[d, two heads, "\delta"] \\
H_{n-1}(X;\mathbb{Z}) \arrow[r, two heads, "s"] \arrow[d] & St_n^{\omega} \\
H_{n-1}(Y;\mathbb{Z}) = 0
\end{tikzcd}$$

<u>FIRST ATTEMPT FOR X</u>: "SAME" CPX AS FOR $St_n(\mathbb{Z})$

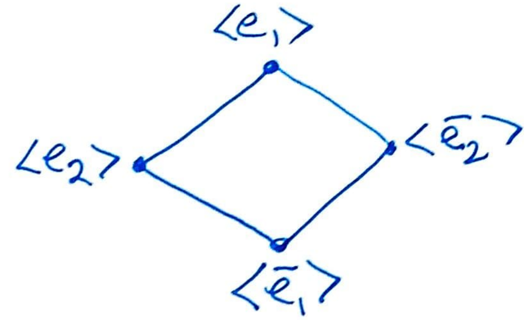
$$\begin{tikzcd}
I_n \arrow[dr] & \\
\cup & \\
\Delta \arrow[r, mapsto] & \mathrm{span}_{\mathbb{Q}} \Delta \in T_n^{\omega}
\end{tikzcd}$$

k-SIMPLEX $\leftrightarrow$ SET OF LINES IN $\mathbb{Z}^{2n}$ $\{\langle \pm v_0 \rangle, \dots, \langle \pm v_k \rangle\}$ S.T. $\{v_0, \dots, v_k\}$ IS A PARTIAL ISOTROPIC BASIS (I.E. SUBSET OF A SYMPL. BASIS NOT CONTAINING A SYMP. PAIR)

FIBERS HIGHLY CONNECTED BY SL_n-CASE [CFP/MAAZEN]

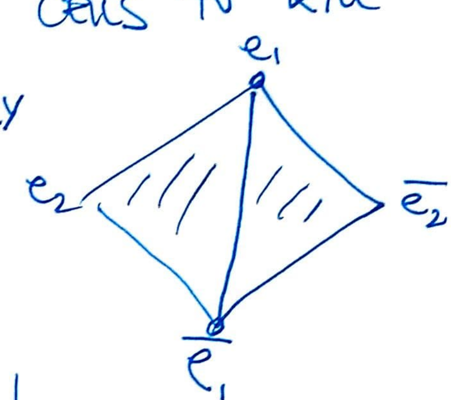
$\Rightarrow H_{n-1}(I_n) \xrightarrow{f} St_n^{\omega}$.

<u>INSIDE I_n</u>: $\underline{n=2}$



<u>ATTEMPT FOR Y</u>: NEED TO ADD CELLS TO KILL THESE APARTMENT CLASSES. TRY

i.e. ALLOW 1 SYMPLECTIC PAIR.

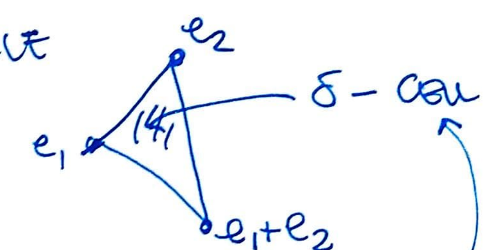


$I_n \subseteq I_n^{\sigma} = \{ \{\langle \pm v_1\rangle, \dots, \langle \pm v_n\rangle\} \mid$ 1 SYMPL PAIR ALLOWED $\}$

<u>PROBLEM</u>: I_n^{σ} IS IN GENERAL NOT $(n-1)$-CONNECTED

(n EVEN): SURVIVING CYCLE

$n=2$



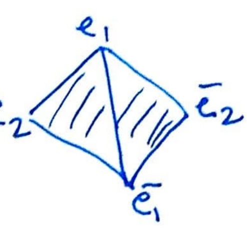
$\rightarrow$ ADD MORE CELLS TO KILL THOSE

<u>THM</u> (PUTMAN): $I_n^{\delta,\sigma}$ $(n-1)$-CONNECTED.

WITH $X = I_n$, $Y = I_n^{\delta,\sigma}$, HAVE $\xrightarrow{S}$, $\xrightarrow{\delta}$ SURJECTIONS

BUT NOT α_n!

$\alpha_n: M = (e_1, e_2, \bar{e}_2, \bar{e}_1) \mapsto$



PROP [BPS] $\exists$ IA_n s.t.

(1) $I^{\sigma,\delta} \hookrightarrow IA_n$ IS n-conn.

(2) $\alpha_n : \mathbb{Z}[Sp_{2n}(\mathbb{Z})] \twoheadrightarrow A_n(IA_n, I_n^{\delta})$

WHICH COMPLETES THE PROOF, WHERE SURJECTIVITY OF THE MAP S NOW USES [CHURCH-PUTMAN] FOR THE FIBERS, FROM THE PROOF OF codim 1 VANISHING FOR $H^*(SL_n\mathbb{Z})$.