THE STRATIFIED HOMOTOPY TYPE OF THE REDUCTIVE BOREL-SERRE COMPACTIFICATION

Mikala Ørsnes Jansen

University of Copenhagen

MASTERCLASS

High dimensional cohomology of moduli spaces.

SETTING THE SCENE

Consider

$$\Gamma \leq SL_n(\mathbb{Z})$$

finite index
neat subgroup

very torsion free.

Γ acts freely and properly discontinuously on the space

$$X = SO(n) \backslash SL_n(\mathbb{R})$$

X/Γ is
* a smooth manifold
* a model of $B\Gamma$

but not compact!

BOREL-SERRE COMPACTIFICATION

Introduced in 1973 by Borel and Serre

$$X/\Gamma \overset{\sim}{\hookrightarrow} \overline{X}^{BS}/\Gamma$$

* compact.
* smooth manifold with corners.
* model for $B\Gamma$.

Important applications:

* Borel-Serre duality
* Algebraic K-theory $K_*(\mathcal{O}_F)$

$\mathcal{O}_F$ ring of integers in number field F. **(Borel, Quillen)**

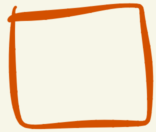
Idea

* Partial compactification $X \subseteq \overline{X}^{BS}$.
* Extend action of Γ to $\overline{X}^{BS}$.

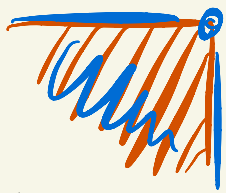
BOREL-SERRE COMPACTIFICATION

Parabolic subgroups in SL_n:

* **Standard parabolic**: block upper triangular.

$P =$  $\leq$  $= SL_n$

* **Rational parabolic**: conjugate of a standard parabolic subgroup by some $\gamma \in SL_n(\mathbb{Z})$

Naturally stratified as manifolds with corners: 

$\overline{X}^{BS}/\Gamma$ has a stratum for every Γ-conjugacy class of rational parabolic subgroups $[P]$:

$$\boxed{Y_P \cong e_P/(\Gamma \cap P) \cong B(\Gamma \cap P)}$$

REDUCTIVE BOREL-SERRE COMPACTIFICATION

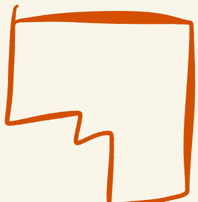
$\overline{X}^{BS}/\Gamma$ is "too big" for L^2-cohomology.

Problem: does not admit partitions of unity

To remedy this, Zucker introduced another compactification in 1982.

Idea: the strata of $\overline{X}^{BS}/\Gamma$ fit into fibrations $\boxed{N_P \longrightarrow Y_P \longrightarrow X_P}$

$N_P \cong B(\Gamma \cap \mathcal{U}_P)$ unipotent radical

$\mathcal{U}_P = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \leq$  $= P$

Collapse these fibres N_P

$$\overline{X}^{BS}/\Gamma \longrightarrow\!\!\!\!\rightarrow \overline{X/\Gamma}^{RBS}$$

REDUCTIVE BOREL-SERRE COMPACTIFICATION

Has come to play a central role with many interesting applications.

- L^p-cohomology. (Zucker)
- Weighted cohomology. (Goresky-Harder-MacPherson)

  Includes: $H^*(\Gamma)$, $H_*(\Gamma)$, $H_{(2)}(X/\Gamma)$, $IH^*(\overline{X/\Gamma}^{BB})$ Baily-Borel.
- $\mathcal{L}$-modules. (Saper)

  combinatorial analogue of constructible complexes of sheaves.

  cohomology theories (weighted, intersection)

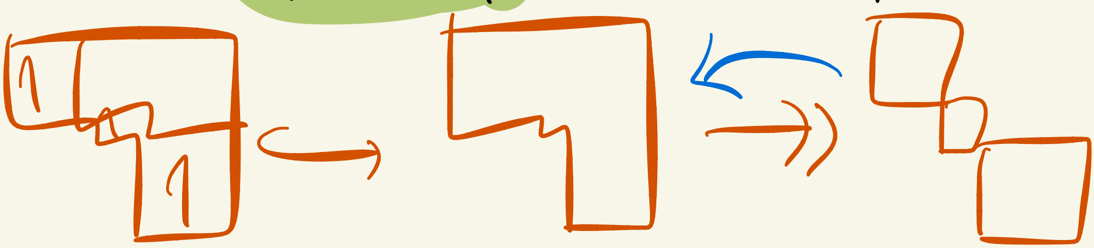
REDUCTIVE BOREL-SERRE COMPACTIFICATION

$$\overline{X}^{BS}/\Gamma \twoheadrightarrow \overline{X/\Gamma}^{RBS}$$

$\overline{X/\Gamma}^{RBS}$ inherits a natural stratification from $\overline{X}^{BS}/\Gamma$.

Strata: $\boxed{X_P \cup B\Gamma_{L_P}}$ $\qquad \Gamma\cap P/\Gamma\cap \mathcal{U}_P$ (up to conjugacy of P)

where $L_P = P/\mathcal{U}_P$ is the Levi quotient.



$$\mathcal{U}_P \hookrightarrow P \twoheadrightarrow L_P$$

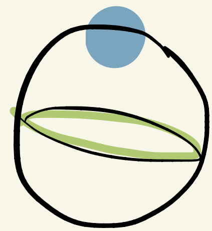
$$N_P \longrightarrow Y_P \longrightarrow X_P$$

STRATIFIED HOMOTOPY THEORY

(MacPherson, Treumann, Lurie)

Stratified space: topological space equipped with (well-behaved) partition.

Intuitively:
* collection of spaces
* glueing data



$$X \rightsquigarrow \Pi_\infty^{\mathrm{exit}}(X)$$

stratified space — exit path ∞-category

Analogue/generalisation of the fundamental ∞-groupoid.

The exit path ∞-category encodes

the homotopy type of this data

CONSTRUCTIBLE SHEAVES

top. sp. spaces

Monodromy equivalence

$$\mathrm{Shv}_{\boxed{\text{loc.const}}}(X, \ell) \simeq \mathrm{Fun}(\Pi_\infty(X), \ell)$$

In a similar way, strat. sp

$$\mathrm{Shv}_{\boxed{\text{constructible}}}(X, \ell) \simeq \mathrm{Fun}(\Pi_\infty^{\text{exit}}(X), \ell)$$

Derived constructible category:

X strat.

R ring

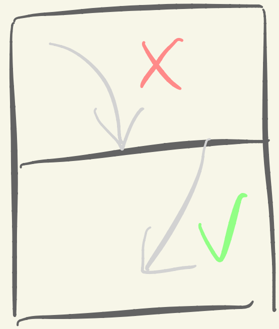
$$\boxed{\mathcal{D}_{cbl}(X,R) \subseteq \mathcal{D}(\mathrm{Shv}(X, \mathrm{Mod}_R))}$$

full subcategory of complexes with constructible cohomology sheaves.

If $\Pi_\infty^{\text{exit}}(X) \simeq N(\mathcal{C})$, $\mathcal{C}$ 1-cat, then

$$\boxed{\mathcal{D}_{cbl}(X,R) \simeq \mathcal{D}(\mathrm{Fun}(\mathcal{C}, \mathrm{Mod}_R))}$$

EXIT PATH ∞-CATEGORY

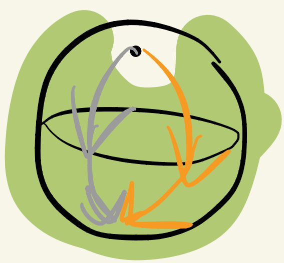


Objects: points of X

Morphisms: "exit paths".

Higher simplices: stratum preserving homotopies, ...

Examples:

	X	$\Pi_\infty^{exit}(X)$
(i)		$\cdot \leftarrow \underset{\circlearrowleft\,\mathbb{Z}}{\cdot} \rightarrow \cdot$
(ii)		$\cdot \xrightarrow[S^1]{} \cdot$

REDUCTIVE BOREL-SERRE

The stratified homotopy type of $\overline{X/\Gamma}^{RBS}$:

Theorem (J)

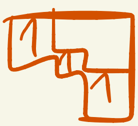
G reductive algebraic group $/\mathbb{Q}$, SL_n

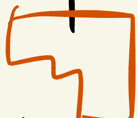
$\Gamma \leq G(\mathbb{Q})$ neat arithmetic group. $\Gamma \leq SL_n(\mathbb{Z})$

$$\Pi_\infty^{exit}\left(\overline{X/\Gamma}^{RBS}\right) \simeq N\left(\mathcal{C}_\Gamma^{RBS}\right)$$

where $\mathcal{C}_\Gamma^{RBS}$ is the following 1-category:

objects: rational parabolic subgroups of G.





+ conj.

hom-sets:

$$\mathrm{Hom}(P,Q) = \{\gamma \in \Gamma \mid \gamma P \gamma^{-1} \leq Q\} / \left(\Gamma \cap \mathcal{U}_P(\mathbb{Q})\right)$$

composition given by multiplication in Γ.

CONSEQUENCES

Homotopy type:

Inverting all morphisms $\Pi_\infty^{exit}(X)$ recovers $\Pi_\infty(X)$.

Corollary $\overline{X/\Gamma}^{RBS} \simeq |\mathcal{C}^{RBS}|$

Recover:

Corollary (Ji-Murty-Saper-Scherk 11)

$$\pi_1\left(\overline{X/\Gamma}^{RBS}\right) \cong \Gamma/E_\Gamma$$

$E_\Gamma \triangleleft \Gamma$ generated by the $\Gamma \cap \mathcal{U}_P$

Compare: $\Gamma = SL_n(\mathbb{Z})$ $n \geq 3$

$E_\Gamma = E_n(\mathbb{Z})$ elementary matrices.

CONSEQUENCES

Constructible sheaves:

Corollary

$$\boxed{\mathcal{S}hv_{cbl}(\overline{X/T}^{RBS}, \mathrm{Mod}_R) \simeq \mathrm{Fun}(\mathcal{C}_{T}^{RBS}, \mathrm{Mod}_R)}$$

1- and ∞-categorically.

Corollary

$$\boxed{\mathcal{D}_{cbl}(\overline{X/T}^{RBS}, R) \simeq \mathcal{D}(\mathrm{Fun}(\mathcal{C}_{T}^{RBS}, \mathrm{Mod}_R))}$$

combinatorial incarnation of constructible complexes of sheaves. $\rightsquigarrow$ cohomology theories!

POINT FREE VERSIONS.

ADDENDUM: ALGEBRAIC K-THEORY

Forget the space! (jt with Clausen)

Generalisation:

$\Gamma = GL_n(\mathbb{Z})$: $\mathcal{C}_n^{RBS}(\mathbb{Z}) := \mathcal{C}^{RBS}_{GL_n(\mathbb{Z})}$

$\Gamma = GL_n(R)$ arbitrary ring R:

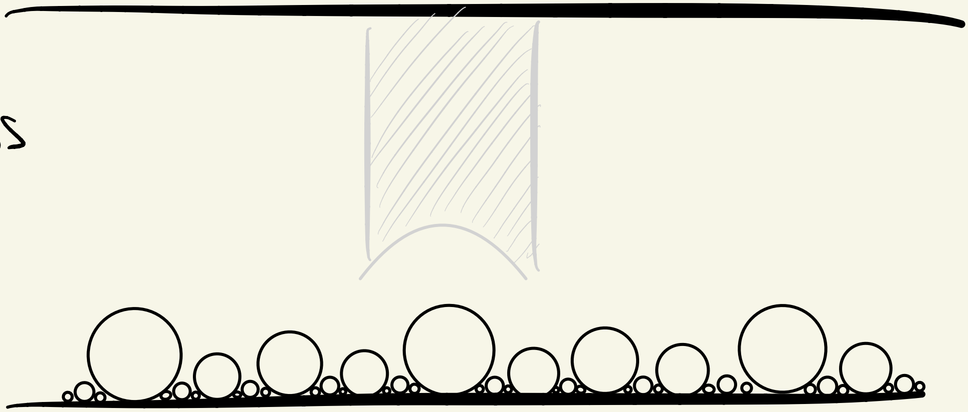
small translation $\mathcal{C}_n^{RBS}(R)$

Unstable algebraic K-theory:

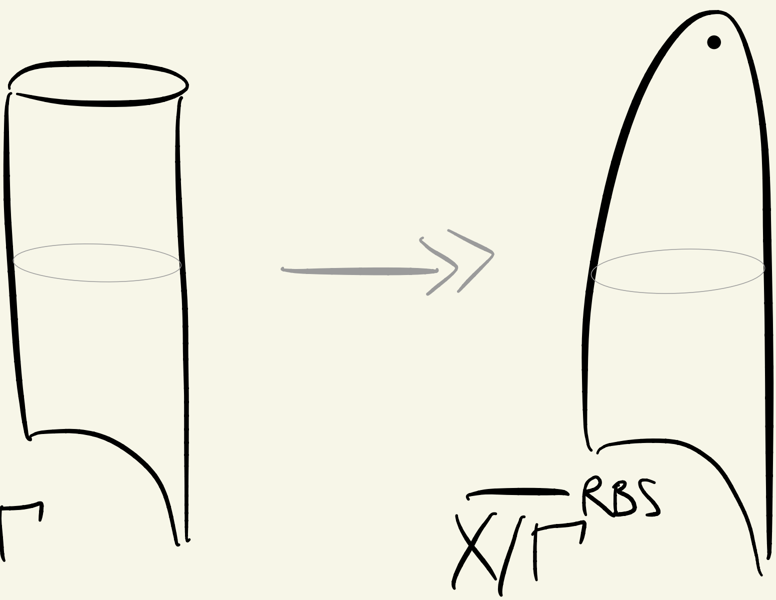
$$\boxed{BGL_n(R) \to |\mathcal{C}_n^{RBS}(R)| \to K(R)}$$

Promising properties....

$\overline{X}^{BS}$



THANK YOU FOR YOUR ATTENTION.



$\overline{X}^{BS}/\Gamma$ $\qquad$ $\overline{X/\Gamma}^{RBS}$