Lecture 1: Representation stability

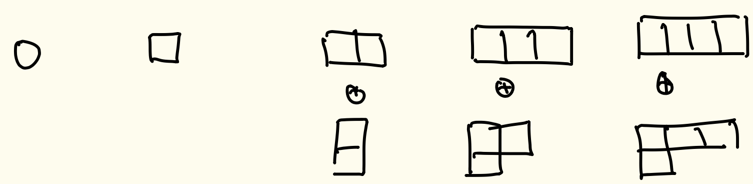
Recall: Irreps of $\mathfrak{S}_n$ over $\mathbb{Q}$ are Specht modules indexed by partitions $\lambda=(\lambda_1\geq\dots\geq\lambda_\ell)$ of size $|\lambda|=\lambda_1+\dots+\lambda_\ell=n$.

My notation: $\mathfrak{S}_n(\lambda)$

Examples of natural sequences of reps:

- Permutation rep: $\mathfrak{S}_n \circlearrowright \mathbb{Q}^n \cong \mathfrak{S}_n(n-1,1)\oplus\mathfrak{S}_n(n)$ if $n\geq 2$

0 1 2 3 4



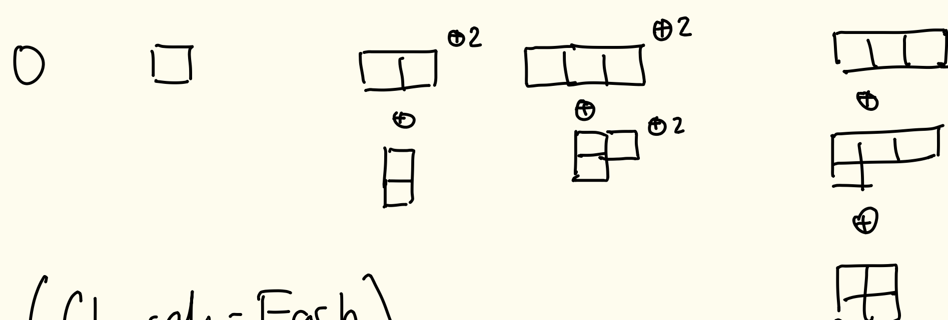
$c_{\emptyset,n}=1$ $n\geq 1$

$c_{\square,n}=1$ $n\geq 2$

- Degree-2 polynomials in n variables: $\mathfrak{S}_n \circlearrowright \mathrm{Sym}^2\mathbb{Q}^n \cong \mathbb{Q}^n\otimes\mathbb{Q}^n/\mathfrak{S}_2$

0 1 2 3 4 $n\geq 4$

basis $\emptyset$ x_1^2 x_1^2, x_2^2, x_1x_2 x_i^2, x_ix_j $i<j$



$\mathfrak{S}_n(n)^{\oplus 2}$ $c_\emptyset$

$\oplus$

$\mathfrak{S}_n(n-1,1)^{\oplus 2}$ $c_\square$

$\oplus$

$\mathfrak{S}_n(n-2,2)$ $c_{\square\square}$

(Church-Farb)

Def A sequence $(V_n)_{n\in\mathbb{N}}$ of $\mathfrak{S}_n$-reps V_n is called <u>(uniformly) multiplicity stable</u> if for

$$V_n \cong \bigoplus_{\lambda} \mathfrak{S}_n(\underbrace{n-|\lambda|, \lambda}_{\lambda[n]})^{\oplus c_{\lambda,n}} \quad \text{for } n-|\lambda| \geq \lambda_1$$

the sequences $(c_{\lambda,n})_{n\in\mathbb{N}}$ stabilize (uniformly).

Ex (Church) M ctd, orient. mfd of dim $d \geq 2$ of finite type

$$\mathrm{Conf}_n M = \mathrm{Emb}([n], M) = \{(x_1,\dots,x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\}$$

$(H^i(\mathrm{Conf}_n M; \mathbb{Q}))_{n\in\mathbb{N}}$ is uniformly mult stable.

Def FI := category of finite sets and injections

FI-mod := functor category FI $\longrightarrow$ $\mathbb{Q}$-mod

(FI is equivalent to a small category)

(morphisms are natural transformations)

Ex FI $\longrightarrow$ $\mathbb{Q}$-mod

$S \longmapsto \mathbb{Q}[S]$ ($\mathbb{Q}$-vs w/ basis S)

Given $V : \mathrm{FI} \longrightarrow \mathbb{Q}\text{-mod}$ let $V_n := V([n])$.

$\rightsquigarrow$ $\mathfrak{S}_n \circlearrowright V_n$ b/c $\mathrm{FI}([n],[n]) = \mathfrak{S}_n$

In fact, for every injection $[m] \hookrightarrow [n]$ we get a map $V_m \to V_n$.

Let $\varphi_n : V_n \longrightarrow V_{n+1}$ denote $V([n] \subset [n+1])$.

Note that every FI-mod V is uniquely determined by the sequence of $\mathfrak{S}_n$-reps $(V_n)_{n\in\mathbb{N}}$ and maps $(\varphi_n)_{n\in\mathbb{N}}$.

Prop A sequence of $\mathfrak{S}_n$-reps $(V_n)_{n\in\mathbb{N}}$ and maps $(\varphi_n : V_n \to V_{n+1})_{n\in\mathbb{N}}$ comes from an FI-module if and only if φ_n is $\mathfrak{S}_n$-equivariant and $1\times\mathfrak{S}_m \subset \mathfrak{S}_{n+m}$ act trivially on the image of the composition

$$V_n \xrightarrow{\varphi_n} V_{n+1} \xrightarrow{\varphi_{n+1}} \cdots \xrightarrow{\varphi_{n+m-1}} V_{n+m}$$

for all $n,m\in\mathbb{N}$.

Proof: Exercise.

Def W is an FI-submodule of V if $W(S)\subset V(S)$ for all finite sets S and for every injection $f: S\hookrightarrow T$ $\mathrm{im}\left(V(f)\big|_{W(S)}\right) \subset W(T)$. (I.e. W_n is an $\mathfrak{S}_n$-subrep of V_n and $\varphi_n\big|_{W_n} : W_n \longrightarrow W_{n+1}$.)

A set $X \subset \coprod_{n\in\mathbb{N}} V_n$ generates V if there is no proper FI-submodule W that contains X.

V is finitely generated if it has a finite generating set.

Thm (Church-Ellenberg-Farb)
A finitely generated FI-module V gives rise to a unif mult stab sequence $(V_n)_{n \in \mathbb{N}}$.

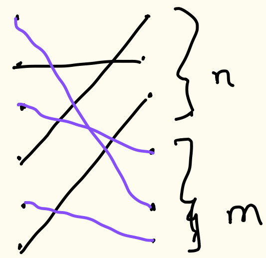
Thm (Church-Ellenberg-Farb (-Nagpal))
An FI-submodule of a finitely generated FI-module is finitely generated.

Ex (CEF) $H^i(\mathrm{Emb}(-, M))$ is a finitly generated FI-mod.
for M $\underset{\text{ctd, oriented}}{\text{mfd}}$ of dim $d \geq 2$ finite type.

(Proof idea : There is a spectral sequence by Totaro which is which has clear generation properties)

Generalizations to other groups

Note that $\mathrm{Inj}([n], [n+m]) \cong G_{n+m} / 1 \times G_m$



Are there other sequences of Groups G_n that define a category w/ morphisms given by

$G_{n+m} / 1 \times G_m$?

Many different setups:

"Homogeneous categories" Randal-Williams–Wahl

"Complemented categories" Putman–Sam

"Stability categories" P.

Def (Stability groupoid)

- Let $(G_n)_{n \in \mathbb{N}}$ be a sequence of groups

$\rightsquigarrow$ $\mathcal{G}$ groupoid w/ objects $\mathbb{N}$ and only automorphisms G_n (cat w/ only isom.)

- Let $\oplus : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ be a $\overset{\text{braided}}{\vee}$ monoidal structure

w/ $\oplus : G_m \times G_n \longrightarrow G_{m+n}$ injective.

- $G_0 = \{1\}$
- $(G_{\ell+m} \times 1) \cap (1 \times G_{m+n}) = 1 \times G_m \times 1 \subset G_{\ell+m+n}$

Ex $\mathcal{G} = (G_n)_{n \in \mathbb{N}}$, $GL(R) = (GL_n(R))_{n \in \mathbb{N}}$, $Sp(R) = (Sp_{2n}(R))_{n \in \mathbb{N}}$, $\beta = (\beta_n)_{n \in \mathbb{N}}$,

$Mod\Sigma = (\operatorname{Mod} \Sigma_{g,1})_{g \in \mathbb{N}}$, $Aut F = (\operatorname{Aut}(F_n))_{n \in \mathbb{N}}$, ...

Def (Quillen) "stability category"

$U\mathcal{G}$ w/ $U\mathcal{G}(m, m+n) \cong G_{m+n} / 1 \times G_n$

Ex $\mathcal{U}G \cong FI$, $\mathcal{U}Sp(R) \cong SI(R)$ f.g. free symplectic R-mods
+ form preserving maps
(automatically inj)

$\mathcal{U}GL(R) \cong VIC(R)$ f.g. free R-mods
+ split injective maps + choice of splitting / complement of image

(exercise)

Thm f.g. $\mathcal{U}\mathcal{G}$-mod $\rightsquigarrow$ unif. mult stab. (Define!)

- $\mathcal{G} = GL(\mathbb{Q})$ with assumption that reps are algebraic (P.)
or $Sp(\mathbb{Q})$
- $\mathcal{G} = GL(\mathbb{F}_q)$ (Gan-Li)

Connection to homological stability

Semi simplicial cat $\Delta_{inj} \longrightarrow \mathcal{U}\mathcal{G}$

$\rightsquigarrow$ $\mathcal{U}\mathcal{G}(-,n)$ semi simplicial set

"H3" (in RWW) : $\tilde{H}_k(\|\mathcal{U}\mathcal{G}(-,n)\|) \cong 0$ for $n \gg k$

Thm H3 $\Rightarrow$ $H_k(G_n) \xrightarrow{\cong} H_k(G_{n+1})$ for $n \gg k$

(Randal-Williams
Wahl / Quillen)

Given a SES

$$1 \to N \to G \longrightarrow Q \to 1$$

we get $Q \circlearrowright H_k(N)$.

<u>Thm</u> (Putman-Sam / P.)

Given a sequence of SES (w/ certain compatibility conditions)

$$1 \to N_n \longrightarrow \underset{\text{stab grp}}{G_n} \longrightarrow \underset{\text{stab grp}}{Q_n} \longrightarrow 1$$

- H3 for G and Q
- Noetherianity for UQ

Then $\left(H_i(N_n)\right)_{n \in \mathbb{N}}$ comes from a f.g UQ-mod.

Lecture 2 Rep stability for diagram algebras

Review: TL_n, Br_n, P_n over $\mathbb{C}$ w/ δ generic

Irreps: $TL_n(\lambda)$ for partitions λ w/ $|\lambda| = n$ $\ell(\lambda) \leq 2$

$Br_n(\lambda)$ for partitions λ w/ $\underset{\text{even}}{n - |\lambda| \geq 0}$

$P_n(\lambda)$ for partitions λ w/ $|\lambda| \leq n$

Write A to mean TL, Br, or P.

Def

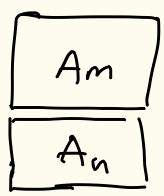
$$V_n = \bigoplus_{\lambda, i} A_n\underbrace{(n-i-|\lambda|, \lambda)}_{\lambda[n-i]}{}^{\oplus c_{n,\lambda,i}} \quad \text{w/} \quad \begin{array}{l} i=0 \text{ for } A=TL \\ i \text{ even for } A=Br \\ i \text{ arb for } A=P \end{array}$$

$n \geq |\lambda| + \lambda_1 + i$

is (unif) multiplicity stable if $(c_{n,\lambda,i})_{n \in \mathbb{N}}$ (unif.) stabilizes.

We want to create stability categories $\mathcal{C}_A$ in analogy to UG:

- $A_m \otimes A_n \hookrightarrow A_{m+n}$



- $A_0 = \mathbb{C}$
- $(A_{\ell+m} \otimes \mathbb{C}) \cap (\mathbb{C} \otimes A_{m+n}) = \mathbb{C} \otimes A_m \otimes \mathbb{C} \subset A_{\ell+m+n}$

Recall: $\mathbb{C}\mathcal{UG}(n, n+m) = \mathbb{C}[G_{n+m}/G_m] \cong \mathrm{Ind}_{G_m}^{G_{n+m}} \mathbb{C}$

What is the trivial rep for A_n?

$\mathbb{C} \cong A_n(n)$ on which invertible diagrams (i.e. permutatio act by the identity and other diagrams by zero.

Define $\mathcal{C}_A$ to be the category on objects $\mathbb{N}$ and

$$\mathcal{C}_A(m,n) = \begin{cases} \mathrm{Ind}_{\mathbb{C}\otimes A_{n-m}}^{A_n} \mathbb{C} = A_n \otimes_{A_{n-m}} \mathbb{C} & \\ 0 & m > n. \end{cases}$$

with composition

$$\mathcal{C}(m,n) \times \mathcal{C}(\ell,m) \to \mathcal{C}(m,n) \otimes_{A_m} \mathcal{C}(\ell,m)$$

$$\to \left(A_n \otimes_{A_{n-m}} \mathbb{C}\right) \otimes_{A_m} \left(A_m \otimes_{A_{m-\ell}} \mathbb{C}\right)$$

$$\cong A_n \otimes_{A_{n-m}\otimes A_m} \left(\mathbb{C} \boxtimes \left(A_m \otimes_{A_{m-\ell}} \mathbb{C}\right)\right)$$

$$\cong A_n \otimes_{A_{n-m}\otimes A_{m-\ell}} (\mathbb{C} \boxtimes \mathbb{C}) \to A_n \otimes_{A_{n-\ell}} \mathbb{C} = \mathcal{C}(\ell,n).$$

Def A $\mathcal{C}_A$-module V is a linear functor $V : \mathcal{C}_A \to \mathbb{C}\text{-mod}$,

i.e. a functor such that

$$\mathcal{C}_A(m,n) \longrightarrow \mathrm{Hom}(V_m, V_n) \quad \text{is a linear map.}$$

$\rightsquigarrow$ sequence $(V_n)_{n\in\mathbb{N}}$ of A_n-modules and $\varphi_n : V_n \longrightarrow V_{n+1}$

coming from $1_{n+1} \underset{A_1}{\otimes} 1 \in A_{n+1} \underset{A_1}{\otimes} \mathbb{C} = \mathcal{C}_A(n, n+1)$.

Exercise: What charaterizes sequences coming from $\mathcal{C}_A$-modules?

Def Let $M(m)$ be the "free" $\mathcal{C}_A$-module $\mathcal{C}_A(m,-)$.

A $\mathcal{C}_A$-module V is finitely generated if there is a surjection

$$\bigoplus_{i\in I} M(m_i) \longrightarrow\!\!\!\!\!\rightarrow V$$

for a finite set I. (Exercise: compare to other definition)

V is finitely presented if there is a right exact sequence

$$\bigoplus_{j\in J} M(m_j) \longrightarrow \bigoplus_{i\in I} M(m_i) \longrightarrow V \longrightarrow 0$$

for finite sets I, J.

Thm (P.) For $A_n = TL_n, Br_n$, or P_n, a f.p. $\mathcal{C}_A$-module V gives rise to a uniformly multiplicity stable sequence $(V_n)_{n\in\mathbb{N}}$.

Thm (Gan-Ta) Submodules of f.g. $\mathcal{C}_A$-modules are f.g.

Homological stability

In a much more general setting over a commutative ring R and an (almost) arbitrary $\delta \in R$ one can show:

<u>Thm</u> (Boyd-Hepworth) $\delta = r + r^{-1}$ for some unit $r \in R^\times$

$$\mathrm{Tor}_d^{TL_n(\delta)}(R,R) = 0 \quad \text{for} \quad n \geq d-1.$$

<u>Thm</u> (Boyd-Hepworth-P.) $R\mathfrak{S}_n \hookrightarrow Br_n(\delta)$ induces an iso

$$H_d(\mathfrak{S}_n; R) \cong \mathrm{Tor}_d^{R\mathfrak{S}_n}(R,R) \xrightarrow{\cong} \mathrm{Tor}_d^{Br_n(\delta)}(R,R)$$

for $n \geq 2d+1$.

<u>Thm</u> (Boyd-Hepworth-P.) $R\mathfrak{S}_n \hookrightarrow P_n(\delta)$ induces an iso

$$H_d(\mathfrak{S}_n; R) \xrightarrow{\cong} \mathrm{Tor}_d^{P_n(\delta)}(R,R)$$

for $n \geq 2d+1$.

Recall: $\mathcal{C}_{Br}(m,n) = Br_n \otimes_{Br_{n-m}} \mathbb{C}$

$$M(m) = \mathcal{C}_{Br}(m,-) : \mathcal{C}_{Br} \longrightarrow \mathbb{C}\text{-mod}$$

Goal: $\displaystyle\bigoplus_{\substack{j\in J \\ \text{fin.}}} M(m_j) \longrightarrow \bigoplus_{\substack{i\in I \\ \text{fin.}}} M(m_i) \longrightarrow V \longrightarrow 0$

$m_j \leq r \qquad m_i \leq g$

$$\Rightarrow \quad V_n = \bigoplus Br_n(\mu[n-2i])^{c_{\mu,i,n}}$$

$(c_{\mu,i,n})_{n\in\mathbb{N}}$ stabilizes for $n \geq 2g + \max(g,r)$

Lecture 3: Proof of representation stability for Brauer algebras

Overview:

Step 1: Define a functor $\tau_{n,m} : Br_n\text{-mod} \longrightarrow Br_m\text{-mod}$ $(m \leq n)$ and show that $(\tau_{n,m} V_n)_{n\in\mathbb{N}}$ stabilizes if V is f.g. (for fixed m)

Step 2: Analyze the multiplicity of $Br_m(\lambda)$ in $\tau_{n,m} Br_n(\mu)$ using branching rules.

Step 3: Use the decomposition $V_n = \bigoplus_{\mu, i} Br_n(n-2i-|\mu|, \mu)^{\oplus c_{\mu,i,n}}$

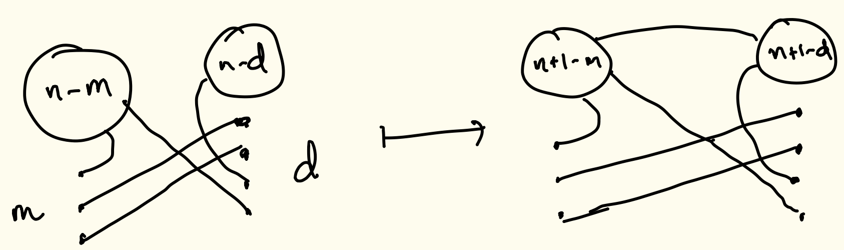
And step 1+2 to show that $c_{\mu,i,n}$ stabilizes.

Step 1: $\tau_{n,m} : Br_n\text{-mod} \longrightarrow Br_m\text{-mod}$

$$V_n \longmapsto \mathbb{C} \otimes_{Br_{n-m}} \mathrm{Res}^{Br_n}_{Br_{n-m}\otimes Br_m} V_n$$

Prop $\tau_{n,m} M(d)_n \cong \mathbb{C} \otimes_{Br_{n-m}} Br_n \otimes_{Br_{n-d}} \mathbb{C}$ stabilizes for $n \geq m+d$.

Proof:



is always injective and surjective if $n \geq m+d$ b/c $m+d$ point cannot connect to $2n+2-m-d > m+d$ blob entries.

$\square$

Prop: If V is f.p. generated in degrees $\leq g$ and related in degrees $\leq r$ then $(\tau_{n,m} V_n)_{n\in\mathbb{N}}$ stabilizes for $n \geq m + \max(g,r)$.

Proof: Exercise:

(a) Check that if V is generated in degrees $\leq g$,

$\tau_{n,m} V_n \longrightarrow \tau_{n+1,m} V_{n+1}$ is surjective for $n \geq m+g$

using the diagram

$$\begin{tikzcd} \tau_{n,m} F_n \arrow[r] \arrow[d] & \tau_{n,m} V_n \arrow[d] \\ \tau_{n+1,m} F_{n+1} \arrow[r] & \tau_{n+1,m} V_{n+1} \end{tikzcd}.$$

(b) Consider SES $0 \to K \to F \to V \to 0$

w/ K f.g. in degrees $\leq r$. Use Four Lemma to show that $\tau_{n,m} V_n \longrightarrow \tau_{n+1,m} V_{n+1}$ is injective for $n \geq m+r$. $\square$

Step 2: $\mathrm{Res}^{Br_n}_{Br_{n-m} \otimes Br_m} V_n$ decomposes into irreducible $Br_{n-m} \otimes Br_m$-modules $\bigoplus W_i \otimes W_i'$ and $\tau_{n,m} V_n \cong \bigoplus_{W_i \text{ trivial}} W_i'$.

$$\rightsquigarrow \left[Br_m(\lambda), \tau_{n,m} Br_n(\mu) \right] \cong \left[\mathbb{C} \otimes Br_m(\lambda), \mathrm{Res}^{Br_n}_{Br_{n-m} \otimes Br_m} Br_n(\mu) \right]$$

multiplicity

Thm (Branching rules)

Littlewood–Richardson coefficients

$$\left[Br_m(\mu) \otimes Br_n(\nu), \mathrm{Res}^{Br_{m+n}}_{Br_m \otimes Br_n} Br_{m+n}(\nu) \right] = d^{\lambda}_{\mu\nu} := \sum_{\alpha,\beta,\gamma} c^{\lambda}_{\alpha\beta} c^{\mu}_{\alpha\gamma} c^{\nu}_{\beta\gamma}$$

Exercise: Derive this from Branching rules of $O(n)$

Littlewood-Richardson coefficients:

$$c^{\lambda}_{\mu\nu} = \left[\mathfrak{S}_{m+n}(\lambda),\ \mathrm{Ind}^{\mathfrak{S}_{m+n}}_{\mathfrak{S}_m \times \mathfrak{S}_n} \mathfrak{S}_m(\mu) \otimes \mathfrak{S}_n(\nu) \right]$$

$$\underset{\text{Frob}}{=} \left[\mathfrak{S}_m(\mu) \otimes \mathfrak{S}_n(\nu),\ \mathrm{Res}^{\mathfrak{S}_{m+n}}_{\mathfrak{S}_m \times \mathfrak{S}_n} \mathfrak{S}_{m+n}(\lambda) \right]$$

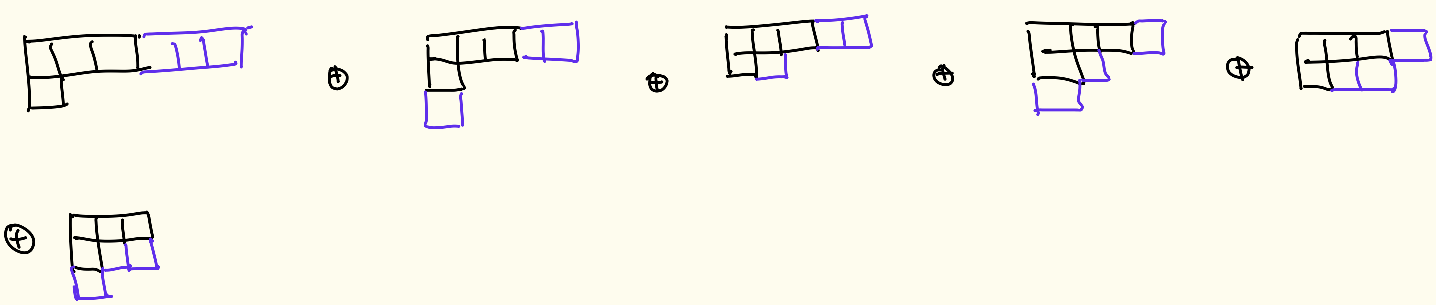
$$\underset{\substack{\text{Schur-}\\ \text{Weyl}}}{=} \left[GL_k(\lambda),\ GL_k(\mu) \otimes GL_k(\nu) \right] \qquad \text{if } k \geq \ell(\lambda), \ell(\mu)+\ell(\nu)$$

$$= \left[GL_m(\mu) \otimes GL_n(\nu),\ \mathrm{Res}^{GL_{m+n}}_{GL_m \times GL_n} GL_{m+n}(\lambda) \right]$$

Properties we need:

- $c^{\lambda}_{\mu\nu} = c^{\lambda}_{\nu\mu}$ • $c^{\lambda}_{\mu\nu} = 0$ if $|\lambda| \neq |\mu| + |\nu|$
- $c^{\lambda}_{\mu\nu} = 0$ if $\mu \not\subseteq \lambda$ ($\mu \subseteq \lambda :=$ $\mu_i \leq \lambda_i$ for all i)
- $c^{\lambda}_{\mu\,(n)} = \begin{cases} 1 & \text{if } \mu \text{ can be obtained from } \lambda \text{ by removing at most one box per column} \\ 0 & \text{otw} \end{cases}$

<u>Ex</u> $\mathrm{Ind}^{\mathfrak{S}_7}_{\mathfrak{S}_4 \otimes \mathfrak{S}_3} \mathfrak{S}_4((3,1)) \otimes \mathfrak{S}_3((3)) \cong$



$\rightsquigarrow \left[Br_m(\lambda), \ T_{n,m} Br_n(\mu) \right] = d^{\mu}_{(n-m),\lambda} = \sum c^{(n-m)}_{\alpha\beta} c^{\lambda}_{\alpha\gamma} c^{\mu}_{\beta\gamma}$

$$= \sum_{\ell,\gamma} c^{\lambda}_{(\ell)\gamma} c^{\mu}_{(n-m-\ell)\gamma}$$

Prop: $m - |\lambda| \geq 0$ even, $n - 2i - |\mu| \geq \mu_1$

$$\left[Br_m(\lambda), T_{n,m} Br_n(n-2i-|\mu|, \mu) \right] = \begin{cases} 0 & \text{if } |\lambda| < m-2i \text{ or } \\ & |\lambda| = m-2i \text{ and } |\mu| \geq m-2i \\ & \text{and } \lambda \neq \mu \\ 1 & \text{if } \lambda = \mu \text{ and } |\lambda| = m-2i. \end{cases}$$

It is always independent of $n \geq m - i + \mu_1$.

Proof example: (a) $c^{(n-2i-|\mu|,\mu)}_{(n-m-\ell)\gamma} \neq 0 \Rightarrow n - 2i = n - m - \ell + |\gamma|$

$$\Rightarrow |\gamma| = m + \ell - 2i$$

$c^{\lambda}_{(\ell),\gamma} \neq 0 \Rightarrow |\lambda| = \ell + |\gamma| = m + 2\ell - 2i \geq m - 2i$.

(b) $|\lambda| = m - 2i \Rightarrow \ell = 0 \Rightarrow \gamma = \lambda$ so

$c^{(n-2i-|\mu|,\mu)}_{(n-m)\lambda} \neq 0 \Rightarrow \underset{\text{(first row)}}{n - 2i - |\mu|} \geq n - m \Rightarrow |\mu| \leq m - 2i$

(b') $|\mu| = m - 2i \Rightarrow n - 2i - |\mu| = n - m$ and so $c^{(n-m,\mu)}_{(n-m)\lambda} = \begin{cases} 0 & \mu \neq \lambda \\ 1 & \mu = \lambda \end{cases}$.

(c) From (a) $|\lambda| = m + 2\ell - 2i$. B/c $|\lambda| \leq m$, $0 \leq \ell \leq i$. We want to show that

$$\sum_{\gamma, 0 \leq \ell \leq i} c^{\lambda}_{(\ell)\gamma} c^{(n-2i-|\mu|,\mu)}_{(n-m-\ell)\gamma}$$

is independent of $n \geq m + i + \mu_1$.

The set of γ w/ $c^{\lambda}_{(\ell)\gamma} \neq 0$ is independent of n.

Exercise: $c^{(n-|\nu|,\nu)}_{(n-k)\,\eta}$ is independent of $n \geq k+\nu_1$

$\Rightarrow c^{(n-2i-|\mu|,\mu)}_{(n-m-\ell)\,\gamma}$ is independent of $n \geq m+\ell-2i+\mu_1$

which is true if $n \geq m-i+\mu_1$

b/c $\ell \leq i$. $\square$

Prop: If V is a f.g. $\mathcal{C}_{Br}$-module generated in deg g and $[Br_n(n-2i-|\mu|,\mu), V_n] \neq 0$, then $|\mu|+i \leq g$.

Proof: It's enough to show this for $V = M(d)$ w/ $d \leq g$. So assume

$$d^{(n-2i-|\mu|,\mu)}_{(n-d)\,\lambda} = \sum_{\gamma,\, 0\leq \ell \leq i} c^{(n-2i-|\mu|,\mu)}_{(n-d-\ell)\,\gamma}\, c^{\lambda}_{(\ell)\,\gamma} \neq 0 \text{ for some } \lambda. \Rightarrow n-2i-|\mu| \geq n-d-\ell$$

$\Leftrightarrow |\mu|+i \leq i-\ell+d \quad \Rightarrow \quad |\mu|+i \leq g.$ $\square$

Step 3:

Thm If V is a f.p. $\mathcal{C}_{Br}$-module generated in deg g and presented in deg r, and

$$V_n = \bigoplus_{i,\mu} Br_n(n-2i-|\mu|,\mu)^{\oplus c_{n,\mu,i}},$$

then $(c_{n,\mu,i})_{n\in\mathbb{N}}$ is independent of $n \geq 2g+\max(g,r)$.

Proof: We only have to consider (μ,i) w/ $|\mu|+i \leq g$. (finitely many)

Induction: $i\downarrow$, $|\mu|\uparrow$ so $(\emptyset, g) \to \cdots \to (\emptyset, 0) \to (\square, g-1) \to \cdots \to (\square, 0) \to \cdots$

Fix (λ, j) and assume

the statement for (μ, i) s.t. $j<i$ or ($j=i$ and $|\lambda|>|\mu|$).

We now want to prove the statement for (λ, j).

Set $m := |\lambda| + 2j$. $(\leq 2g)$ $\qquad (|\lambda| = m - 2j)$

Consider the decomposition

independent of $n \geq m + \max(g,r)$

$$\left[Br_m(\lambda),\ \tau_{n,m} V_n \right]$$

zero

$$= \sum_{\substack{i<j \\ \mu}} \left[Br_m(\lambda),\ \tau_{n,m} Br_n(n-2i-|\mu|, \mu) \right] \cdot c_{\mu,i,n}$$

independent of $n \geq 2g + \max(g,r)$ by induction

$$+ \sum_{\substack{i>j \\ \mu}} \left[Br_m(\lambda),\ \tau_{n,m} Br_n(n-2i-|\mu|, \mu) \right] \cdot c_{\mu,i,n}$$

independent of $n \geq m - i + \mu_1$

zero

$$+ \sum_{\substack{i=j \\ |\mu|>|\lambda|}} \left[Br_m(\lambda),\ \tau_{n,m} Br_n(n-2i-|\mu|, \mu) \right] \cdot c_{\mu,i,n}$$

independent of $n \geq 2g + \max(g,r)$ by induction

$$+ \sum_{\substack{i=j \\ |\mu|<|\lambda|}} \left[Br_m(\lambda),\ \tau_{n,m} Br_n(n-2i-|\mu|, \mu) \right] \cdot c_{\mu,i,n}$$

independent of $n \geq m - i + \mu_1$

$$+ \sum_{\substack{i=j \\ |\mu|=|\lambda|}} \left[Br_m(\lambda),\ \tau_{n,m} Br_n(n-2i-|\mu|, \mu) \right] \cdot c_{\mu,i,n}$$

$$\begin{cases} 1 & \lambda = \mu \\ 0 & \lambda \neq \mu \end{cases}$$

$$\Rightarrow c_{\lambda,j,n} \text{ independent of } n \geq \max\big(m + \max(g,r),\ 2g + \max(g,r),\ m - i + \mu_1 \big)$$

$$= 2g + \max(g,r). \qquad \square$$

More:

Ideas for partition algebras:

$$\left[\mathfrak{S}_n(\lambda),\ \mathfrak{S}_n(\mu)\otimes\mathfrak{S}_n(\nu)\right] =: g_{\lambda,\mu,\nu}$$ Kronecker coefficients

independent of order

Murnaghan's Thm: λ,μ,ν partitions (not necessarily the same size)

$\left(g_{\lambda[n],\mu[n],\nu[n]}\right)_n$ stabilizes.

Define $\bar{g}_{\lambda,\mu,\nu}$ to be the limit.

$$\left[P_{m+n}(\lambda),\ \mathrm{Ind}_{P_m\otimes P_n}^{P_{m+n}} P_m(\mu)\otimes P_n(\nu)\right] = \bar{g}_{\lambda,\mu,\nu}.$$

$g_{\lambda,\mu,\nu}$ and $\bar{g}_{\lambda,\mu,\nu}$ are in general very difficult!

But we can use:

- $\bar{g}_{\lambda,\mu,\nu} = 0$ if $|\lambda|+|\mu| < |\nu|$
- $\bar{g}_{\lambda,\mu,\nu} = c^{\nu}_{\lambda\mu}$ if $|\lambda|+|\mu| = |\nu|$
- $\bar{g}_{\lambda,\mu,(k)} = \sum c^{\lambda}_{\pi,\alpha,(k_1)}\, c^{\mu}_{\pi,\alpha,(k_2)}$

where $c^{\lambda}_{\alpha,\beta,\gamma} = \left[\mathfrak{S}_a(\alpha)\otimes\mathfrak{S}_b(\beta)\otimes\mathfrak{S}_c(\gamma),\ \mathrm{Res}^{\mathfrak{S}_{a+b+c}}_{\mathfrak{S}_a\times\mathfrak{S}_b\times\mathfrak{S}_c}\mathfrak{S}_{a+b+c}(\lambda)\right]$

Open Problems:

1. Find an interesting f.g. $\mathcal{C}_A$-module for which we don't know how to compute completely.
2. Extend the theory to other diagram algebras.
3. What can one say in the non-semisimple setting?

Exercises for Representation Stability and Diagram algebras

Lectures 1+2:

① Given a sequence $(V_n)_{n\in\mathbb{N}}$ of $\mathfrak{S}_n$-representations and a sequence of maps $(\varphi_n : V_n \to V_{n+1})_{n\in\mathbb{N}}$, there is an FI-module V with $V([n]) = V_n$ and $V([n]\subset[n+1]) = \varphi_n$ if and only if

(a) φ_n is $\mathfrak{S}_n$-equivariant,

(b) the stabilizer of $[n]$ in $\mathfrak{S}_{n+m}$ acts trivially on the image of the composition

$$V_n \xrightarrow{\varphi_n} V_{n+1} \xrightarrow{\varphi_{n+1}} \cdots \xrightarrow{\varphi_{n+m-1}} V_{n+m}.$$

② Let R be a commutative ring. Consider $GL_m R$ as a subgroup of $GL_{n+m} R$ be the block embedding $A \longmapsto \left(\begin{array}{c|c} I_n & 0 \\ \hline 0 & A \end{array}\right)$.

Show that $GL_{n+m} R / GL_m R$ as a isomorphic to

(a) $\{ (f, s) \mid f : R^n \longrightarrow R^{n+m},\ s : R^{n+m} \longrightarrow R^n,\ s \circ f = I_n \}$

(b) $\{ (f, C) \mid f : R^n \hookrightarrow R^{n+m}$ monomorphism, C submodule of R^{n+m}, $\operatorname{im} f \oplus C = R^{n+m},\ C \cong R^m \}$

as left $GL_{n+m} R$-sets and as right $GL_n R$-sets. (What are the actions?)

(3) (a) Let $\mathcal{C}$ be a category and $V: \mathcal{C} \to \mathbb{C}\text{-mod}$ (a "$\mathcal{C}$-module") a functor. Given a set S of pairs (v, c) with $v \in V(c)$, there is a functor

$$F = \bigoplus_{(v,c) \in S} \mathbb{C}[\mathcal{C}(c, -)] : \mathcal{C} \longrightarrow \mathbb{C}\text{-mod}$$

and a natural transformation $F \longrightarrow V$ that sends id_c to v for every pair $(v,c) \in S$. The image of this natural transformation is the smallest $\mathcal{C}$-submodule of V containing a v's in S.
(This is essentially Yoneda's Lemma.)

(b) Formulate and prove an analogous statement for linear categories $\mathcal{C}$.

Exercises for Representation Stability and Diagram algebras

Lectures 3+4:

(4) (a) Check that if V is a $\mathcal{C}_{Br}$-module that is generated in degrees $\leq g$, then

$\tau_{n,m} V_n \longrightarrow \tau_{n+1,m} V_{n+1}$ is surjective for $n \geq m+g$

using the diagram

$$\begin{tikzcd} \tau_{n,m} F_n \arrow[r] \arrow[d] & \tau_{n,m} V_n \arrow[d] \\ \tau_{n+1,m} F_{n+1} \arrow[r] & \tau_{n+1,m} V_{n+1} \end{tikzcd}.$$

(b) Consider SES $0 \to K \to F \to V \to 0$ w/ K f.g. in degrees $\leq r$. Use Four Lemma to show that $\tau_{n,m} V_n \longrightarrow \tau_{n+1,m} V_{n+1}$ is injective for $n \geq m+r$.

(5) Given

Theorem 3.7 ([HTW05, 2.1.2]). *Given nonnegative integer partitions λ, μ, ν such that $\ell(\lambda) \leq \lfloor n/2 \rfloor$ and $\ell(\mu) + \ell(\nu) \leq \lfloor n/2 \rfloor$, then*

$$[\mathrm{O}_n(\mu) \otimes \mathrm{O}_n(\nu), \mathrm{O}_n(\lambda)] = \sum_{\alpha,\beta,\gamma} c^{\lambda}_{\alpha\beta} c^{\mu}_{\alpha\gamma} c^{\nu}_{\beta\gamma} =: d^{\lambda}_{\mu\nu}.$$

Prove

Corollary 3.8. *Let $\delta \in \mathbb{C} \setminus \{m \in \mathbb{Z} \mid 4 - 2(e+f) \leq m \leq e+f-2\}$. Then*

$$[\mathrm{Res}^{\mathrm{Br}_{e+f}}_{\mathrm{Br}_e \otimes \mathrm{Br}_f} \mathrm{Br}_{e+f}(\lambda), \mathrm{Br}_e(\mu) \otimes \mathrm{Br}_f(\nu)] = d^{\lambda}_{\mu\nu} = \sum_{\alpha,\beta,\gamma} c^{\lambda}_{\alpha\beta} c^{\mu}_{\alpha\gamma} c^{\nu}_{\beta\gamma}.$$

Using Schur-Weyl duality.

(6) $c^{(n-|\nu|,\nu)}_{(n-k)\eta}$ is independent of $n \geq k + \nu_1$ using Pieri rule.