KØBENHAVNS UNIVERSITET

Bachelor thesis in mathematics

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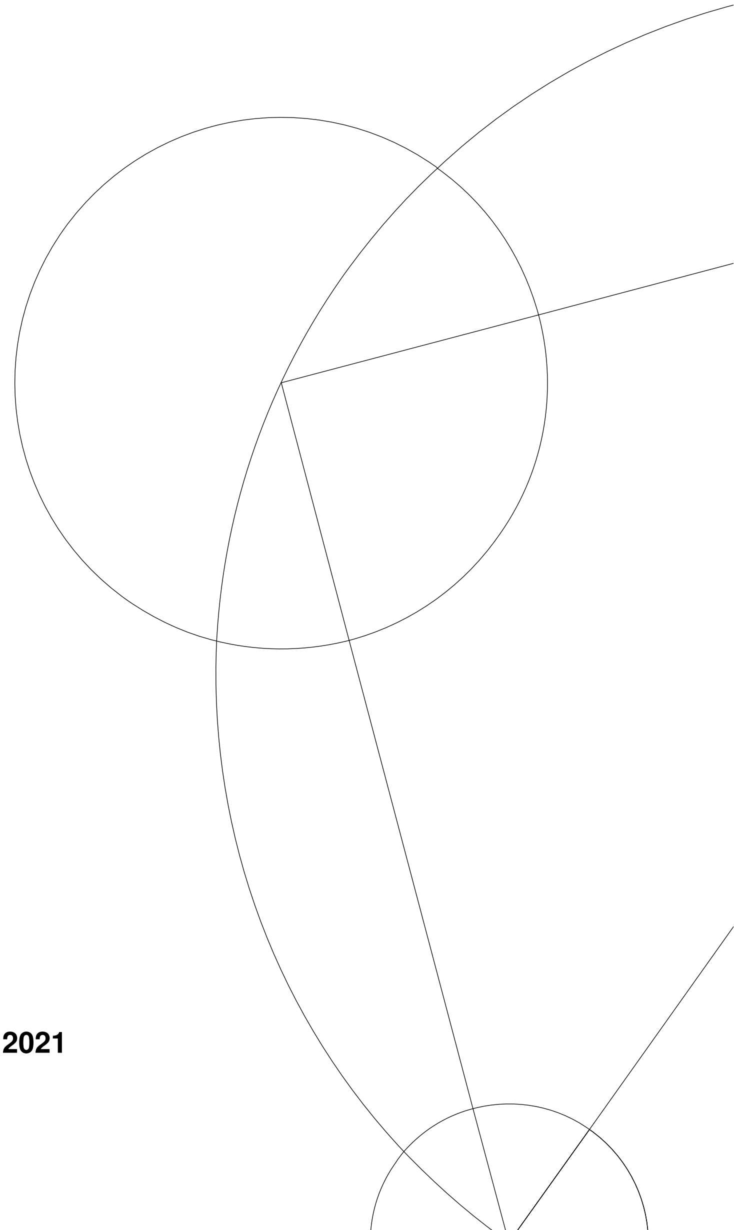
Perfect forms and Voronoi Domains



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Abstract

This thesis introduces the perfect quadratic forms, which are quadratic forms on $\mathbb{R}^n$, whose minimal vectors x_i have the property that the set of $x_i x_i^t$ spans the Voronoi space of perfect quadratic forms. We see that a perfect form is uniquely defined by its minimal vectors up to proportionality, which leads us to the result that there are finitely many classes of perfect forms modulo $GL_n(\mathbb{Z})$ up to proportionality. The thesis then goes on to define the Voronoi domains of positive definite quadratic forms. The Voronoi domains are polyhedral cones in the Voronoi space that we come to see constitute a polyhedral complex, which covers the interior of the Voronoi cone of positive semi-definite quadratic forms.

Contents

Introduction

This thesis concerns perfect forms and the polyhedral cones called Voronoi domains in the Voronoi space of quadratic forms. Perfect forms are used in study of sphere packings and their Voronoi domains have applications to group homology, especially concerning the group homology of $GL_n(\mathbb{Z})$.

Section 0 is a quick run through of the preliminaries for the thesis. Here we define a quadratic form and a lattice, and mention important results in linear algebra that are used throughout the thesis.

We work towards two big theorems; one in section 1 and one in section 2. The first of these statements is that *there are finitely many classes of perfect quadratic forms modulo* $GL_n(\mathbb{Z})$, which is the statement presented in the end of section 1. In order to prove this, we need to define and understand what a perfect quadratic form is. We therefore go through a couple of examples and introduce some of the most basic properties of a perfect form. We then prove that *a perfect quadratic form is uniquely defined by its minimal set up to proportionality.* From earlier in the section, we know that not all sets can be a minimal set, but provided that a subset $S \subseteq \mathbb{R}^n$ is a minimal set that satisfies the perfection property, there is only one perfect quadratic form that has this minimal set up to a scaling by a positive real. Finally we introduce the classes of perfect quadratic forms modulo $GL_n(\mathbb{Z})$, as well as the Hadamard and Hermite inequalities in order to prove the last statement in section 1.

Section 2 centers about the Voronoi domains of positive definite quadratic forms and aims towards proving that they constitute a polyhedral complex in the Voronoi space of quadratic forms. The section builds on the results in the previous section, as we see that the Voronoi domains of perfect forms are the only full dimensional Voronoi domains in the Voronoi space. In order to prove that the Voronoi domains constitute a polyhedral complex, we first get more familiar with Voronoi domains, by looking at the properties and relations between the Voronoi domains of different quadratic forms. One important theorem that we prove in the section, is the one that states *for a given positive definite quadratic form* Q_0*, there are finitely many perfect forms up to proportionality that satisfy* $\langle Q, Q_0 \rangle \leq K$*, for a given* $K > 0$. We need this to prove that the polyhedral complex of Voronoi domains cover the interior of the Voronoi cone, which is the set of positive semi-definite quadratic form in the Voronoi space.

Included here is a short list of notation and conventions used throughout the thesis.

- A column vector $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is in a text environment denoted $(x_1, .., x_n)$.
- $Sym_n(\mathbb{R})$ denotes the vector space of symmetric $n \times n$-matrices with real entries.
- $GL_n(\mathbb{Z})$ is the general linear group over the integers. $GL_n(\mathbb{Z})$ consists of the set of $n \times n$-matrices with integer entries and determinant either 1 or -1, together with the operation which is the regular matrix multiplication.
- The additive identity is generally denoted 0. This could be the $n \times n$-matrix with 0 in all entries or the 0-vector in $\mathbb{R}^n$, depending on context.
- ♣ marks the end of an example.

0 Preliminaries

0.1 Important definitions and results

This section is a collection of definitions and propositions that will be used frequently throughout the thesis. The main references for the definitions and propositions are the book "Lineær Algebra" [Hesselholt and Wahl(2017)] and the book by Martinet [Martinet(2003)]. No proofs will be included in this section, but they can be read in the books mentioned.

As this thesis only treats Euclidean spaces, all definitions are made for $\mathbb{R}^n$.

Definition 0.1. A *bilinear form* B on $\mathbb{R}^n$ is a map $B : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ that is linear in both arguments, which means that any $x, y, z \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ satisfy the following,

$$\begin{aligned} B(x, y+z) &= B(x,y) + B(x,z) \text{ and } B(\alpha x, y) = \alpha B(x,y), \\ B(x+z, y) &= B(x,y) + B(z,y) \text{ and } B(x, \alpha y) = \alpha B(x,y). \end{aligned}$$

Definition 0.2. A *quadratic form* on $\mathbb{R}^n$ is a map $Q : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $Q(x) = B(x,x)$ for all $x \in \mathbb{R}^n$ and a bilinear form B on $\mathbb{R}^n$.

Remark. Notice that $Q(\alpha x) = \alpha^2 Q(x)$ for any quadratic form Q, $x \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$.

Proposition 0.3. Let Q be a quadratic form on $\mathbb{R}^n$. There exists a unique symmetric $n \times n$-matrix A such that $Q(x) = x^t A x$ for all $x \in \mathbb{R}^n$.

Definition 0.4.

(i) A *positive (semi-)definite* matrix A, is a matrix with positive (non-negative) eigenvalues.

(ii) A *positive (semi-)definite* quadratic form Q, is a form that satisfies $Q(x) > 0$ ($Q(x) \geq 0$) for all $x \in \mathbb{R}^n \backslash \{0\}$.

Proposition 0.5. The matrix corresponding to a positive (semi-) definite quadratic form is positive (semi-) definite.

Proposition 0.6. A symmetric matrix A is diagonizable by an orthogonal matrix. We can therefore write $A = UDU^t$ where D is a diagonal matrix with real entries, U is an orthogonal matrix and U^t is the transposed matrix of U.

Definition 0.7. The *trace* of a matrix A, denoted $Tr(A)$, is the sum of the diagonal elements.

Proposition 0.8. Let A and B be matrices of dimension $n \times m$ and $m \times n$ respectively. Then $Tr(AB) = Tr(BA)$.

Remark. Note that the above result means that the trace is invariant up to a change of basis. Hence, if A is diagonizable, the trace of A is the sum of the eigenvalues of A.

Definition 0.9. A set $A \subseteq \mathbb{R}^n$ is called *convex* if for every two points a, b in A the straight line between the two points are also contained in A.

Definition 0.10. The *convex hull* of a subset A of $\mathbb{R}^n$ is defined as follows:

$$\text{convex hull of of } A = \bigcap_{\substack{A \subseteq C \subseteq \mathbb{R}^n, \\ C \text{ convex}}} C$$

0.2 Lattices

A lot of the theory about perfect forms that will be introduced in the next section has a direct translation to the theory about perfect lattices, which will not be introduced in this thesis (see [Martinet(2003)] if interested). *However* lattices can still provide a useful way of thinking about quadratic forms and the following short introduction is therefore included. This subsection is based on [Martinet(2003)].

Definition 0.11. A *lattice* Λ in $\mathbb{R}^n$ is the set of integer linear combinations of the elements in its basis. And a *basis of a lattice* in $\mathbb{R}^n$ is a basis of a subspace of $\mathbb{R}^n$ of dimension $k \leq n$. A lattice Λ is called a *full* lattice or simply a lattice if its basis span $\mathbb{R}^n$ and a *relative* lattice otherwise.

Definition 0.12. From now on we shall refer to the *norm* $N(x)$ of an element $x \in \mathbb{R}^n$, given as

$$N(x) = \langle x, x \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$.

Note that this norm is the square of the usual norm.

Definition 0.13.

(i) The *minimum* of a lattice Λ, denoted $m(\Lambda)$, is the minimal norm of the non-zero elements in Λ:

$$m(\Lambda) = \min_{x \in \Lambda \setminus \{0\}} N(x).$$

(ii) The *minimal* set of a lattice Λ is the set

$$S(\Lambda) = \{x \in \Lambda \mid N(x) = m(\Lambda)\}.$$

The elements of $S(\Lambda)$ are called *minimal elements.*

In Figure 1 we see two examples of lattices in $\mathbb{R}^2$, where the minimal sets are seen as the lattice points on the circles. Note the difference in the order of the two minimal sets.

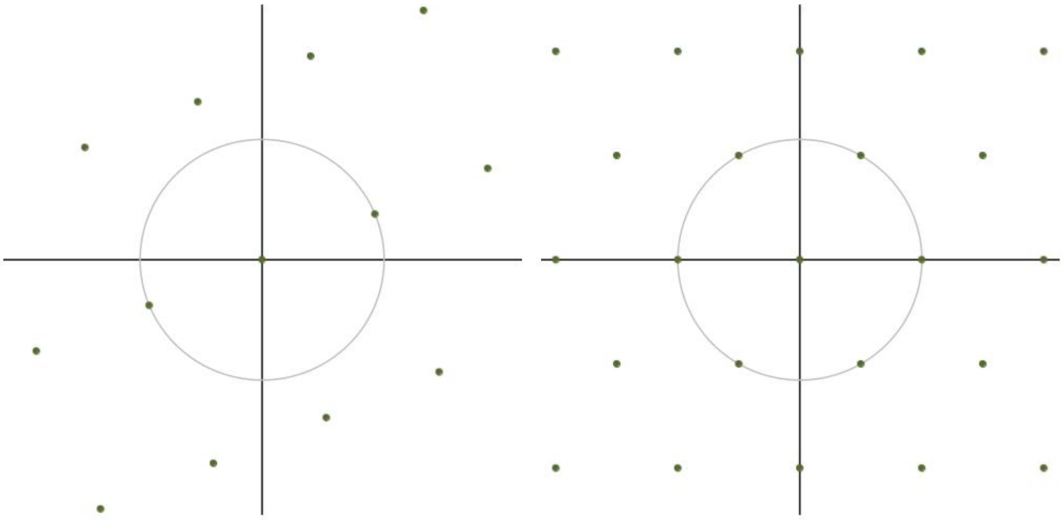

Figure 1: Random and hexagonal lattice in $\mathbb{R}^2$. The gray circles have the minimum of the lattice as radius squared. The minimal elements are the lattice points on the circle.

1 Perfect forms

This section introduces the minimum and minimal set of a quadratic form. It then defines a perfect quadratic form, whose minimal elements x_i satisfy that the matrices $x_i x_i^t$ span the Voronoi space. Furthermore, it includes a theorem on how a perfect form is uniquely defined by its minimal vectors up to proportionality. The final subsection proves that there are finitely many classes of perfect forms. The primary source for this section is Chapter 1-3 in [Martinet(2003)]. The last subsection has also been assisted by the book by Schürmann [Schürmann(2009)] and discussions with Peter Patzt, the supervisor on this project.

Note that Q (among other letters) is often used to describe both a quadratic form and its corresponding symmetric matrix.

1.1 Perfect forms

Definition 1.1. Let Q be a positive quadratic form on $\mathbb{R}^n$

(i) The *minimum* of Q, denoted $m(Q)$, is the minimal value of $Q(x)$ for $x \in \mathbb{Z}^n \backslash \{0\}$:

$$m(Q) = \min_{x \in \mathbb{Z}^n \backslash \{0\}} Q(x).$$

(ii) The *minimal set* of Q, denoted $S(Q)$, is the set $\{x \in \mathbb{Z}^n \backslash \{0\} \mid Q(x) = m(Q)\}$. The elements of the minimal set are referred to as *minimal elements* or *minimal vectors.*

Definition 1.2. Let $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ be a column vector. We define the matrix $P_x = xx^t$ and the set $P_{S(Q)} = \{P_x \mid x \in S(Q)\}$.

Let us note a few things about P_x,

$$\begin{aligned} P_x = xx^t &= \begin{bmatrix} x_1^2 & x_1x_2 & \dots & x_1x_n \\ x_2x_1 & x_2^2 & \dots & x_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_nx_1 & x_nx_2 & \dots & x_n^2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 & 0 & \dots & 0 \\ x_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}. \end{aligned}$$

Since multiplication on $\mathbb{R}$ is commutative, P_x is a symmetric matrix. This means that P_x can be considered as a quadratic form on $\mathbb{R}^n$. Furthermore, we can see from the above that P_x has rank 1 and is positive semi-definite.

Definition 1.3. A positive quadratic form Q is called *perfect* if $P_{S(Q)}$ spans $Sym_n(\mathbb{R})$.

Let us take a look at one of the first things we can say about perfect forms, namely that if the quadratic form Q is perfect then all the forms αQ for $\alpha > 0$ are perfect.

Proposition 1.4. The quadratic form Q is perfect if and only if the quadratic form αQ is perfect, when $\alpha > 0$.

Proof. Letting Q be a perfect quadratic form with minimum m. As αQ is a scaling of Q and $\alpha > 0$, the minimum of Q is αm. We thus see that Q and αQ have the same minimal set. ☐

A thing to note about quadratic forms is that $Q(x) = Q(-x)$. This means that if x is a minimal element, then $-x$ is also a minimal element. Furthermore, we note that $P_x = P_{-x}$, because $P_{-x} = (-x)(-x)^t = (-1)^2xx^t = P_x$. In order for a form Q to be perfect, the set $P_{S(Q)}$ must span $Sym_n(\mathbb{R})$, which has dimension $^{n(n+1)}/_2$. A necessary condition for Q being perfect is therefore that $|P_{S(Q)}| \geq {}^{n(n+1)}/_2$ and thus that $|S(Q)| \geq n(n+1)$. From this we can conclude that for $n \geq 2$, the quadratic form given by the identity matrix cannot be a perfect form, as the minimal elements would be the elements in the standard basis and their negative counterparts, hence $|S(I_n)| = 2n < n(n+1)$.

Another noticeable thing about the minimal elements is that the coordinates do not have a common divisor other than ± 1. This can be seen by assuming that zx is a minimal element, where $z \in \mathbb{Z}$ and $x \in \mathbb{Z}^n \backslash \{0\}$. We then get that $Q(zx) = z^2Q(x) \leq Q(x)$, hence $z \in \{1, -1\}$. For $n = 1$ this means that the minimal elements are always ± 1, and for $n = 2$ the coordinates will always be co-primes.

And now we take a look at an example of a perfect quadratic form on $\mathbb{R}^2$.

Example 1.5 (A perfect quadratic form).

The quadratic form Q given by the matrix $Q = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ is perfect.

Finding the minimum is rarely a simple process. In this case the minimum is 2. We can look at the value of Q for some integral vectors:

$$Q\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 2, Q\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 2, Q\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 6 \text{ and } Q\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 2$$

When we write the quadratic form $Q(x)$ out for some general $x = (a, b)$, we get :

$$Q\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = 2a^2 + 2b^2 + 2ab.$$

From this it is fairly easy to convince oneself that 2 is actually the minimum, and that there are no other minimal vectors - noting, of course, that for any quadratic form $Q(x) = Q(-x)$, hence there are six minimal vectors in total:

$$S(Q) = \left\{\pm\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \pm\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \pm\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} \text{ and corresponding } P_{S(Q)} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right\}.$$

We see that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and therefore that the set $P_{S(Q)}$ spans $Sym_2(\mathbb{R})$. ♣

1.2 Constructing a quadratic form from a lattice

In this section we see how to construct a quadratic form from a lattice. To do this, we note that for any matrix B the matrix B^tB is always symmetric as $(B^tB)^t = B^t(B^t)^t = B^tB$. In fact, any positive semi-definite symmetric matrix can be written this way.

Proposition 1.6. *Given a symmetric, positive semi-definite $n \times n$-matrix A, there exists an $n \times n$-matrix B such that $A = B^tB$.*

Proof. By Theorem 0.6 we can diagonalize A by an orthogonal matrix U; $A = UDU^t$. A is positive definite, so its eigenvalues are positive and we can take the square root of all the diagonal elements in D. This gives us a diagonal matrix D', where $D'D' = D$. Hence, $A = UD'D'U^t = UD'(UD')^t$. The matrix $B = (UD')^t$ then satisfies our claim. $\square$

Remark. If A and B are as in the previous proposition, note that any matrix UB, where U is an orthogonal $n \times n$-matrix, satisfies $(UB)^t(UB) = B^tB = A$. B is in particular not unique.

Let Q be a positive semi-definite quadratic form on $\mathbb{R}^n$. We know that there exists an $n \times n$-matrix B such that $Q = B^tB$, and we see that multiplying by the identity matrix in the middle does nothing: $Q = B^tI_nB$. The norm N is the quadratic form corresponding to the identity matrix, and thus the quadratic form Q of an element $x \in \mathbb{R}^n$ can be written as

$$Q(x) = x^tQx = x^tB^tI_nBx = N(Bx)$$

In this way we can view the process of taking Q of x as first doing a basis change of x and then taking the norm. When we look at the minimum and minimal set of the quadratic form Q, we look at the value of the elements $x \in \mathbb{Z}^n \backslash \{0\}$. This corresponds to the norm of the elements $Bx \in \mathbb{Z}^n \backslash \{0\}$, which we recognize as the elements of a lattice Λ whose basis is the columns of B.

Furthermore, we see that the minimum of Q is the minimum of Λ, and the minimal set of Λ is $\{Bx \mid x \in S(Q)\}$. In addition, we do indeed see that B is not unique since any norm preserving action on B will leave $Q(x)$ unchanged.

Example 1.7. In the previous example, we looked at the quadratic form Q given by the matrix

$$Q = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \text{ which can be written as Q=} \begin{bmatrix} 1/\sqrt{2} & \sqrt{3}/\sqrt{2} \\ -1/\sqrt{2} & \sqrt{3}/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ \sqrt{3}/\sqrt{2} & \sqrt{3}/\sqrt{2} \end{bmatrix}.$$

We see that we can perceive values of $Q(x)$ for $x \in \mathbb{Z}^n$ as the norm of the elements in the lattice Λ with basis $(1/\sqrt{2}, \sqrt{3}/\sqrt{2})$, $(-1/\sqrt{2}, \sqrt{3}/\sqrt{2})$. Λ is a hexagonal lattice and thus has 6 minimal elements, which means that there are 6 minimal elements of Q, which we already knew in this case. We also see that flipping and/or rotating Λ does not affect the norm. As a result, both lattices depicted in Figure 2 can describe Q. $\clubsuit$

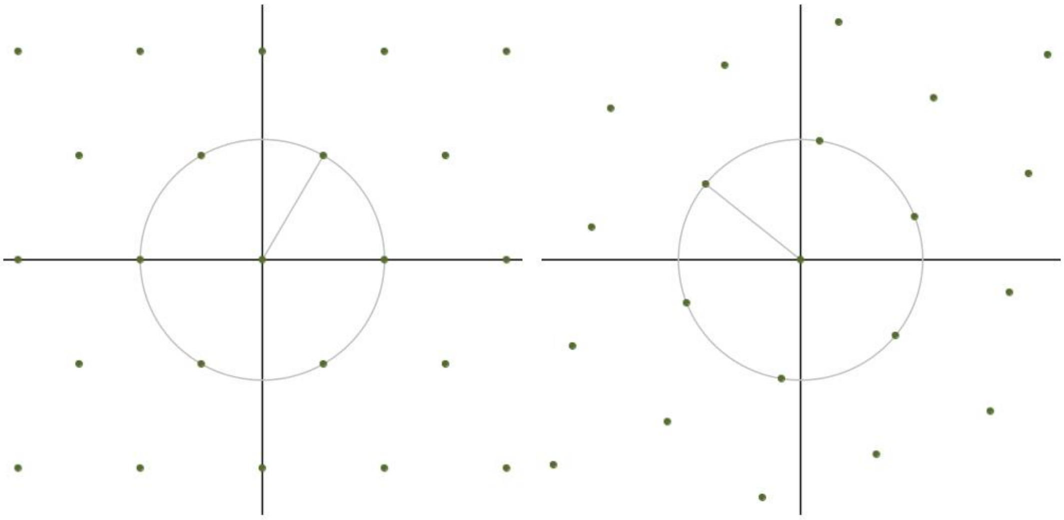

Figure 2: These lattices both describe the quadratic form in Example 1.7. The lattice on the right is a rotation of the one on the left, which has the basis $(1/\sqrt{2}, \sqrt{3}/\sqrt{2})$, $(-1/\sqrt{2}, \sqrt{3}/\sqrt{2})$. As in the last figure the gray circles have the minimum of the lattices as radius squared, and the minimal elements are the lattice points on the circle.

1.3 A perfect quadratic form is uniquely defined by its minimal set

This subsection looks at how a perfect quadratic form is uniquely defined, up to proportionality, by its minimal set. We've seen that any scaling of a perfect form Q by a positive real is also perfect. We can therefore focus on the quadratic forms with minimum m or up to proportionality. This section also introduces the Voronoi space, which is the space of quadratic forms on $\mathbb{R}^n$ and an inner product on this space.

Proposition 1.8. The set $\mathcal{Q}$ of quadratic forms on $\mathbb{R}^n$ is a vector space with addition $(Q+Q')(x) = x^t(Q+Q')x$ and scalar multiplication $(\alpha Q)(x) = x^t(\alpha Q)x$ for $\alpha \in \mathbb{R}$, quadratic forms Q and Q', and an element $x \in \mathbb{R}^n$.

Proof. The addition and scalar multiplication in question is in fact the addition and scalar multiplication on the level of the corresponding matrices. Since the space of symmetric matrices is closed under addition and scalar multiplication, meaning that the sum of two quadratic forms or a scaling of a form will in fact be a quadratic form. Both the addition and multiplication is carried out on each entry in the matrix. Furthermore, as $\mathbb{R}$ is a commutative ring, we do indeed have associative and commutative addition, an additive identity and inverse elements, as well as associative scalar multiplication, the desired distribution, and an identity scalar. ☐

As pointed out in the proof, there's not much difference in looking at the vector space $Sym_n(\mathbb{R})$ and the vector space $\mathcal{Q}$. From now on the two spaces will therefore be used interchangeably.

Note that because of distributivity of matrix multiplication over matrix addition and properties of scalar multiplication on matrices, the operations defined above behave nicely, i.e. $(Q+Q')(x) = Q(x) + Q'(x)$ and $(\alpha Q)(x) = x^t(\alpha Q)x$.

Proposition 1.9. The binary operation $\langle \cdot, \cdot \rangle : \mathcal{Q} \times \mathcal{Q} \longrightarrow \mathbb{R}$ given by $\langle Q, Q' \rangle = Tr(QQ')$ is an inner product on the vector space of quadratic forms.

Proof. Trace being a linear map gives the linearity in both arguments. We have symmetry of the inner product because of Proposition 0.8. Furthermore, since Q is a symmetric matrix with real eigenvalues, and the trace is invariant under basis change, $Tr(QQ)$ is the sum of the (real) eigenvalues squared. We see that for $Q \neq 0$ we have $\langle Q, Q\rangle = Tr(QQ) > 0$. □

Definition 1.10. The *Voronoi space* of quadratic forms is the vector space $\mathcal{Q}$ of quadratic forms equipped with the inner product $\langle \cdot, \cdot \rangle$ defined in Proposition 1.9.

Note that with this inner product, $\langle P_x, Q\rangle = Q(x)$ for any $x \in \mathbb{R}^n$ and quadratic form Q since:

$$\langle P_x, Q\rangle = Tr(P_x Q) = Tr(xx^t Q) = Tr(x^t Q x) = x^t Q x = Q(x),$$

where we use the fact that the trace of a real number is simply the number itself.

Keep in mind that an inner product is a non-degenerate bilinear form. In this case, this means that if $\langle Q, R\rangle = 0$ for all $R \in \mathcal{Q}$, then $Q = 0$. The argument is that $\langle Q, Q\rangle = 0$ when $R = Q$, which implies that $Q = 0$.

Theorem 1.11. A perfect quadratic form Q with minimum m is uniquely defined by its minimal set $S(Q)$.

Proof. Let Q_1, Q_2 be two perfect quadratic forms such that

$$m = m(Q_1) = m(Q_2) \text{ and } S = S(Q_1) = S(Q_2).$$

Furthermore let $Q = Q_1 - Q_2$. Notice that for $x \in S$, $Q(x) = m - m = 0$. Since the space of quadratic forms is a vector space, Q is also a quadratic form. As a result, we can write the statement above as $Q(x) = \langle P_x, Q\rangle = 0$ for $x \in S$. The set P_S spans $Sym_n(\mathbb{R})$ as it is the minimal set of a perfect form, and we therefore see that Q is 0, by the following:

Let $M \in Sym_n(\mathbb{R})$, as P_S spans $Sym_n(\mathbb{R})$, M can be written as $M = \sum_{x\in S} \lambda_x P_x$. Using the linearity of the inner product, we get

$$\langle Q, M\rangle = \sum_{x\in S} \lambda_x \langle Q, P_x\rangle = \sum_{x\in S} \lambda_x \cdot 0 = 0,$$

and since the inner product is non-degenerate, Q must be 0. We can thereby conclude that Q_1 and Q_2 are identical. □

1.4 Classes of perfect quadratic forms

The aim of this subsection is to prove that there are finitely many classes of perfect forms modulo $GL_n(\mathbb{Z})$ up to proportionality. We therefore introduce these classes and the Hadarmard and Hermite inequalities needed to prove Theorem 1.16, which states that there are finitely many classes of perfect forms.

Proposition 1.12. Let Q be a perfect quadratic form and $C \in GL_n(\mathbb{Z})$. Then $Q' = Q \circ C$ is perfect.

Proof. Note that the map that sends $x \mapsto Cx$ is a bijection on $\mathbb{Z}^n \backslash \{0\}$. Hence, $m(Q) = m(Q')$ and $S(Q') = \{C^{-1}x \mid x \in S(Q)\}$. To see that $P_{S(Q')}$ spans $Sym_n(\mathbb{R})$, let $A \in Sym_n(\mathbb{R})$ and $B = CAC^t$. Notice that B is a symmetric matrix. Because Q is perfect we can write
$B = \sum_{x\in S(Q)} b_x P_x$ and therefore

$$A = C^{-1}B(C^{-1})^t = C^{-1}(\sum_{x\in S(Q)} b_x P_x)(C^{-1})^t = \sum_{x\in S(Q)} b_x C^{-1}xx^t(C^{-1})^t = \sum_{x\in S(Q)} b_x P_{C^{-1}x}.$$

Thus, we see that $P_{S(Q')}$ spans $Sym_n(\mathbb{R})$. □

Remark. As $GL_n(\mathbb{Z})$ is a group with respect to multiplication, we see that Q is perfect if and only if $Q' = Q \circ C$ is perfect.

Because of this proposition, we have a well defined equivalence relation on the perfect forms, and we can consider classes of perfect forms.

Definition 1.13. We say that two perfect quadratic forms Q and Q' on $\mathbb{R}^n$ are in the same class modulo $GL_n(\mathbb{Z})$ if there exists a $C \in GL_n(\mathbb{Z})$ such that $Q = Q' \circ C$.

Remark. The equivalence relation $Q \sim Q \circ C$, $C \in GL_n(\mathbb{Z})$ is indeed symmetric, reflexive and transitive because of the group structure on $GL_n(\mathbb{Z})$.

For the following, note that $\det(Q)$ denotes the determinant of the matrix corresponding to the quadratic form Q.

Theorem 1.14 (Hadamard inequality)**.** Let Q be a positive semi-definite quadratic form on $\mathbb{R}^n$ and let $v_1, v_2, ..., v_n$ be n linear independent vectors in $\mathbb{Z}^n$. Then

$$\det(Q) \leq Q(v_1)Q(v_2)...Q(v_n).$$

Theorem 1.15 (Hermite inequality)**.** Let Q be a positive semi-definite quadratic form on $\mathbb{R}^n$. There exists $C \in GL_n(\mathbb{Z})$ with column vectors $c_1, c_2, .., c_n$ such that:

$$Q(c_1)Q(c_2)...Q(c_n) \leq \left(\frac{4}{3}\right)^{n(n-1)/2} \det(Q).$$

The proofs of the Hermite and Hadamard inequalities will not be made here, but can be read in chapter 2 of [Martinet(2003)]. We are now ready to look at the last statement of this section.

Theorem 1.16. For a given dimension n, there are finitely many classes of perfect forms up to proportionality.

Proof. Let Q' be a perfect quadratic form on $\mathbb{R}^n$ with $m(Q') = 1$. Assume that Q' satisfies:

$$Q'(e_1)Q'(e_2)...Q'(e_n) \leq \left(\frac{4}{3}\right)^{n(n-1)/2} \det(Q'),$$

where $e_1, e_2, ..., e_n$ is the standard basis of $\mathbb{R}^n$.

Let $Q' = B^tB$, where we let $b_1, b_2, ..., b_n$ be the column vectors of B. Secondly, let $a \in \mathbb{Z}^n$ and $x = Ba$. Thirdly, let $\widetilde{B}$ be the matrix with column vectors $b_1, ..., b_{i-1}, x, b_{i+1}, ..., b_n$, notice that $\widetilde{B}$ is equal to B, except the i'th column is x in stead of b_i. Lastly, let $\widetilde{Q}$ be the quadratic form with the matrix $\widetilde{B}^t\widetilde{B}$. Then

$$a_i = \frac{\det(\widetilde{B})}{\det(B)}.$$

Which implies

$$a_i^2 = \frac{\det(\widetilde{B}^t\widetilde{B})}{\det(Q)} \leq k_n \frac{\widetilde{Q}(e_i)}{Q'(e_i)} \leq k_n \langle x, x \rangle = k_n Q'(a),$$

where we use the Hadamard inequality on the numerator and the assumption about Q' on the denominator as well as letting $k_n = \left(\frac{4}{3}\right)^{n(n-1)/2}$.

If we let $a \in S(Q')$ then we have the following inequality:

$$a_i^2 \leq k_n.$$

Since $a_i \in \mathbb{Z}$ is bounded by a constant that only depends on the dimension n, there are finitely many possible minimal vectors for Q'. Hence, by Theorem 1.11, there are finitely many Q' up to proportionality for a given dimension.

Now consider the class of perfect quadratic forms represented by Q. By the Hermite inequality, there exists a $C \in GL_n(\mathbb{Z})$ with column vectors $c_1, c_2, ..., c_n$ such that $Q(c_1)Q(c_2)...Q(c_n) \leq k_n \det(Q)$. If we let $Q' = Q \circ C$, we see that Q' satisfies $Q'(e_1)Q'(e_2)...Q'(e_n) \leq k_n \det(Q')$, and that Q' is in the same class as Q. Since we know there are finitely many Q' up to proportionality, there are finitely many classes of perfect quadratic forms up to proportionality. □

2 Voronoi domains

This section aims to prove that the Voronoi domains in the Voronoi space of quadratic forms are a polyhedral complex. Firstly, we introduce the Voronoi cone of positive semi-definite quadratic forms and the Voronoi domains, which we will see are polyhedral cones in the Voronoi cone attached to a positive definite form. Secondly, we show that only the Voronoi domains of perfect forms are full dimensional polyhedra, which means that they are not contained in any hyperplane. Thus the previous section tells us that there are only finitely many full dimensional Voronoi domains up to pre-composition with an element of $GL_n(\mathbb{Z})$. This section is largely based on [Martinet(2003)] except the very last part from Theorem 2.20 and out, which have been made with assistance from Peter Patzt, the supervisor on this project. The parts on vector-spaces, inner products and matrices build on the results in [Hesselholt and Wahl(2017)]. The section also includes parts on polyhedra, hyperplanes and faces; assumptions on which are mentioned in the note in appendix, but for a comprehensive introduction to polyhedra, the reader is referred to [Ziegler(1995)].

This section builds on the notation in the previous section. Q thus still denotes both a quadratic form and its corresponding matrix, and the Voronoi space and $Sym_n(\mathbb{R})$ are used interchangeably.

2.1 Voronoi domains

Firstly, we will define the Voronoi cone in the Voronoi space. For the following, note that a half-line with origin in 0 is a set $\{\alpha x \mid \alpha \geq 0\}$ for a given x in the space.

Definition 2.1. (i) A *cone* is a union of half-lines with origin in 0.

(ii) The *Voronoi cone* in the Voronoi space is the subset of positive semi-definite quadratic forms.

Notice that the Voronoi cone is in fact a cone. Given a positive semi-definite form Q, we see that $(\alpha Q)(x) = \alpha Q(x) \geq 0$ for all $\alpha \geq 0$ and $x \in \mathbb{R}^n$.

We will now define the Voronoi domain of a positive definite quadratic form on $\mathbb{R}^n$, which is a subset of the Voronoi space. But first we shall note that the Voronoi space is a euclidean space and therefore isomorphic to some $\mathbb{R}^m$. This is important as most definitions in this thesis refer to $\mathbb{R}^n$. An example of this is the definition of a convex hull, which we shall be using now.

Definition 2.2. The *Voronoi domain* D_Q of a positive definite quadratic form Q is the convex hull of the half-lines $\{\alpha P_x \mid \alpha \geq 0\}$ for $x \in S(Q)$ in the Voronoi space.

Proposition 2.3. The Voronoi domain D_Q of a positive definite quadratic form Q is the set

$$D_Q = \left\{ \sum_{x \in S(Q)} \lambda_x P_x \mid \lambda_x \geq 0 \right\}.$$

This does not immediately follow from the Definition 0.10 of a convex hull stated in section 0. A proof is therefore included.

Proof. We prove the two inclusions, and begin with $D_Q \subseteq A = \{\sum_{x\in S(Q)} \lambda_x P_x \mid \lambda_x \geq 0\}$. Firstly, note that $\{\alpha P_x \mid \alpha \geq 0, x \in S(Q)\}$ is a subset of A, which we, in the following, also find to be convex.

Let $R, U \in A$, i.e. $R = \sum_{x\in S(Q)} \lambda_x P_x$ and $U = \sum_{x\in S(Q)} \sigma_x P_x$ for $\lambda_x, \sigma_x \geq 0$. We define the line $l : [0,1] \longrightarrow \mathcal{Q}$, where $l(t) = tR + (1-t)U$ and see that the line segment between R and U is the image $l([0,1])$, which is contained in A since

$$l(t) = tR + (1-t)U = \sum_{x\in S(Q)} (t\lambda_x + (1-t)\sigma_x)P_x$$

and the constants $t\lambda_x + (1-t)\sigma_x \geq 0$ for all $t \in [0,1]$ and $x \in S(Q)$. Since R and U were random elements of A we see that A is convex. By the definition of the convex hull we therefore get that D_Q is in A.

We now prove the other inclusion, $A \subseteq D_Q$, by induction. Let $R = \sum_{x\in S(Q)} \lambda_x P_x$. Note that if all or all but one λ_x are equal to 0, then R is in the set of half-lines, hence in D_Q.

If all but two λ_x are equal to zero, R can be written as $R = \lambda_{x_1} P_{x_1} + \lambda_{x_2} P_{x_2}$. Notice that

$$R \in \{2t\lambda_{x_1} P_{x_1} + 2(1-t)\lambda_{x_2} P_{x_2} \mid t \in [0,1]\},$$

which is the line segment between two points on the half-lines that define D_Q. We see that R must be in any convex set containing the half-lines and hence $R \in D_Q$.

If all but n λ_x are equal to zero, R can be written as $R = \sum_{i=1}^{n} \lambda_{x_i} P_{x_i}$. Let $U = \sum_{i=1}^{n-1} \lambda_{x_i} P_{x_i}$. By the induction hypothesis U is in any convex set containing the half-lines. Now note that

$$R \in \{2t\lambda_{x_n} P_{x_n} + 2(1-t)U \mid t \in [0,1]\},$$

which is the line segment between $2U$ and $2\lambda_{x_n} P_{x_n}$. By the induction hypothesis $2U = \sum_{i=1}^{n-1} 2\lambda_{x_i} P_{x_i}$ is in any convex set containing the half-lines, and $2\lambda_{x_n} P_{x_n}$ is on a half-line. We can now conclude that R is in any convex set containing the half-lines and therefore in D_Q. ☐

Example 2.4 (Voronoi domain)**.** When $n = 2$ the Voronoi space is 3-dimensional, which makes it possible to illustrate in a way that is understandable. We will therefore look at the domain of the perfect form Q from Example 1.5. We know that $Q = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ is a perfect quadratic form with minimal set

$S(Q) = \left\{ \pm \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \pm \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \pm \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ and corresponding $P_{S(Q)} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right\}$.

The Voronoi domain D_Q therefore consists of all quadratic forms R that can be written as

$$R = \lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \lambda_i \geq 0 \text{ for all } i \in \{1,2,3\}.$$

D_Q is illustrated in Figure 3. The figure also shows that Q itself is not in D_Q and therefore that positive definite quadratic forms aren't necessarily in their Voronoi domain as the name might lead one to think. ♣

In the proof of Proposition 1.4 we saw that all Q has the same minimal set as any αQ where $\alpha > 0$, which implies that the Voronoi domains are same. In Figure 3, we therefore see that all the quadratic forms on the gray line have the illustrated Voronoi domain. One should also notice that the Voronoi domain illustrated is full dimensional.

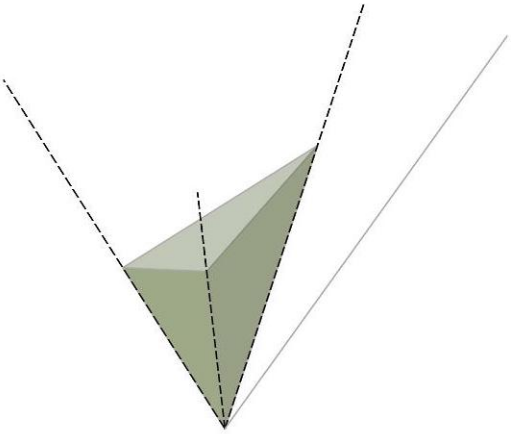

Figure 3: Illustration of the Voronoi domain of $Q = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. The gray line is the half-line from 0 through Q. Notice that Q itself is not inside its Voronoi domain.

Proposition 2.5. The Voronoi domain of the positive definite quadratic form Q is of full dimension, i.e. not contained in any hyperplane, if and only if Q is perfect.

Proof. Let Q be a positive definite quadratic form on $\mathbb{R}^n$ that is not perfect. We note that $P_{S(Q)}$ does not span $Sym_n(\mathbb{R})$, which means that there are at most $\frac{n(n+1)}{2} - 1$ linearly independent P_x'es, and the set of $P_{S(Q)} \cup 0$ is contained in a hyperplane. Therefore, $D_Q \subseteq span\{P_{S(Q)}\}$ is also contained in the hyperplane. With a similar argument, we see that if Q is perfect $P_{S(Q)}$ spans $Sym_n(\mathbb{R})$ and D_Q is not contained in a hyperplane. $\square$

From Theorem 1.16 we see that there are finitely many full dimensional Voronoi domains up to a pre-composition of the perfect quadratic form with an element in $GL_n(\mathbb{Z})$. In terms of the Voronoi domains pre-composing the quadratic form results in the following change of domain:

$$\text{If } A \in D_Q \text{ and } C \in GL_n(\mathbb{Z}), \text{ then } C^{-1}A(C^{-1})^t \in D_{Q \circ C}.$$

In order to see this, note that the minimal set of $Q \circ C$ is $\{C^{-1}x \mid x \in S(Q)\}$ as we saw in the proof of Proposition 1.12.

Example 2.6 (The half-line $\{\alpha P_{(1,0)} \mid \alpha \geq 0\}$ in $Sym_2(\mathbb{R})$ is a Voronoi domain)**.**

To see that $\{\alpha P_{(1,0)} \mid \alpha \geq 0\}$ is a Voronoi domain, we can find a quadratic form whose minimal set is $\{\pm(1,0)\}$. The quadratic form Q with the following matrix is an example of this:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

We see that $Q((x,y)) = x^2 + 3y^2$ does indeed have the minimal set $\{\pm(1,0)\}$. Furthermore we notice that any quadratic form with matrix

$$P = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \text{ where } 0 < a < b,$$

has the minimal set in question. This is also an example of how a non-perfect form Q has the same Voronoi domain as many other quadratic forms that are not necessarily proportional to Q.

The general statement that a half-line $\{\alpha P_x \mid \alpha \geq 0\}$ for an $x \in \mathbb{Z}^n$ is a Voronoi domain is proven as a part of the proof of Lemma 2.21. ♣

2.2 More on Voronoi domains

In this section we look at the relation between Voronoi domains. We begin by seeing that they are all inside of the Voronoi cone.

Proposition 2.7. A Voronoi domain is contained in the Voronoi cone.

Proof. Let Q be a positive definite quadratic form, and $R \in D_Q$. By Proposition 2.3 R can be written as $R = \sum_{x \in S(Q)} \lambda_x P_x$. Since P_x is a positive semi-definite quadratic form, we have $P_x(y) \geq 0$ for all $y \in \mathbb{R}^n$ and all $x \in S(Q)$. We also know that $\lambda_x \geq 0$ for all $x \in S(Q)$, and therefore $R(y) \geq 0$ for all $y \in \mathbb{R}^n$, which means that R is in the Voronoi cone. □

The inner product on the Voronoi space induces a metric, which in turn induces a topology on the space. We can therefore speak of the interior of a set A in the Voronoi space, which we denote $Int(A)$. We will also need a variant of the interior of the Voronoi domains, which we call the relative interior.

Definition 2.8. The *relative interior* of a Voronoi domain D_Q is the interior of the Voronoi domain regarded in the subspace of the Voronoi space spanned by $P_{S(Q)}$. The relative interior of D_Q is denoted $Rel(D_Q)$.

This leads us to this theorem on the relation between Voronoi domains.

Theorem 2.9. If a Voronoi domain contains a point in the relative interior of another Voronoi domain, then the first domain contains the second one.

Remark. If two domains have overlapping relative interiors, they are the same since by 2.9 each domain would contain the other.

Proof of Theorem 2.9. Let D_Q, $D_{Q'}$ be Voronoi domains and let Q, Q' be fitting positive definite quadratic forms. Because $m(\alpha Q) = \alpha m(Q)$ for all $\alpha \geq 0$ and the Voronoi domain of Q is the same as the Voronoi domain of αQ, we can assume without loss of generality that $m = m(Q) = m(Q')$.

Let $R \in D_Q \cap D_{Q'}$. Because $R \in D_Q$, R can be written as the sum $R = \sum_{x \in S(Q)} \lambda_x P_x$. Now look at the inner product $\langle R, Q \rangle$. By using the linearity of the inner product, we see:

$$\langle R, Q \rangle = \sum_{x \in S(Q)} \lambda_x \langle P_x, Q \rangle = \sum_{x \in S(Q)} \lambda_x Q(x) = m \sum_{x \in S(Q)} \lambda_x.$$

We obtain the last equality by realizing that we are summing over the minimal elements of Q. We can consider $\langle R, Q' \rangle$ in the same manner:

$$\langle R, Q' \rangle = \sum_{x \in S(Q)} \lambda_x \langle P_x, Q' \rangle = \sum_{x \in S(Q)} \lambda_x Q'(x) \geq m \sum_{x \in S(Q)} \lambda_x.$$

For the inequality, we use that $S(Q) \subseteq \mathbb{Z}^n \backslash \{0\}$ and that m is the minimum of Q, which leads to $Q'(x) \geq m$. As a result, we can conclude that $\langle R, Q' \rangle \geq \langle R, Q \rangle$. Repeating the argument where we instead write R as a linear combination of the elements in $P_{S(Q')}$, we get the opposite equality and thus $\langle R, Q'\rangle = \langle R, Q\rangle$. This is equivalent to

$$0 = \langle R, Q'\rangle - \langle R, Q\rangle = \sum_{x\in S(Q)} \lambda_x(\langle P_x, Q'\rangle - \langle P_x, Q\rangle) = \sum_{x\in S(Q)} \lambda_x(Q'(x) - Q(x)).$$

Assume, as in the theorem, that $R \in Rel(D_Q)$. This implies that $\lambda_x > 0$ for all $x \in S(Q)$. In order for the sum above to be equal to zero, we therefore need $Q'(x) = Q(x) = m$ for all $x \in S(Q)$. Since m is the minimum of Q', we have $S(Q) \subseteq S(Q')$, which implies $P_{S(Q)} \subseteq P_{S(Q')}$. This leads to the desired result that $D_Q \subseteq D_{Q'}$. □

2.3 Polyhedra and contiguous forms

In this section we see that a Voronoi domain is a polyhedral cone. We also introduce the contiguous form, which is a perfect quadratic form whose Voronoi domain share a facet with another perfect quadratic form.

Definition 2.10.

(i) A *polyhedron* in $\mathbb{R}^n$ is the intersection of finitely many closed half-spaces of $\mathbb{R}^n$.

(ii) A *face* of a polyhedron is the intersection of the polyhedron with any number of the hyperplanes that define the mentioned half-spaces.

(iii) A *facet* is a face that generates an affine subspace of co-dimension 1 in the affine space generated by the polyhedron.

Definition 2.11. A *polyhedral cone* C in $\mathbb{R}^n$ is the set of elements $c = \sum_{i=1}^{k} \lambda_i y_i$ with $\lambda_i \geq 0$ generated by a finite set of vectors $y_1, y_2, ..., y_k \in \mathbb{R}^n$.

A polyhedral cone is as the name suggest a polyhedron. This is expressed as a theorem in [Ziegler(1995), Theorem 1.3]. We note, by Proposition 2.3, that a Voronoi domain of a positive definite form Q is a polyhedral cone, as $S(Q)$ is finite.

Definition 2.12. Let Q be a positive definite quadratic form such that the subspace V generated by D_Q is at least two-dimensional. Furthermore, let F be a facet of the Voronoi domain D_Q. A facet vector R of F is a nonzero element in V that is orthogonal to F and satisfies $\langle R, U\rangle \geq 0$ for all $U \in D_Q$.

Note that because F generates a hyperplane of V, R is unique up to a scaling by a positive real. Furthermore we see that,

$$\begin{aligned} \langle R, P_x\rangle &= 0 \text{ for all } P_x \in P_{S(Q)} \cap F \\ \langle R, P_x\rangle &> 0 \text{ for all } P_x \in P_{S(Q)} \backslash F. \end{aligned}$$

Note also that the facet vector R is an element of the Voronoi space, which makes it a quadratic form.

Definition 2.13. A *contiguous* form of a perfect quadratic form Q is another perfect quadratic form Q' whose Voronoi domain shares a facet with the Voronoi domain of Q.

It turns out that every perfect form has a unique - up to proportionality - contiguous form for every facet. This is expressed in the following theorem, the proof of which will not be made here but can be read in [Martinet(2003), Theorem 7.2.1].

Theorem 2.14. Let Q be a perfect quadratic form with minimum m, F a facet of D_Q and R a facet vector of F in D_Q. There exists a $\rho > 0$ such that the quadratic form $Q_\rho = Q + \rho R$ is perfect with minimum m and contiguous to D_Q along the facet F. Q_ρ and Q are the unique perfect forms with minimum m whose Voronoi domains contain F.

The rest of this subsection aims to prove that any positive quadratic form is in a Voronoi domain, which is a part of Theorem 2.17. In the proof we move between the Voronoi domains of contiguous forms eventually arriving to a Voronoi domain that contains the positive quadratic form in question. In order to do this, we need a bit more knowledge about the perfect forms, seen in the following two statements.

Lemma 2.15. Let Q, Q_0 be two positive definite quadratic forms on $\mathbb{R}^n$ and μ the minimum of Q_0 on the unit sphere in $\mathbb{R}^n$. The eigenvalues λ_i of Q are bounded as follows:

$$0 < \lambda_i \leq \frac{1}{\mu}\langle Q, Q_0\rangle, \text{ for all } i \in \{1, ..., n\}$$

Proof. The first part of the inequality, $0 < \lambda_i$, is given because Q is positive definite. Since the matrix Q is symmetric, it can be diagonalized by an orthogonal matrix B: $Q = BDB^t$. Letting b_i be the i'th column of B, the set of vectors $b_1, b_2, ..., b_n$ is an orthonormal basis of $\mathbb{R}^n$. Now, let $M_0 = B^t Q_0 B$ and look at the inner product:

$$\langle Q, Q_0\rangle = Tr(QQ_0) = Tr(BDB^tBM_0B^t) = Tr(DM_0) = \sum_{i=1}^{n} \lambda_i m_{i,i},$$

where $m_{i,j}$ is the entry of M_0 in the i'th row and the j'th column. In order to reach the sum above we use that D is a diagonal matrix, which means that multiplying the two matrices just adds a factor of λ_i in every entry on the i'th row. Let $e_1, e_2, ..., e_n$ be the standard basis on $\mathbb{R}^n$ and note

$$m_{i,i} = e_i^t M_0 e_i = e_i^t B^t Q_0 B e_i = b_i^t Q_0 b_i = Q_0(b_i), \text{ for all } i \in \{1, ..., n\}.$$

Since b_i is a vector of an orthonormal basis, it is on the unit sphere. Hence, $Q(b_i) \geq \mu$ for all $i \in \{1, ..., n\}$, which reveals that $\langle Q, Q_0\rangle \geq \mu \sum_{i=1}^{n} \lambda_i$. Because all the eigenvalues are positive, we now have

$$\lambda_i \leq \frac{1}{\mu}\langle Q, Q_0\rangle, \text{ for all } i \in \{1, ..., n\}.$$

□

Theorem 2.16. Let Q_0 be a positive definite quadratic form. There is a finite number of perfect forms Q with minimum m that satisfies:

$$\langle Q, Q_0\rangle \leq K, \text{ for a given } K > 0.$$

Proof. Let Q, Q_0, m, and K be as above. In the first part of the proof, we establish a lower bound $a > 0$ for the eigenvalues of Q. Lemma 2.15 gives us an upper bound for the eigenvalues $\lambda_1, .., \lambda_n$ of Q:

$$\lambda_i \leq \frac{1}{\mu}\langle Q, Q_0\rangle \leq \frac{K}{\mu}, \text{ for all } i \in \{1, .., n\}.$$

The Hermite inequality in Theorem 1.15 tells us that there exists a $C \in GL_n(\mathbb{Z})$ with column vectors $c_1, c_2, ..., c_n$, and a constant k_n only depending on n such that:

$$Q(c_1)Q(c_2)...Q(c_n) \leq k_n \det(Q).$$

We note that $c_i \in \mathbb{Z}^n \backslash \{0\}$, hence $Q(c_i) \geq m$ for all $i \in \{1, .., n\}$. Thus:

$$\frac{m^n}{k_n} \leq \det(Q) = \prod_{j=1}^{n} \lambda_j \leq \left(\frac{K}{\mu}\right)^{n-1} \lambda_i, \text{ for all } i \in \{1, .., n\},$$

where the last inequality is obtained by evaluating all λ_j with $j \neq i$ at the upper bound. We now have a lower bound for the eigenvalues of Q:

$$a = \frac{m^n}{k_n} \left(\frac{\mu}{K}\right)^{n-1} \leq \lambda_i, \text{ for all } i \in \{1, .., n\}.$$

Notice two things about a. Firstly that $a > 0$ since m, K, k_n, and μ all are strictly positive. Secondly, a depends on the dimension n as well as the choice of Q_0, K, and m but not on Q.

In the second part of the proof, we establish an upper bound for the norm of the minimal vectors of Q. Firstly, let $x \in S(Q)$, a minimal vector and let $Q = BDB^t$ be the diagonalization of Q by an orthogonal matrix B with column vectors $b_1, b_2, ..., b_n$. Secondly, let $x = By$ for $y = (y_1, ..., y_n)$. Then

$$m = Q(x) = Q(By) = y^t B^t Q B y = y^t D y = \sum_{i=1}^{n} \lambda_i y_i^2 \geq a \sum_{i=1}^{n} y_i^2 = aN(y) = aN(x),$$

where the last equality uses that B is orthogonal, hence $N(x) = N(y)$. We now have the inequality $N(x) \leq \frac{m}{a}$, and since $x \in \mathbb{Z}^n$ there are finitely many possible minimal vectors for Q. By Theorem 1.11, a perfect form Q with minimum m is defined by its minimal vectors, which means that there are finitely many possible Qs. ☐

Theorem 2.17. Given a positive definite quadratic form Q and a perfect quadratic form Q_1, there exists a finite sequence of perfect forms $Q_1, Q_2, ..., Q_k$ such that Q_i is contiguous to Q_{i+1}, and Q is in the Voronoi domain of Q_k.

In order to prove the theorem we need the following lemma.

Lemma 2.18. Let Q be a quadratic form and let Q' be a perfect form. Then $Q \in D_{Q'}$ if and only if $\langle Q, R_i \rangle \geq 0$ for all facet vectors R_i of $D_{Q'}$.

Proof. Assuming $Q \in D_{Q'}$ we see by the definition of the facet vector, that $\langle Q, R_i \rangle \geq 0$ for all facet vectors R_i. If we on the contrary assume $\langle Q, R \rangle \geq 0$ for all facet vectors R_i, then Q must be in the intersections of the half spaces $\{A \in \mathcal{Q} \mid \langle R_i, A \rangle \geq 0\}$. This intersection is exactly Voronoi domain $D_{Q'}$ [1]. ☐

Proof of Theorem 2.17. Let Q and Q_1 be as in the theorem. If $Q \in D_{Q_1}$ we see that $k = 1$ and there's nothing to prove. If on the other hand $Q \notin D_{Q_1}$ let we can by Lemma 2.18 choose a facet vector R_1 of D_{Q_1} such that $\langle Q, R_1 \rangle < 0$ and define $Q_2 = Q_1 + \rho_1 R_1$ to be the contiguous form to Q_1 along the facet vector R_1. Note using linearity of the inner product, we see

$$\langle Q, Q_2 \rangle = \langle Q, Q_1 \rangle + \rho_1 \langle Q, R_1 \rangle < \langle Q, Q_1 \rangle.$$

Now if $Q \in D_{Q_2}$ we see that $k = 2$ and otherwise we define $Q_3 = Q_2 + \rho_2 R_2$ for a facet vector R_2 of D_{Q_2} with $\langle Q, R_2 \rangle < 0$. Continuing this inductively we get a strictly decreasing sequence

$$\langle Q_1, Q \rangle > \langle Q_2, Q \rangle > ... > \langle Q_k, Q \rangle,$$

---

[1] See appendix A.

where $Q_1, .., Q_k$ are perfect forms with the same minimum m, and where Q_i is contiguous to Q_{i+1} for $i \in \{1, .., k-1\}$. Because of the strictly decreasing sequence we also notice that $Q_i \neq Q_j$ for $i \neq j$. Theorem 2.16 now yields that there are finitely many perfect forms P with minimum m that satisfy $\langle Q_1, Q\rangle > \langle P, Q\rangle$, we therefore know that the sequence is finite. This also implies that there does not exist a Q_{k+1} where

$$\langle Q, Q_{k+1}\rangle < \langle Q, Q_k\rangle.$$

Implying that $\langle Q, R\rangle \geq 0$ for all facet vectors R of D_{Q_k}, and by Lemma 2.18 we see $Q \in D_{Q_k}$. $\square$

2.4 The Voronoi domains constitute a polyhedral complex

In this section we prove that that the Voronoi domains constitute a polyhedral complex.

Definition 2.19. A *polyhedral complex* K is a set of polyhedra such that:

(i) Every face of $P \in K$ is in K

(ii) The intersection of any $P_1, P_2 \in K$ is a face of both P_1 and P_2.

Theorem 2.20. *The Voronoi domains of the positive definite quadratic forms constitute together with the set $\{0\}$ a polyhedral complex that covers the interior of the Voronoi cone and is contained in the Voronoi cone.*

The proof of Theorem 2.20 relies on the following three lemmas:

Lemma 2.21. *A face of a Voronoi domain is a Voronoi domain or $\{0\}$.*

Proof. Let Q be a positive definite quadratic form on $\mathbb{R}^n$ with minimum m, D_Q its Voronoi domain, and F a facet of D_Q. If the Voronoi domain is a half-line, then $F = \{0\}$.

For higher dimensional Voronoi domains we can let R be a facet vector of F. We see that $x^t R x = \langle R, P_x\rangle$ for any $x \in \mathbb{R}^n$. Let $Q_\theta = Q + \theta R$ for a $\theta > 0$. Because R is a facet vector we have

$$R(x)\begin{cases} = 0 & P_x \in P_{S(Q)} \cap F \\ > 0 & P_x \in P_{S(Q)} \backslash F \end{cases} \text{ and thus } Q_\theta(x)\begin{cases} = Q(x) & P_x \in P_{S(Q)} \cap F \\ > Q(x) & P_x \in P_{S(Q)} \backslash F \end{cases}.$$

For a sufficiently small $\theta > 0$, the minimum of Q_θ is the same as the minimum of Q. This is because $Q_\theta(x) = m$ for $P_x \in P_{S(Q)} \cap F$, which means that the minimum is unchanged, unless there is another integral vector z that now has $Q_\theta(z) < m$. But by choosing a sufficiently small $\theta > 0$ we can ensure that all z with $P_z \in \mathbb{Z}^n \backslash (\{0\} \cup S(Q) \cap F)$ will satisfy $Q_\theta(z) > m$, and therefore that Q_θ will have the minimal set $S(Q_\theta) = S(Q) \cap F$. We thereby see that F is the Voronoi domain of Q_θ. Inductively, we see that all faces of a Voronoi domain are Voronoi domains. $\square$

Lemma 2.22. *The intersection of two Voronoi domains is a face of both domains.*

Proof. Let D_{Q_1} and D_{Q_2} be the Voronoi domains of Q_1 and Q_2, respectively, and consider their intersection. By Theorem 2.9, we can divide the situation into three cases: (i) There's a point which is in the relative interior of both domains, (ii) the intersection is in the boundary of one of the domains and contains a relative interior point of the other domain, and (iii) the intersection is in the boundary of both domains.

Assuming (i), we see with Theorem 2.9 that the two domains are in fact the same. The intersection is therefore everything, and hence a face of both domains.

Assuming (ii), Theorem 2.9 reveals that the one domain is a subset of the other. Without loss of generality, we can assume that $D_{Q_2} \subseteq D_{Q_1}$, and now the intersection is all of D_{Q_2}. But as the intersection is in the boundary of D_{Q_1}, D_{Q_2} is a subset of the boundary of D_{Q_1}, and as it is convex, it must be a subset of a face of D_{Q_1}. Choosing this face minimal we get that $D_{Q_2} = \{0\}$ or that it contains a point in the relative interior of the face, which by the previous lemma is a Voronoi domain. By 2.9, we see that D_{Q_2} is in fact equal to the minimal face of D_{Q_1} containing D_{Q_2}.

Assuming (iii), we get, because the intersection is convex, that it is contained in faces of both D_{Q_1} and D_{Q_2}. We see, as above, that by choosing these faces minimal, the intersection must contain a point of the relative interior of the two faces, and by 2.9 the two faces are the same. ☐

Lemma 2.23. A quadratic form in the interior of the Voronoi cone is positive definite.

Proof. We prove this lemma by contraposition. Let Q be a quadratic form in the Voronoi cone i.e. a positive semi-definite quadratic form, and assume that Q is not positive definite. Clearly Q has an eigenvalue of zero. Let this be the i'th eigenvalue.

Since the matrix corresponding to the quadratic form Q is symmetric, it can be diagonalized such that $Q = PDP^t$. Let E be the diagonal matrix with entries 0 except for the i'th diagonal entry being -1, and let $Q_r = Q + rPEP^t$ for $r > 0$. Note that Q_r is in every ball around Q - for a suiting r, and that every Q_r has a negative eigenvalue, hence isn't in the Voronoi cone. We see that Q is not in the interior of the Voronoi cone. ☐

Proof of Theorem 2.20. By Lemma 2.21 and 2.22, the set of Voronoi domains constitute a polyhedral complex. Since by Theorem 2.17 every positive definite quadratic form is in the Voronoi domain of a perfect form, we can conclude, by Lemma 2.23, that the interior of the Voronoi cone is in the union of the Voronoi domains. And by Theorem 2.7 the Voronoi domains are contained in the Voronoi cone. ☐

Concluding remarks

In this thesis we have looked at perfect quadratic forms, and their Voronoi domains and seen that there are many interesting points of view to this topic. We showed that there are finitely many classes of perfect quadratic forms modulo $GL_n(\mathbb{Z})$ up to proportionality. But it would have been interesting to linger a bit longer at these classes, and maybe have shown that there are only one class of perfect quadratic forms on $\mathbb{R}^2$ up to proportionality. Contiguity actually also behaves well when looking at classes of perfect forms, and the theme of contiguity could definitely be explored more.

Given the natural constraints of a bachelor thesis, it can always be improved. In my opinion this thesis would benefit from having an integrated and more explicit foundation for the results regarding polyhedra, as well as including the proofs for the Hadamard and Hermite inequalities.

Through the work of making this thesis, I stumbled upon a counterexample for a proposition in Martinets book. It did not make sense to include the proposition in the project, but I have included it in the appendix.

I have also spent time looking into the statement, that in every Voronoi domain of a perfect form Q, the form in 'the middle', meaning $Q' = \sum_{x \in S(Q)} P_x$, is another perfect form, and this is the only perfect form in D_Q up to proportionality. Especially for $n = 2$ it seems that given a perfect form Q, the form given by Q^{-1} is also perfect and the form in the middle of the domain D_Q.

As mentioned, the perfect forms are quite interesting and a lot more observations could be made on them and their Voronoi domains. For me, the set $S(Q)$ would be interesting to look more into, with questions like what is the maximal order of $S(Q)$ for a given n?

And finally, a natural extension of this project would be to look into group homology in particular the group homology of $GL_n(\mathbb{Z})$. As we now have seen that $GL_n(\mathbb{Z})$ acts on the Voronoi cone by sending the Voronoi domains to other Voronoi domains, and that there are only finitely many domains up to an action with $GL_n(\mathbb{Z})$.

Appendix A: Note on assumptions about polyhedra

This note is an overview of all definitions and assumptions about polyhedra and half spaces, that have been stated and used in the thesis.

Definitions from the thesis

Definition 2.10.

(i) A *polyhedron* in $\mathbb{R}^n$ is the intersection of finitely many closed half-spaces of $\mathbb{R}^n$.

(ii) A *face* of a polyhedron is the intersection of the polyhedron with any number of the hyperplanes that define the mentioned half-spaces.

(iii) A *facet* is a face that generates an affine subspace of co-dimension 1 in the affine space generated by the polyhedron.

Definition 2.12. Let Q be a positive definite quadratic form such that the subspace V generated by D_Q is at least 2-dimensional. Furthermore let F be a facet of the Voronoi domain D_Q. A facet vector R of F is a nonzero element in V that is orthogonal to F and satisfies $\langle R, U \rangle \geq 0$ for all $U \in D_Q$.

The above definition is made for the Voronoi domains in the Voronoi space, but could as well have been made for a polyhedron P in $\mathbb{R}^n$. Keep this in mind for Assumption 2.

Definition 2.11. A *polyhedral cone* C in $\mathbb{R}^n$ is the set of elements $c = \sum_{i=1}^{k} \lambda_i y_i$ with $\lambda_i \geq 0$ generated by a finite set of vectors $y_1, y_2, ..., y_k \in \mathbb{R}^n$.

Assumptions

Assumption 1. Let R be a nonzero element of the Voronoi space $\mathcal{Q}$. The set $\{A \in \mathcal{Q} \mid \langle R, A \rangle = 0\}$ is a hyperplane of the Voronoi space and the sets $\{A \in \mathcal{Q} \mid \langle R, A \rangle \leq 0\}$ and $\{A \in \mathcal{Q} \mid \langle R, A \rangle \geq 0\}$ are closed half spaces of the Voronoi space.

Assumption 2. Let P be a polyhedron in $\mathbb{R}^n$ such that $span\{P\} = \mathbb{R}^n$. Let $F_1, .., F_k$ be the facets of P and $R_1, .., R_k$ the corresponding facet vectors. Then the intersection of the closed half spaces $H_i = \{x \in \mathbb{R}^n \mid \langle R_i, x \rangle \geq 0\}$ for $i \in 1, .., k$ is the polyhedron P.

Assumption 3. A polyhedral cone is a polyhedron.

This last statement is the content of Theorem 1.3 in [Ziegler(1995)].

Appendix B: Comment on proposition 7.1.4 in [Martinet(2003)]

The proposition 7.1.4 as stated in [Martinet(2003)] is not correct.

The proposition states that the Voronoi cone is the convex hull of the projections cone, which is equal to the set of positive semi-definite quadratic forms of rank 1 or 0.

But the projections cone includes more than just the positive semi-definite quadratic forms of rank 1 or 0. The statement is therefore not correct.

To see this, I will define the angle and projections cone as presented in the book and then point to an element of the projections cone that does not have rank 1 or 0. Note that the book actually neither defines the angle nor circular cone. Some of the following is therefore a product of discussions with Peter Patzt, my supervisor.

- The inner product in the Voronoi space gives an angle between two elements $A, B \in \mathcal{Q}$ given as:
  $\text{angle}(A, B) = \arccos\left(\frac{\langle A,B\rangle}{\sqrt{\langle A,A\rangle}\sqrt{\langle B,B\rangle}}\right)$.
- A circular cone C is a set where all $c \in C$ satisfy $\text{angle}(c, A) = k$ for a given $k \in \mathbb{R}$ and a given axis, defined by an element A in the space.
- The projections cone is the circular cone with angle $\arccos(\frac{1}{\sqrt{n}})$ around the axis given by the identity matrix, I_n.

Now take a look at the quadratic form on $\mathbb{R}^3$ given by

$$Q = \begin{bmatrix} -2/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Q is clearly not positive semi-definite. Hence, it is not in the Voronoi cone and furthermore it does not have rank 1 or 0. But we see that Q is, in fact, in the projections cone, by calculating the angle:

$$\begin{aligned} \text{angle}(Q, I_3) &= \arccos\left(\frac{\langle Q, I_3\rangle}{\sqrt{\langle Q,Q\rangle}\sqrt{\langle I_3, I_3\rangle}}\right) = \arccos\left(\frac{Tr(Q)}{\sqrt{Tr(Q^2)}\sqrt{3}}\right) \\ &= \arccos\left(\frac{7/3}{\sqrt{49/9}\sqrt{3}}\right) = \arccos\left(\frac{1}{\sqrt{3}}\right). \end{aligned}$$

This particular example can be expanded to work for all $n \geq 3$ by adding zeroes.

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