MSRI Summer School: Representation stability

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1 Free groups and their automorphisms

Definition 1.1. Let T be a set. A *word* from the *alphabet* T is a map $[n] := \{1, \dots, n\} \to T$ for some $n \in \mathbb{N}_0$. We denote the *empty word* $[0] \to T$ by ε. We call n the *length* of a word $[n] \to T$.

Example 1.2. For an alphabet $T = \{a, b, c\}$, words are $\varepsilon, ab, aaa, abaca$. The have the lengths $0, 2, 3, 5$, respectively.

Definition 1.3. Let S be a set. Let $T = S \times \{-1, 1\}$, where by abuse of notation we identify $S \cong S \times \{1\} \subset T$ and write simply s for $(s, 1) \in T$. We also write s^{-1} for $(s, -1) \in T$. We call s^{-1} the inverse of s.

Define an equivalence relation on the set of words from the alphabet T by adding/removing a pair of adjacent s and s^{-1}.

The *free group* F_S is the set of all equivalence classes of words from T. Group multiplication is given by concatenation.

For $S = \{x_1, \dots, x_n\}$ denote F_S by F_n.

Exercise 1.4. Show that this describes a well defined group.

Theorem 1.5. *Let G be a group. A set map $S \to G$ uniquely determines a group homomorphism $F_S \to G$ extending the set map via $S \subset F_S$.*

Proof. Exercise. □

Definition 1.6. Let S be a subset of a group G. We say that S *generates* the smallest subgroup of G that contains S.

Observe that $S \subset G$ generates the group G if and only if the induced map $F_S \to G$ is surjective. More generally, the image of $F_S \to G$ is the subgroup generated by S.

Definition 1.7. Let G be a group. An *automorphism* of G is a group homomorphism $f\colon G \to G$, i.e. $f(gh) = f(g)f(h)$, that is bijective. The set of automorphisms of G is denoted by $\mathrm{Aut}(G)$ and it forms a group.

Note that an element of $f \in \mathrm{Aut}(F_n)$ is determined by the images $f(x_1), \dots, f(x_n)$.

Example 1.8. There is an inclusion of the symmetric group S_n into $\mathrm{Aut}(F_n)$ by sending $\sigma \in S_n$ to the automorphism defined by $x_i \mapsto x_{\sigma(i)}$.

Inverting the ith generator is an automorphism:

$$\mathrm{inv}_i \colon x_j \longmapsto \begin{cases} x_i^{-1} & j = i \\ x_j & j \neq i \end{cases}$$

Multiplying the ith generator to the jth (from the left or the right) is an automorphism:

$$\mathrm{leftmul}_{ij} \colon x_k \longmapsto \begin{cases} x_i x_j & k = j \\ x_k & k \neq j \end{cases}$$

$$\mathrm{rightmul}_{ij} \colon x_k \longmapsto \begin{cases} x_j x_i & k = j \\ x_k & k \neq j \end{cases}$$

Theorem 1.9 (Nielsen, 1924)**.** *$\mathrm{Aut}(F_n)$ is generated by permutations, inv_1, and $\mathrm{leftmul}_{12}$.*

Definition 1.10. Let G be a group. For $a, b \in G$, the *commutator* is $[a, b] = aba^{-1}b^{-1}$. The *commutator subgroup* G' of G is generated by all commutators. More generally, let H be a subgroup of G, denote $[G, H]$ to be the subgroup generated by commutators $[g, h]$ with $g \in G$ and $h \in H$.

The *lower central series* of G is a series of subgroups $\gamma_i G$ of G, defined recursively by $\gamma_1 G = G$ and $\gamma_{i+1} G = [G, \gamma_i G]$.

We call $G^{\mathrm{ab}} := G/G'$ the *abelianization* of G.

Proposition 1.11. *There is a surjective group homomorphism*

$$\mathrm{Aut}(F_n) \longrightarrow \mathrm{GL}_n(\mathbb{Z}).$$

Proof. Note that $F_n^{\mathrm{ab}} \cong \mathbb{Z}^n$ (exercise). Because abelianizing is functorial (exercise), we get a group homomorphism

$$\mathrm{Aut}(F_n) \longrightarrow \mathrm{Aut}(\mathbb{Z}^n).$$

Observe that $\mathrm{Aut}(\mathbb{Z}^n) \cong \mathrm{GL}_n(\mathbb{Z})$ is the group of invertible integral $n \times n$ matrices (exercise). $\mathrm{GL}_n(\mathbb{Z})$ is generated by $\begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$ and all matrices with ones on the diagonal and one one off the diagonal (exercise). All of these generators are in the image of the map $\mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$. More precisely, they are images of inv_1 and $\mathrm{leftmul}_{ij}$. □

Definition 1.12. The *Torelli subgroup* of $\mathrm{Aut}(F_n)$ is the kernel

$$\mathrm{IA}_n := \ker(\mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z})).$$

Exercises

1.) The free group: Let S be a set and S^{-1} the symbols of inverses of S. Adding and removing ss^{-1} or $s^{-1}s$ for $s \in S$ defines an equivalence relation on the set of words of $S \cup S^{-1}$. The *free group F_S* is the set of equivalence classes with concatenation as group multiplication.

(a) Prove that in every equivalence class there is exactly one fully canceled word, i.e. one word that doesn't contain an ss^{-1} or $s^{-1}s$ for $s \in S$.

(b) Prove that F_S is a group.

(c) Prove the universal property of F_S: Let G be a group. For every set map $f\colon S \to G$, there is a unique group homomorphism $F_S \to G$ extending f.

2.) The generators of $\mathrm{Aut}(F_n)$: Let $S = \{x_1, \dots, x_n\}$ and denote F_S by F_n. A group homomorphism $f\colon F_n \to F_n$ is given by the images $f(x_1), \dots, f(x_n)$. Define the *length* $|f|$ of f be the sum of the lengths of (the completely canceled words) $f(x_i)$.

(a) Prove that $|f| \geq n$ if $f \in \mathrm{Aut}(F_n)$.

(b) Observe that every permutation $\sigma \in S_n$ defines an automorphism $x_i \mapsto x_{\sigma(i)}$.

(c) Let inv_i be the automorphism of F_n defined by $x_j \mapsto x_j$ for $j \neq i$ and $x_i \mapsto x_i^{-1}$. Prove that if $|f| = n$ and $f \in \mathrm{Aut}(F_n)$, then f is generated by permutations and inv_1.

(d) Let $\mathrm{leftmul}_{ij}$ be the automorphism of F_n defined by $x_k \mapsto x_k$ for $k \neq j$ and $x_j \mapsto x_i x_j$. Let $\mathrm{rightmul}_{ij}$ be the automorphism of F_n defined by $x_k \mapsto x_k$ for $k \neq j$ and $x_j \mapsto x_j x_i$. Observe that the permutations, inv_1, and $\mathrm{leftmul}_{12}$ generate all $\mathrm{leftmul}_{ij}$ and $\mathrm{rightmul}_{ij}$.

(e) Let f be an automorphism of F_n with $|f| > n$. Let w_i, w_i' be the reduced words defined by $f(x_i), f^{-1}(x_i)$, resp. By replacing the x_i's in w_j' with w_i, we get a word that cancels to x_j. Observe that if $|w_j'| > 1$, one of the ${w_i}^{\pm 1}$ must be completely canceled only by its neighbors.

(f) If a ${w_i}^{\pm 1}$ is canceled completely by its neighbors where one neighbor cancels more letters than the other, use $\mathrm{leftmul}_{ij}$ or $\mathrm{rightmul}_{ij}$ to reduce the length of f.

(g) If all ${w_i}^{\pm 1}$ that are canceled completely by its neighbors are canceled exactly in the middle, let ${w_i}^{\pm 1}$ be one of these with minimal length and let ${w_i}^{\pm 1} = ab$ with $|a| = |b|$, use $\mathrm{leftmul}_{ij}$ or $\mathrm{rightmul}_{ij}$ to replace b^{-1}'s in the beginning and b's in the end of a w_j by a's and a^{-1} respectively. Prove that this can only be done finitely many times before the length of f can be reduced using (f).

(h) Conclude that there is a four element generator set of $\mathrm{Aut}(F_n)$.

3.) Prove that abelianizing is functorial. That means it is a functor from the category of groups to the category of abelian groups. Most importantly, for group homomorphisms $G \to H$, there exist homomorphisms between the abelianizations that behave well under composition.

4.) Prove the universal property of the abelianization: Let G be a group and A be an abelian group. Every group homomorphism $G \to A$ factors uniquely through the abelianization of G.

5.) Show that the abelianization of F_n is $\mathbb{Z}^n$.

6.) Observe that the group of group automorphisms of $\mathbb{Z}^n$ is precisely $\mathrm{GL}_n(\mathbb{Z})$.

2 VIC–modules

Definition 2.1. Let $\mathcal{C}$ be a category whose isomorphism classes of objects from a set. A $\mathcal{C}$–module is a functor from $\mathcal{C}$ to the category of abelian groups Ab. The category $\mathcal{C}-\mathsf{mod}$ of $\mathcal{C}$–modules has natural transformations as morphisms.

Example 2.2. Let G be a group and $\mathcal{C}$ be the one-object category whose morphisms are given by G. Then $\mathcal{C}$–modules is the same as $\mathbb{Z}G$–modules (which we will also call G–representations). Let $F\colon \mathcal{C} \to \mathsf{Ab}$ be a functor and let $*$ denote the single object of $\mathcal{C}$. Then $F(*)$ is an abelian group and for every $g \in G$, we get an endomorphism of $F(*)$ given by $F(g)$.

Example 2.3. Let $\mathcal{C}$ be a groupoid, i.e. all morphisms are isomorphisms. Let F be a $\mathcal{C}$–module. If objects C_1 and C_2 are isomorphic, so are $F(C_1)$ and $F(C_2)$. Thus, the category of $\mathcal{C}$–modules is equivalent to the product category of $\mathbb{Z}\operatorname{Aut}(C)$–modules for all C in a set of representatives of the isomorphism classes of objects of $\mathcal{C}$. That is the same as a collection of $\mathbb{Z}\operatorname{Aut}(C)$–modules.

For example, let R be a commutative ring and let $\mathcal{C}$ be the groupoid of all finitely generated free abelian groups$R$–modules and isomorphisms. Then $C\!-\!\mathsf{mod}$ is equivalent to the category of sequences $(M_n)_{n\in\mathbb{N}_0}$ where M_n is a $\mathbb{Z}\operatorname{GL}_n(R)$–modiule.

Definition 2.4. Let R be a commutative ring. Define $\mathsf{VIC}(R)$ to be the category whose objects are finitely generated free R–modules and whose morphisms are

$$\operatorname{Hom}_{\mathsf{VIC}(R)}(V,W) := \{(f,C) \mid f\colon V \hookrightarrow W, C \text{ is free}, \operatorname{im} f \oplus C = W\}.$$

For $(f,C) \in \operatorname{Hom}_{\mathsf{VIC}(R)}(V,W)$ and $(g,D) \in \operatorname{Hom}_{\mathsf{VIC}(R)}(U,V)$, the composition is given by

$$(f,C)\circ(g,D) := (f\circ g, C \oplus f(D)) \in \operatorname{Hom}_{\mathsf{VIC}(R)}(U,W).$$

We want to make some easy observations:

- $\mathsf{VIC}(R)$ is equivalent to the induced subcategory on only the objects R^n for $n \in \mathbb{N}_0$.
- The endomorphisms $\operatorname{Hom}_{\mathsf{VIC}(R)}(R^n,R^n)$ are all isomorphisms and $\operatorname{Aut}_{\mathsf{VIC}(R)}(R^n) \cong \operatorname{GL}_n(R)$.
- Let M be a $\mathsf{VIC}(R)$–module, then it gives rise to a sequence $(M_n)_{n\in\mathbb{N}_0}$ of $\mathbb{Z}\operatorname{GL}_n(R)$–modules.
- The standard decomposition $R^n \oplus R \to R^{n+1}$ induces a $\operatorname{GL}_n(R)$–equivariant map $\phi_n\colon M_n \to M_{n+1}$.

Proposition 2.5. *A sequence $(M_n)_{n\in\mathbb{N}_0}$ of $\mathbb{Z}\operatorname{GL}_n(R)$–modules together with $\operatorname{GL}_n(R)$–equivariant maps $\phi_n\colon M_n \to M_{n+1}$ comes from a $\mathsf{VIC}(R)$–module if and only if $\operatorname{GL}_m(R)$ acts trivially on the image of $\phi_{n+m-1}\circ\cdots\circ\phi_n\colon M_n \to M_{n+m}$. Such a $\mathsf{VIC}(R)$–module is then uniquely determined.*

Definition 2.6. Let $M(m)$ denote the representable functor $\mathbb{Z}\operatorname{Hom}_{\mathsf{VIC}(R)}(R^m,-)$. We call a direct sum of representable functors *free.*

More about free $\mathsf{VIC}(R)$–modules in the exercises.

Exercises

1.) Prove that $\operatorname{End}_{\mathsf{VIC}(R)}(R^n) = \operatorname{Aut}_{\mathsf{VIC}(R)}(R^n) \cong \operatorname{GL}_n(R)$.

2.) Show that $\operatorname{Hom}_{\mathsf{VIC}(R)}(R^m,R^n) \cong \operatorname{GL}_n(R)/\operatorname{GL}_{n-m}(R)$ as a $\operatorname{GL}_n(R)$-set.

3.) Let $F\colon \mathsf{VIC}(R) \to \mathsf{Ab}$ be a functor.

   (a) Show that $M_n := F(R^n)$ is a $\operatorname{GL}_n(R)$–representation.

(b) Show that $(f, C) \in \mathrm{Hom}_{\mathsf{VIC}(R)}(R^n, R^{n+1})$ given by $f(e_i) = e_i$ for $1 \leq i \leq n$ and $C = \mathrm{span}(e_{n+1})$ induces a $\mathrm{GL}_n(R)$–equivariant map $\phi_n \colon M_n \to M_{n+1}$.

(c) Show that $\mathrm{GL}_m(R)$ included into $\mathrm{GL}_{n+m}(R)$ by the block inclusion

$$A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

acts trivially on the image of the composition

$$M_n \xrightarrow{\phi_n} M_{n+1} \xrightarrow{\phi_{n+1}} \cdots \xrightarrow{\phi_{n+m-1}} M_{n+m}.$$

(d) Conversely, let $(M_n)_{n \in \mathbb{N}_0}$ be a sequence of $\mathrm{GL}_n(R)$–representations and let there be $\mathrm{GL}_n(R)$–equivariant maps $\phi_n \colon M_n \to M_{n+1}$. If $\mathrm{GL}_m(R)$ acts trivially on the image of the composition $M_n \to M_{n+m}$, there is a $\mathsf{VIC}(R)$–module $F \colon \mathsf{VIC}(R) \to \mathsf{Ab}$ such that $F(R^n) = M_n$ and $(f, C) \in \mathrm{Hom}_{\mathsf{VIC}(R)}(R^n, R^{n+1})$ given by $f(e_i) = e_i$ for $1 \leq i \leq n$ and $C = \mathrm{span}(e_{n+1})$ induces $\phi_n \colon M_n \to M_{n+1}$.

4.) Let $M(m) := \mathbb{Z}\,\mathrm{Hom}_{\mathsf{VIC}(R)}(R^m, -)$ define a free $\mathsf{VIC}(R)$–module.

(a) Show that $M(m)$ is generated by one element.

(b) Show that $\mathrm{Hom}_{\mathsf{VIC}(R)-\mathsf{mod}}(M(m), M) \cong M_m$.

(c) Show that if M is generated in degrees $\leq d$, there is a set I, numbers $m_i \leq d$ for $i \in I$ and a surjection $\bigoplus_{i \in I} M(m_i) \to M$.

5.) The following functors from $\mathsf{VIC}(R)$–modules can be considered forgetful functors. Find their left adjoints.

(a) Fix $m \in \mathbb{N}_0$. Let $\mathsf{VIC}(R)-\mathsf{mod} \to \mathsf{Set}$ be the functor sending M to the underlying set of M_m.

(b) Let $\mathsf{VIC}(R)-\mathsf{mod} \to \mathsf{Set}^{\mathbb{N}_0}$ be the functor sending M to the sequence of underlying sets of $(M_m)_{m \in \mathbb{N}_0}$.

(c) Fix $m \in \mathbb{N}_0$. Let $\mathsf{VIC}(R)-\mathsf{mod} \to \mathrm{GL}_m(R) - \mathsf{Set}$ be the functor sending M to the underlying $\mathrm{GL}_m(R)$–set of M_m.

(d) Let $\mathsf{VIC}(R)-\mathsf{mod} \to \prod_{m \in \mathbb{N}_0} \mathrm{GL}_m(R) - -\mathsf{Set}$ be the functor sending M to the sequence of underlying $\mathrm{GL}_m(R)$–sets of $(M_m)_{m \in \mathbb{N}_0}$.

(e) Fix $m \in \mathbb{N}_0$. Let $\mathsf{VIC}(R)-\mathsf{mod} \to \mathsf{Ab}$ be the functor sending M to the underlying abelian group M_m.

(f) Let $\mathsf{VIC}(R)-\mathsf{mod} \to \mathsf{Ab}^{\mathbb{N}_0}$ be the functor sending M to the sequence of underlying abelian groups $(M_m)_{m \in \mathbb{N}_0}$.

(g) Fix $m \in \mathbb{N}_0$. Let $\mathsf{VIC}(R)-\mathsf{mod} \to \mathrm{GL}_m(R)-\mathsf{mod}$ be the functor sending M to the $\mathrm{GL}_m(R)$–representation M_m.

(h) Let $\mathsf{VIC}(R)-\mathsf{mod} \to \prod_{m \in \mathbb{N}_0} \mathrm{GL}_m(R)-\mathsf{mod}$ be the functor sending M to the sequence of $\mathrm{GL}_m(R)$–modules $(M_m)_{m \in \mathbb{N}_0}$.

3 The homology of IA_n

We first recall how group homology is constructed. Let G be a group and M be a $\mathbb{Z}G$–module. Let E_*G be a projective $\mathbb{Z}G$–resolution of the trivial representation. This is unique up to chain homotopy (exercise). Group homology $H_i(G; M)$ is the homology of the chain complex

$$E_*G \otimes_G M.$$

Group homology is functorial in the following sense. Let G and H be two groups and $\phi\colon G \to H$ be group homomorphism. Let M be a $\mathbb{Z}G$–module and N a $\mathbb{Z}H$–module. Through ϕ the module N can be considered as a $\mathbb{Z}G$–module that is denoted by ϕ^*N. Let $\psi\colon M \to \phi^*N$ be a G–equivariant map. Let $\xi\colon E_*G \to \phi * E_*H$ be a G-equivariant map that induces the identity map on the trivial representation. Such a map exists and is unique up to chain homotopy because E_*G is a projective resolution. Then

$$E_*G \otimes_G M \xrightarrow{\xi\otimes\psi} E_*H \otimes_H N$$

is a map of chain complexes and induces a homomorphism

$$H_*(G; M) \longrightarrow H_*(H; N).$$

Let

$$1 \to K \to G \to Q \to 1$$

be a short exact sequence of groups. Fix $g \in G$ and let $c_g \in \mathrm{Aut}(K)$ be the conjugation $c_g(k) = gkg^{-1}$. Let M be a $\mathbb{Z}G$–module. Then $\psi(m) = gm$ gives an K–equivariant map $\psi\colon \mathrm{Res}^G_K M \to \mathrm{Res}^G_K \phi^*M$. Similarly, $\xi(x) = xg^{-1}$ gives an K–equivariant map $\xi\colon \mathrm{Res}^G_K E_*G \to \mathrm{Res}^G_K \phi^*E_*G$. Therefore,

$$\mathrm{Res}^G_K E_*G \otimes_K \mathrm{Res}^G_K M \xrightarrow{\xi\otimes\psi} \mathrm{Res}^G_K E_*G \otimes_K \mathrm{Res}^G_K M$$

gives rise to an automorphism of $H_*(K; M)$. Because K acts trivially by this action, $H_*(K; M)$ is in fact a $\mathbb{Z}Q$–module.

Now consider the sequence of short exact sequences

$$1 \to \mathrm{IA}_n \to \mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z}) \to 1.$$

Then $(H_*(\mathrm{IA}_n; \mathbb{Z}))_{n\in\mathbb{N}_0}$ is a sequence of $\mathbb{Z}\,\mathrm{GL}_n(\mathbb{Z})$–modules. The inclusion $\mathrm{IA}_n \subset \mathrm{IA}_{n+1}$ induces a $\mathrm{GL}_n(\mathbb{Z})$–equivariant map

$$\phi_n\colon H_*(\mathrm{IA}_n; \mathbb{Z}) \longrightarrow H_*(\mathrm{IA}_{n+1}; \mathbb{Z}).$$

This data comes in fact from a $\mathsf{VIC}(\mathbb{Z})$–module, that we will denote by $H_i(\mathrm{IA})$ (exercise).

Let us concentrate now on $H_1(\mathrm{IA})$.

Theorem 3.1 (Andreadakis 1965 (for $n \leq 3$), Bachmuth 1966)**.**

$$H_1(\mathrm{IA}_n) \cong \mathrm{Hom}(\mathbb{Z}^n, \bigwedge^2 \mathbb{Z}^n) \cong (\mathbb{Z}^n)^* \otimes \bigwedge^2 \mathbb{Z}^2$$

Proposition 3.2. *There is a $\mathsf{VIC}(\mathbb{Z})$–module M given by $M_n = (\mathbb{Z}^n)^* \otimes \bigwedge^2 \mathbb{Z}^2$ and*

$$\phi_n\colon (\mathbb{Z}^n)^* \otimes \bigwedge^2 \mathbb{Z}^{n+1} \longrightarrow (\mathbb{Z}^{n+1})^* \otimes \bigwedge^2 \mathbb{Z}^{n+1}$$
$$e_i^* \otimes (e_j \wedge e_k) \longmapsto e_i^* \otimes (e_j \wedge e_k),$$

where $e_1, \ldots, e_m$ denotes the standard basis of $\mathbb{Z}^m$ and $e_1^*, \ldots, e_m^*$ its dual basis of $(\mathbb{Z}^m)^*$.

Corollary 3.3. *The $\mathsf{VIC}(\mathbb{Z})$–module $H_1(\mathrm{IA})$ is isomorphic to M from the previous proposition.*

Exercises

1.) Let G be a group and let M and N be $\mathbb{Z}G$–modules. Let $P_* \to M \to 0$ and $Q_* \to N \to 0$ be projective G–resolutions. Given a G–equivariant map $M \to N$, show that up to chain homotopy there exists a unique G–equivariant map of chain complexes $P_* \to Q_*$ inducing the given map.

2.) Let G be a group. Show $H_1(G;\mathbb{Z}) \cong G^{\text{ab}}$.

3.) Let H be a subgroup of G. Show that a projective resolution $E_*G \to \mathbb{Z} \to 0$ of $\mathbb{Z}G$–modules is also a projective resolution of $\mathbb{Z}H$–modules.

4.) Let

$$1 \to K \to G \to Q \to 1$$

be a short exact sequence of groups. This exercise proves that Q acts on the homology of K.

(a) G acts on K by conjugation. Let $E_*G \to \mathbb{Z} \to 0$ be a projective (right) G–resolution of the trivial representation. Check that multiplication by g^{-1} induces a map of chain complexes $E_*G \otimes_K \mathbb{Z} \to E_*G \otimes_K \mathbb{Z}$. Thus induces an action of G on the homology of K.

(b) Check that K acts trivially through this action and deduce that Q acts.

5.) We want to show that there is a $\mathsf{VIC}(\mathbb{Z})$–module structure on the sequence $(H_i(\mathrm{IA}_n))_{n\in\mathbb{N}_0}$ for every fixed $i \in \mathbb{N}_0$. The $\mathrm{GL}_n(\mathbb{Z})$–action on $H_i(\mathrm{IA}_n)$ follows from the previous exercise.

(a) The inclusion $\mathrm{IA}_n \subset \mathrm{IA}_{n+1}$ induces a map on homology. Check that this map is $\mathrm{GL}_n(\mathbb{Z})$–equivariant.

(b) Show that $\mathrm{GL}_m(\mathbb{Z})$ acts trivially on the image of $H_i(\mathrm{IA}_n) \to H_i(\mathrm{IA}_{n+m})$.

6.) We want to prove Corollary 3.3: Recall that the Johnson homomorphism sends $f \in \mathrm{IA}_n$ to $x \cdot F_n' \mapsto f(x)x^{-1} \cdot [F_n, F_n']$ which is a map in $\mathrm{Hom}(\mathbb{Z}^n, \bigwedge^2\mathbb{Z}^n)$. Show that this gives a morphism of $\mathsf{VIC}(\mathbb{Z})$–modules.

7.) We want to prove that $H_1(\mathrm{IA}_n) \cong \mathrm{Hom}(\mathbb{Z}^n, \bigwedge^2 \mathbb{Z}^n)$. Recall that $\gamma_2 F_n = [F_n, F_n]$ and $\gamma_3 F_n = [F_n, [F_n, F_n]]$. Let $x_1, \dots, x_n$ denote the basis of F_n and $e_i = x_i \cdot \gamma_2 F_n$ the basis of $\mathbb{Z}^n \cong F_n/\gamma_2 F_n$.

(a) Prove that $\bigwedge^2 \mathbb{Z}^n \to \gamma_2 F_n/\gamma_3 F_n$ defined by $e_i \wedge e_j \mapsto [x_i, x_j] \cdot \gamma_3 F_n$ is an isomorphism.

(b) Let $f \in \mathrm{IA}_n$ and $c \in \gamma_2 F_n$. Show that $f(c) \in \gamma_3 F_n$.

(c) Prove that $\mathrm{IA}_n \to \mathrm{Hom}(F_n/\gamma_2 F_n, \gamma_2 F_n/\gamma_3 F_n)$ given by sending f to $w \cdot \gamma_2 F_n \mapsto f(w)w^{-1}\gamma_3 F_n$.

(d) Prove that the map is surjective.

(e) Show that IA_n can be generated by $\frac{1}{2}n^2(n-1)$ elements implies that $\mathbb{Z}^{\frac{1}{2}n^2(n-1)}$ surjects onto $H_1(\mathrm{IA}_n)$.

(f) Finish the proof.

4 Central stability homology

Central stability homology is supposed to detect in which degrees syzygies of $\mathsf{VIC}(\mathbb{Z})$–modules are generated. Let us first construct it:

Definition 4.1. Let M be a $\mathsf{VIC}(\mathbb{Z})$–module. For later notation, let $e_1, \dots, e_n$ denote the standard basis of $\mathbb{Z}^n$. Define

$$CS_p(M)_n := \bigoplus_{(f,C) \in \operatorname{Hom}_{\mathsf{VIC}(\mathbb{Z})}(\mathbb{Z}^p, \mathbb{Z}^n)} M_C.$$

Let $d_i \colon CS_p(M)_n \to CS_{p-1}(M)_n$ map the summand M_C corresponding to (f, C) to the summand $M_{C \oplus \operatorname{span}(e_i)}$ corresponding to

$$(f, C) \circ (\operatorname{inc}_i, \operatorname{span}(e_i)) = (f \circ \operatorname{inc}_i, C \oplus \operatorname{span}(e_i)),$$

where $\operatorname{inc}_i \colon \mathbb{Z}^{p-1} \to \mathbb{Z}^p$ with

$$\operatorname{inc}_i(e_j) = \begin{cases} e_j & j < i \\ e_{j+1} & j \geq i. \end{cases}$$

Define $\partial := \sum_{i=1}^p (-1)^i d_i \colon CS_p(M)_n \to CS_{p-1}(M)_n$.

We call $CS_*(M)$ the *central stability chain complex* of M and its homology $HS_*(M) := H_*(CS_*(M))$ the *central stability homology* of M.

Theorem 4.2 (Maazen 1979, Randal-Williams–Wahl 2017)**.** *$HS_i(M(0))_n \cong 0$ for all $n > 2i$.*

This theorem can be reinterpreted to a connectivity statement. There is a semi-simplicial set (δ–complex) $W(n)$ whose p-simplicies $W_p(n) = \operatorname{Hom}_{\mathsf{VIC}(\mathbb{Z})}(\mathbb{Z}^{p+1}, \mathbb{Z}^n)$. Face maps are given by precomposition of $(\operatorname{inc}_i, \operatorname{span}(e_i)) \in \operatorname{Hom}_{\mathsf{VIC}(\mathbb{Z})}(\mathbb{Z}^p, \mathbb{Z}^{p+1})$. The reduced homology of $W(n)$ is can be computed using its simplicial chain complex

$$\tilde{C}_p(W(n)) = \mathbb{Z}W_p(n) = \mathbb{Z} \operatorname{Hom}_{\mathsf{VIC}(\mathbb{Z})}(\mathbb{Z}^{p+1}, \mathbb{Z}^n).$$

It is easy to observe that

$$\tilde{C}_*(W(n)) \cong CS_{*+1}(M(0))_n.$$

Theorem 4.3 (P.)**.** *Let M be a $\mathsf{VIC}(\mathbb{Z})$–module, $N \in \mathbb{N}_0$, and $d_0, \dots, d_N \in \mathbb{N}_0$ with $d_{i+1} - d_i \geq 2$. Then the following two statements are equivalent.*

1. *There is a partial resolution*
$$P_N \to \cdots \to P_0 \to M \to 0$$
of free $\mathsf{VIC}(\mathbb{Z})$–modules P_i generated in degrees $\leq d_i$.

2. *$HS_i(M)_n \cong 0$ for all $n > d_i$.*

Proof. Exercise. □

Exercises

1.) Show that $CS_*(M)_n$ together with ∂ is a chain complex.

2.) A semi-simplicial set is a sequence of sets $(X_p)_{p \in \mathbb{N}_0}$ together with maps $d_i \colon X_p \to X_{p-1}$ for $i = 0, \dots, p$ such that $d_i \circ d_j = d_{j-1} \circ d_i$ for all pairs $i < j$.

   (a) Show that every semi-simplicial set is isomorphic to a set of the form, where $X_p \subset X_0^{p+1}$ and $d_i \colon X_p \to X_{p-1}$ will drop the $(i-1)$th entry from the sequence.

   (b) Check that $W_p(n) = \operatorname{Hom}_{\mathsf{VIC}(\mathbb{Z})}(\mathbb{Z}^{p+1}, \mathbb{Z}^n)$ gives a semi-simplicial set.

(c) The simplicial chain complex of a semi-simplicial set X is given by $C_p(X) = \mathbb{Z}X_p$ and $\partial = \sum(-1)^i d_i$. Check that $C_*(W(n)) \cong CS_{*-1}(M(0))_n$.

3.) This exercise shall show that central stability homology gives bounds on the generation degree of syzygies. You may use that $HS_i(M(0))_n = 0$ for all $n > 2i$.

(a) Observe that a $\mathsf{VIC}(\mathbb{Z})$–module M is generated in degrees $\leq d$ if and only if $HS_0(M)_n = 0$ for all $n > d$.

(b) Show $HS_i(M(m))_n = 0$ for all $n > 2i + m$.

(c) Let $d_0, \ldots, d_N \in \mathbb{N}_0$ with $d_{i+1} - d_i \geq 2$ and

$$P_N \to \cdots \to P_0 \to M \to 0$$

be a resolution of free $\mathsf{VIC}(\mathbb{Z})$–modules P_i generated in degrees $\leq d_i$. Show that $HS_i(M)_n = 0$ for all $n > d_i$.

(d) Let d_i be as above and $HS_i(M)_n \cong 0$ for all $n > d_i$. Show that there exists a partial resolution

$$P_N \to \cdots \to P_0 \to M \to 0$$

of free $\mathsf{VIC}(\mathbb{Z})$–modules P_i generated in degrees $\leq d_i$.

5 Highly connected simplicial complexes

Definition 5.1. An (abstract) simplicial complex X on a vertex set V is a set of nonempty subsets of V that is closed under subsets and contains all singletons. We call a subset in X a simplex of X. If a simplex has $(p + 1)$ elements it is called an p-simplex or p-dimensional. A proper subset of a simplex is called a face.

A (topological) p-simplex is the topological space given by the convex hull of the standard basis vectors in $\mathbb{R}^{p+1}$. The simplex spanned by a proper subset of standard basis vectors is called a face. The (topological) realization $|X|$ of an abstract simplicial complex X is the space of topological simplicies for each simplex in X glued along their faces.

Definition 5.2. An (abstract) Δ-complex X (or semisimpicial set) is a sequence of sets $(X_p)_{p\in\mathbb{N}_0}$ together with face maps $d_i \colon X_{p+1} \to X_p$ for each $i \in \{0, \ldots, p + 1\}$ and $p \geq 1$, such that

$$d_i \circ d_j = d_{j-1} \circ d_i \quad \text{for } i < j.$$

The (topological) realization $|X|$ of an abstract Δ-complex X is the space of topological p-simplicies for each element in X_p for all $p \geq 1$ glued together along the face maps.

Exercise 5.3. Given an abstract simplicial complex, find an abstract Δ-complex with the same realization.

Definition 5.4. The simplicial chain complex $C_*(X)$ of a Δ-complex X is given by $C_p(X) = \mathbb{Z}X_p$ and the boundary map $\partial = \sum(-1)^i d_i$. Denote the homology of this chain complex by $H_*(X)$. (It is isomorphic to the (singular) homology of the realization.)

Definition 5.5. A simplicial map $X \to Y$ between simplicial complexes is a map between the vertex sets such that the image of a simplex of X is a simplex of Y.

For a simplicial complex X, let $[S^p, X]$ be the set of equivalence classes of all simplicial maps $Y \to X$ for all simplicial complexes Y whose realization is homeomorphic to the p-sphere S^p under the following equivalence relation.[1] $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ are (freely homotopy) equivalent if there is a simplicial complex Z whose realization is homeomorphic to $S^p \times [0,1]$ and whose two boundaries are Y_1 and Y_2 together with a simplicial map $Z \to X$ that restricts to f_1 and f_2 on the boundary. ($[S^p, X]$ is in bijection to the set of free homotopy classes of continuous maps $S^p \to |X|$.)

A simplicial complex X is called n-connected if $[S^p, X]$ contains only the trivial class for all $p \leq n$.

Theorem 5.6 (Hurewicz)**.** *If a simplicial complex is n-connected than $\tilde{H}_i(X) \cong 0$ for all $i \leq n$.*

Definition 5.7. Let X be a simplicial complex. The link of a simplex σ in X is the union of all simplicies that are disjoint from σ and whose union with σ is also a simplex in X. It is denoted by $\mathrm{Lk}_X(\sigma)$.

A simplicial simplex X is called weakly Cohen-Macaulay of dimension n if X is $(n-1)$-connected and $\mathrm{Lk}_X(\sigma)$ is $(n-p-2)$-connected for every p-simplex σ of X.

Definition 5.8. Let PB_n be the partial basis complex of $\mathbb{Z}^n$, i.e. a set of nonzero vectors in $\mathbb{Z}^n$ form a simplex if they can be completed to a basis of $\mathbb{Z}^n$.

Theorem 5.9 (Maazen 1979)**.** *PB_n is $(n-2)$–connected.*

Proof. Exercise. ☐

Definition 5.10. Let us define the simplicial complex PBC_n. Its vertex set contains all pairs (v, H), where $v \in \mathbb{Z}^n$ is nonzero and $H \subset Z^n$ is a summand such that $\mathrm{span}(v) \oplus H = \mathbb{Z}^n$. The subset $\{(v_0, H_0), \ldots, (v_p, H_p)\}$ is a simplex if $\{v_0, \ldots, v_p\}$ is a partial basis of $\mathbb{Z}^n$ and $v_i \in H_j$ for all $i \neq j$.

Definition 5.11. A join complex over a simplicial complex X is a simplicial complex Y together with a simplicial map $\pi : Y \to X$, satisfying the following properties:

1. π is surjective.
2. π is simplexwise injective.
3. A collection of vertices $y_0, \ldots, y_p$ spans a simplex of Y whenever there exists simplices $\theta_0, \ldots, \theta_p$ such that for all i, y_i is a vertex of θ_i and the simplex $\pi(\theta_i)$ has vertices $\pi(y_0), \ldots, \pi(y_p)$.

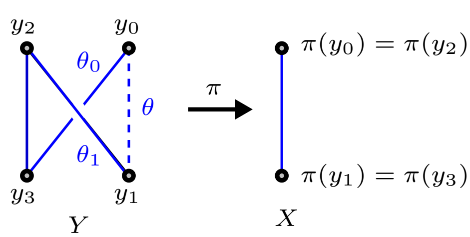

Figure 1: The map π does not exhibit Y as a join complex over X unless θ is a simplex of Y.

Theorem 5.12 (Hatcher–Wahl 2010)**.** *Let Y be a join complex over X via $\pi : Y \to X$. Assume X is weakly Cohen–Macaulay of dimension n. Further assume that for all p-simplices τ of Y, the image of the link $\pi(\mathrm{Lk}_Y(\tau))$ is weakly Cohen–Macaulay of dimension $(n-p-2)$. Then Y is $\frac{n-2}{2}$–connected.*

[1] You may assume that the link (defined below) of a q-simplex is homeomorphic to S^{q-p-1}.

Theorem 5.13 (Randal-Williams–Wahl 2017)**.** PBC_n *is* $\frac{n-3}{2}$*–connected.*

Proof. In the exercises, it is shown that PBC_n is a join complex over PB_n. The other conditions for the previous theorem are also shown. □

Definition 5.14. Let X be a simplicial complex. Define $X^{\mathrm{ord}} = (X_p^{\mathrm{ord}})_{p\in\mathbb{N}_0}$ to be the Δ-complex whose p-simplices are

$$X_p^{\mathrm{ord}} = \{(x_0, \dots, x_p) \in X_0^{p+1} \mid \{x_0, \dots, x_p\} \text{ is a } p\text{-simplex in } X\}.$$

Proposition 5.15 (Randal-Williams–Wahl 2017)**.** *Let X be a simplicial complex that is weakly Cohen–Macaulay of dimension n then X^{ord} is $(n-1)$–connected.*

Proof. Exercise. □

Corollary 5.16. $HS_i(M(0))_n \cong 0$ *for all* $n > 2i$.

Proof. Exercise. □

Exercises

1.) Given an abstract simplicial complex, find an abstract Δ-complex with the same realization.

2.) In this exercise, we want to show that PB_n is weakly Cohen–Macaulay of dimension $n-1$, following a proof by Church–Putman 2017.

  (a) Let V be a summand of $\mathbb{Z}^n$ and $v_0, \dots, v_p$ a basis of V. Observe that $\mathrm{Lk}_{\mathrm{PB}_n}(\{v_0, \dots, v_p\})$ is independent of the choice of basis of V. Denote this link by $\mathrm{Lk}_{\mathrm{PB}_n}(V)$.

  (b) Let $\mathrm{PB}_n^m = \mathrm{Lk}_{\mathrm{PB}_{n+m}}(\mathbb{Z}^m)$. Show that $\mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma) \cong \mathrm{PB}_{n-p}^{m+p}$ for every $(p-1)$-simplex σ of PB_n^m.

  (c) Fix an $F\colon \mathbb{Z}^{m+n} \to \mathbb{Z}$ and $N > 0$. For a subcomplex X of PB_n^m, define $X^{<N}$ to be the full subcomplex of X spanned by the vertices v with $|F(v)| < N$. Let σ be a simplex of PB_n^m that has a vertex v with $F(v) = N$. Show that $\mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma)$ can be retracted to $\mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma)^{<N}$.

  (d) We will prove that PB_n^m is $(n-2)$–connected by induction over n. For $n = 0$ there is nothing to show. For $n = 1$, prove that PB_1^m is non-empty for $m \geq 0$.

  (e) For the induction step, fix a map $\phi\colon S^p \to \mathrm{PB}_n^m$ with $0 \leq p \leq n-2$. (We may assume that there is a triangulation of S^p such ϕ is simplicial.) We want to nullhomtope ϕ. Let $F\colon \mathbb{Z}^{m+n} \to \mathbb{Z}$ be the map that returns the last coordinate and let

$$R(\phi) = \max(F(v) \mid v \text{ a vertex of } \mathrm{PB}_n^m \text{ in the image of } \phi).$$

  Show that the sphere can be coned off if $R(\phi) = 0$.

  (f) If $R = R(\phi) > 0$ then there is a simplex σ of S^p of maximal dimension (with respect to the following condition) such that $F(\phi(x)) = R$ for all $x \in \sigma$. Check that ϕ maps $\mathrm{Lk}_{S^p}(\sigma)$ into $\mathrm{Lk}_{\mathrm{PB}_n^m}(\phi(\sigma))^{<R}$.

  (g) Assume that σ is k–dimensional. You may use that $\mathrm{Lk}_{S^p}(\sigma)$ homeomorphic to S^{p-k-1}. Assume that $\phi(\sigma)$ is ℓ–dimensional. (Note that $k \geq \ell$.) Prove that $\mathrm{Lk}_{\mathrm{PB}_n^m}(\phi(\sigma))^{<R}$ is $(n-\ell-3)$–connected.

  (h) Homotope ϕ to replace $\phi(\sigma)$ by a subcomplex in $\mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma)^{<N}$.

  (i) Observe that we can get reduce $R(\phi)$ this way. And finish the proof.

3.) Show that the link of a simplex in PBC_n is isomorphic to a PBC_m for some $m \leq n$.

4.) Show that PBC_n is a join complex over PB_n by the map π that forgets the complement.

5.) Let τ be a p–simplex of PBC_n. Show that $\mathrm{Lk}_{\mathrm{PB}_n}(\phi(\tau))$ is weakly Cohen–Macaulay of dimension $n-p-3$.

6.) Use Proposition 5.15 and the previous exercise to show that $HS_i(M(0))_n \cong 0$ for $n > 2i$.

6 Quillen's spectral sequence argument for homological stability

In this lecture, we want to revisit Quillen's argument for homological stability with twisted coefficients and look at a similar argument for representation stability.

Theorem 6.1. *Let M be a $\mathsf{VIC}(\mathbb{Z})$–module. The map $\phi_n\colon M_n \to M_{n+1}$ induces a stabilization map on homology*

$$H_*(\mathrm{GL}_{n-1}(\mathbb{Z}); M_{n-1}) \longrightarrow H_*(\mathrm{GL}_n(\mathbb{Z}); M_n).$$

Assume that $HS_i(M) \cong 0$ for all $n > ki + a$ for some $k \geq 2$. Then the stabilization map is an isomorphism for all $n > k(i+1) + a + 1$ and surjective for all $n > k(i+1) + a$.

Proof. Consider the double complex

$$E_* \,\mathrm{GL}_n(\mathbb{Z}) \otimes_{\mathrm{GL}_n(\mathbb{Z})} CS_*(M)_n.$$

There are two spectral sequences coming from this double complex, which both converge to the homology of the total complex of the double complex. The first spectral sequence takes central stability homology first:

$$E^1_{pq} \cong E_p \,\mathrm{GL}_n(\mathbb{Z}) \otimes_{\mathrm{GL}_n(\mathbb{Z})} HS_q(M)_n$$

Since $HS_i(M)_n \cong 0$ for all $n > ki + a$, we get in particular $E^1_{pq} \cong 0$ for $n > k(p+q) + a$. The spectral sequence therefore converges to zero in that range.

The second spectral sequence takes group homology first:

$$E^1_{pq} \cong H_q(\mathrm{GL}_n(\mathbb{Z}); CS_p(M)_n) \cong H_q(\mathrm{GL}_n(\mathbb{Z}); \mathrm{Ind}^{\mathrm{GL}_n(\mathbb{Z})}_{\mathrm{GL}_{n-p}(\mathbb{Z})} M_{n-p}) \cong H_q(\mathrm{GL}_{n-p}(\mathbb{Z}); M_{n-p})$$

This spectral sequence converges to zero in the same range. It turns out that $d^1\colon E^1_{pq} \to E^1_{p-1,q}$ is the stability map $H_*(\mathrm{GL}_{n-p}(\mathbb{Z}); M_{n-p}) \longrightarrow H_*(\mathrm{GL}_{n-p+1}(\mathbb{Z}); M_{n-p+1})$ if p is odd and zero if p is even.

We prove the theorem by induction over i. The statement is trivial for $i < 0$. Let us assume that the statement is true for $q < i$. Observe that $E^2_{pq} \cong 0$ for $q < i$ and $n > kq + p + a + 1$. This implies that $E^2_{0,i} \cong E^\infty_{0,i} \cong 0$ for $n > k(i+1) + a$ and thus the stability map is surjective in that range. Likewise, it implies that $E^2_{1,i} \cong E^\infty_{pq} \cong 0$ for $n > k(i+1) + a + 1$ and thus the stability map is injective in that range. ☐

For representation stability, let us consider the spectral sequence

$$E_* \,\mathrm{Aut}(F_n) \otimes_{\mathrm{IA}_n} \tilde{C}_{*-1}(Y(n)),$$

where $Y(n)$ is a semi-simplicial set similar to $W(n)$ but for $\mathrm{Aut}(F_n)$. In particular, $\tilde{C}_{p-1}(Y(n)) \cong \mathrm{Ind}^{\mathrm{Aut}\,F_n}_{\mathrm{Aut}\,F_{n-p}} \mathbb{Z}$ and the following connectivity result is true.

Theorem 6.2 (Hatcher-Vogtmann 1998, Randal-Williams–Wahl 2017)**.** $\tilde{H}_{i-1}(Y(n)) \cong 0$ *for $n > 2i$.*

Thus both spectral sequences associated to this double complex converge to zero for $n > 2(p+q)$. The spectral sequence taking group homology first turns out to be

$$E^2_{pq} \cong HS_p(H_q(\mathrm{IA}))_n,$$

where $H_q(\mathrm{IA})$ is the $\mathsf{VIC}(\mathbb{Z})$–module introduced in Lecture 4. Clearly, $H_0(\mathrm{IA}) \cong M(0)$ and therefore $E^2_{p,0} \cong 0$ for $n > 2p$. This implies that $E^2_{0,1} \cong HS_0(H_1(\mathrm{IA}))_n \cong 0$ for $n > 4$ and $E^2_{1,1} \cong HS_1(H_1(\mathrm{IA}))_n \cong 0$ for $n > 6$. This means that $H_1(\mathrm{IA})$ is generated in degrees ≤ 4 and related in degrees ≤ 6.

We know the first homology of IA_n exactly already, though. And it will turn out that these bounds are not sharp. Together with bounds on when $E^2_{p,1} \cong HS_p(H_1(\mathrm{IA}))_n$ vanishes for $p \in \{2,3\}$, we can make statements about $H_2(\mathrm{IA})$. These will likely also not be sharp, but very little is known about the second homology of IA_n.

Exercises

1.) Let us fill in the details of the proof of Theorem 6.1:

(a) Check that

$$CS_p(M)_n = \bigoplus_{(f,C)\in \mathrm{Hom}_{\mathsf{VIC}(\mathbb{Z})}(\mathbb{Z}^p,\mathbb{Z}^n)} M_C \cong \mathrm{Ind}^{G_n}_{G_{n-p}} M_{n-p} \cong \mathbb{Z}\,\mathrm{GL}_n(\mathbb{Z}) \otimes_{\mathrm{GL}_{n-p}(\mathbb{Z})} M_{n-p}.$$

(b) Prove Shapiro's Lemma: Let H be a subgroup of a group G and let M be a $\mathbb{Z}H$–module. Then

$$H_*(G;\mathrm{Ind}^G_H M) \cong H_*(H;M).$$

(c) Show that every face map $d_i\colon CS_p(M)_n \to CS_{p-1}(M)_n$ induces the stability map

$$\begin{aligned} E^1_{pq} &\cong H_q(\mathrm{GL}_{n-p}(\mathbb{Z});M_{n-p}) \cong H_q(E_*\,\mathrm{GL}_n(\mathbb{Z}) \otimes_{\mathrm{GL}_n(\mathbb{Z})} CS_p(M)_n \\ &\longrightarrow E^1_{p-1,q} \cong H_q(\mathrm{GL}_{n-p+1}(\mathbb{Z});M_{n-p+1}) \cong H_q(E_*\,\mathrm{GL}_n(\mathbb{Z}) \otimes_{\mathrm{GL}_n(\mathbb{Z})} CS_{p-1}(M)_n. \end{aligned}$$

As a consequence $d^1\colon E^1_{pq} \to E^1_{p-1,q}$ is the stability map if p is odd and zero if p is even.

2.) For the spectral sequence given by the double complex

$$E_*\,\mathrm{Aut}(F_n) \otimes_{\mathrm{IA}_n} \tilde{C}_{*-1}(Y(n)),$$

we want to prove that

$$E^2_{pq} \cong HS_p(H_q(\mathrm{IA}))_n.$$

(a) Show that

$$E_*\,\mathrm{Aut}(F_n) \otimes_{\mathrm{IA}_{n-p}} \mathbb{Z}$$

is a $\mathbb{Z}\,\mathrm{GL}_{n-p}(\mathbb{Z})$–module.

(b) Find the isomorphism

$$E^0_{pq} \cong E_q\,\mathrm{Aut}(F_n) \otimes_{\mathrm{IA}_n} \mathrm{Ind}^{\mathrm{Aut}(F_n)}_{\mathrm{Aut}(F_{n-q})} \mathbb{Z} \cong \mathrm{Ind}^{\mathrm{GL}_n(\mathbb{Z})}_{\mathrm{GL}_{n-p}(\mathbb{Z})} E_*\,\mathrm{Aut}(F_n) \otimes_{\mathrm{IA}_{n-p}} \mathbb{Z}.$$

(c) Prove that face maps of $Y(n)$ precisely induce the face maps of $CS_*(H_*(\mathrm{IA}))_n$.

(d) Finish the proof.

7 Polynomial functors

Polynomial functors have a long history and come in various forms. The prototypical polynomial functor $\mathsf{Ab} \to \mathsf{Ab}$ sends an abelian group A to its kth tensor power $A^{\otimes k}$. These and similar play an important role in algebraic geometry and representation theory of group schemes. In 1950, Eilenberg–Maclane defined them in terms of "cross effects" to compute the homology of Eilenberg–Maclane spaces $K(\pi, n)$. In homological stability, polynomial functors were introduced by Dwyer in 1980 to compute algebraic K-theory groups. We will use a variant of Dwyer's definition that seems to be most general.

Definition 7.1. Let M be a $\mathsf{VIC}(\mathbb{Z})$–module. Define ΣM to be the $\mathsf{VIC}(\mathbb{Z})$–module that is M precomposed with the functor $\mathbb{Z}\oplus - \colon \mathsf{VIC}(\mathbb{Z}) \to \mathsf{VIC}(\mathbb{Z})$ that sends $(f, C) \in \operatorname{Hom}_{\mathsf{VIC}(\mathbb{Z})}(A, B)$ to $(\mathrm{id}_{\mathbb{Z}} \oplus f, C) \in \operatorname{Hom}_{\mathsf{VIC}(\mathbb{Z})}(\mathbb{Z} \oplus A, \mathbb{Z} \oplus B)$.
There is a canonical $\mathsf{VIC}(\mathbb{Z})$–homomorphism $M \to \Sigma M$ given by the maps $M_A \to M_{\mathbb{Z}\oplus A}$ induced by $(A \subset \mathbb{Z} \oplus A, \mathbb{Z}) \in \operatorname{Hom}_{\mathsf{VIC}(\mathbb{Z})}(A, \mathbb{Z} \oplus A)$.
Let us denote

$$(\mathrm{co})\ker(M) := (\mathrm{co})\ker(M \to \Sigma M).$$

Let $d \in \mathbb{N}_0 \cup \{-\infty\}$. We say M has *polynomial degree* $-\infty$ *in ranks* $> d$ if $M_n \cong 0$ for all $n > d$.
We say M has *polynomial degree* ≤ 0 *in ranks* $> d$ if $(\ker M)_n \cong (\operatorname{coker} M)_{n+1} \cong 0$ for all $n > d$.
Let $r \geq 1$. We say M has *polynomial degree* $\leq r$ *in ranks* $> d$ if $(\ker M)_n \cong 0$ for all $n > d$ and $\operatorname{coker} M$ has polynomial degree $\leq r - 1$ in ranks $> d - 1$.
(For simplicity, we will define $0 - 1 = -\infty$.)

Proposition 7.2. *If a* $\mathsf{VIC}(\mathbb{Z})$*–module* M *has polynomial degree* $\leq r$ *in ranks* $> d$*, then there is a polynomial* $p \in \mathbb{Q}[X]$ *of degree* $\leq r$ *such that* $\operatorname{rk} M_n = p(n)$ *for all* $n > d$*.*

Theorem 7.3 (Dwyer 1980)**.** *If* M *has finite polynomial degree then*

$$H_i(\mathrm{GL}_{n-1}(\mathbb{Z}); M_{n-1}) \longrightarrow H_i(\mathrm{GL}_n(\mathbb{Z}); M_n)$$

is an isomorphism for $n \gg i$*.*

Exercise 7.4. Check that $H_1(\mathrm{IA})$ has polynomial degree ≤ 3 in ranks $> -\infty$.

Theorem 7.5 (Miller-P.-Petersen)**.** *Let* M *be a* $\mathsf{VIC}(\mathbb{Z})$*–module that has polynomial degree* $\leq r$ *in ranks* $> d$*. Then*

$$HS_i(M)_n \cong 0 \qquad \textit{for } n > \max(d + i, 2i + r).$$

Corollary 7.6 (Miller-P.-Wilson, Miller-P.-Petersen)**.** $H_2(\mathrm{IA})$ *is presented in degrees* ≤ 9*.*

Exercises

1.) We want to compare different definitions of polynomial functors.

   (a) Show that there is a $\mathsf{VIC}(\mathbb{Z})$–module that sends a finitely generated free $\mathbb{Z}$–module to its underlying abelian group that has polynomial degree ≤ 1 in ranks $> -\infty$.

   (b) Show that there is a $\mathsf{VIC}(\mathbb{Z})$–module that sends a finitely generated free $\mathbb{Z}$–module to the dual of its underlying abelian group that has polynomial degree ≤ 1 in ranks $> -\infty$. (Here $\mathrm{GL}_n(\mathbb{Z})$ acts via its transpose.)

(c) Let $F\colon \mathsf{Ab} \to \mathsf{Ab}$ be a functor. Show that there is a functor $\mathrm{cr}_1(F)\colon \mathsf{Ab} \to \mathsf{Ab}$ called the first cross effects of F such that $F(A) = F(0) \oplus \mathrm{cr}_1(F)(A)$ for all abelian groups A.

(d) Let $F\colon \mathsf{Ab} \to \mathsf{Ab}$ be a functor. Show that there is a functor $\mathrm{cr}_2(F)\colon \mathsf{Ab} \times \mathsf{Ab} \to \mathsf{Ab}$ called the second cross effects of F such that $F(A \oplus B) = F(0) \oplus \mathrm{cr}_1(F)(A) \oplus \mathrm{cr}_1(F)(B) \oplus \mathrm{cr}_2(F)(A, B)$ for all pairs of abelian groups A, B.

(e) Let $\mathsf{VIC}(\mathbb{Z}) \to \mathsf{Ab}$ the functor that forgets about the complement. Let $F\colon \mathsf{Ab} \to \mathsf{Ab}$ be a functor whose second cross effects vanish. Consider F as a $\mathsf{VIC}(\mathbb{Z})$–module. Show that it has polynomial degree ≤ 1 in ranks $> -\infty$.

(f) Prove that if a $\mathsf{VIC}(\mathbb{Z})$–module has polynomial degree $\leq r$ in ranks $> d$ then there is a polynomial $p \in \mathbb{Q}[X]$ such that $\operatorname{rk} M_n = p(n)$ for all $n > d$.

2.) We want to show that $H_1(\mathrm{IA})$ has polynomial degree ≤ 3 in ranks $> -\infty$.

(a) Let M, M', M'' be $\mathsf{VIC}(\mathbb{Z})$–modules and $M' \to M \to M''$ morphisms such that

$$0 \to M'_n \to M_n \to M''_n \to 0$$

is a short exact sequence for $n > d$. Prove that if N' has polynomial degree $\leq r$ in ranks $> d$ and N'' has polynomial degree $\leq r$ in ranks $> d - 1$, then N has polynomial degree $\leq r$ in ranks $> d$.

(b) Let M and N be $\mathsf{VIC}(\mathbb{Z})$–modules and assume that M has polynomial degree $\leq r$ in ranks $> d$ and N has polynomial degree $\leq s$ in ranks $> e$. Prove that $M \otimes N$ has polynomial degree $\leq r + s$ in ranks $> \max(d, e)$.

(c) Show that there is a $\mathsf{VIC}(\mathbb{Z})$–module M with $M_n \cong \mathrm{Hom}_{\mathsf{Ab}}(\mathbb{Z}^n, \bigwedge^2 \mathbb{Z}^n)$ that has polynomial degree ≤ 3 in ranks $> -\infty$.

(d) Show that M coincides with the $\mathsf{VIC}(\mathbb{Z})$–module $H_1(\mathrm{IA})$.

8 Central stability homology for polynomial $\mathsf{VIC}(\mathbb{Z})$–modules

Proof of Theorem 7.5. We prove the theorem by a double induction over r and i. If $r = -\infty$ or $i < 0$ the theorem is true. We thus may assume that if M has polynomial degree $\leq s$ in ranks $> d$,

$$HS_q(M)_n \cong 0 \qquad \text{for } n > \max(d + q, 2q + s)$$

as long as $s < r$ or $q < i$.

Consider two double complexes:

$$\begin{aligned} X_{pq} &= \bigoplus_{(f,C) \in \mathrm{Hom}_{\mathsf{VIC}(Z)}(\mathbb{Z}^p, \mathbb{Z}^n)} \bigoplus_{(g,D) \in \mathrm{Hom}_{\mathsf{VIC}(Z)}(\mathbb{Z}^q, C)} M_{\mathrm{im}\, f \oplus D} \\ &\cong CS_p(CS_q(\Sigma^p M))_n \\ &\cong CS_q(CS_p(M(0)) \otimes M)_n \end{aligned}$$

and

$$\begin{aligned} Y_{pq} &= \bigoplus_{(f,C) \in \mathrm{Hom}_{\mathsf{VIC}(Z)}(\mathbb{Z}^p, \mathbb{Z}^n)} \bigoplus_{(g,D) \in \mathrm{Hom}_{\mathsf{VIC}(Z)}(\mathbb{Z}^q, C)} M_D \\ &\cong CS_p(CS_q(M))_n \\ &\cong CS_q(CS_p(M))_n. \end{aligned}$$

Let

$$E^1_{pq} = CS_p(HS_q(\Sigma^p M))_n$$

denote the spectral sequence associated to X. It converges to zero in the range $n > 2(p+q)$.

Let us denote the spectral sequence associated to Y by $\widehat{E}^r_{pq}$. It turns out that $d^1\colon \widehat{E}^1_{1,q} \to \widehat{E}^1_{0,q}$ is always the zero map.

The map of double complexes

$$Y_{pq} \longrightarrow X_{pq}$$

induces maps

$$\widehat{E}^1_{pq} \longrightarrow E^1_{pq}$$

that are surjective for $n > \max(d+p+q-1, p+2q+r-1)$ and injective for $n > \max(d+p+q, p+2q+r+1)$. This uses the induction hypothesis.

Therefore

$$E^2_{0,i}(M)_n = E^1_{0,i} \cong HS_i(M)_n \qquad \text{for } n > \max(d+i, 2i+r).$$

The theorem follows because by induction

$$E^1_{pq} \cong CS_p(HS_q(\Sigma^p M))_n \cong 0$$

for $q < i$ and $n > \max(d+q, p+2q+r)$. This implies that

$$HS_i(M)_n \cong E^1_{0,1} \cong E^2_{0,i} \cong E^\infty_{0,i}$$

in the given range, which vanishes for $n > 2i$. □

Exercises

1.) Fill in the details about X_{pq} and Y_{pq}:

(a) Show that

$$Y_{pq} \cong CS_p(CS_q(M))_n \cong CS_q(CS_p(M))_n.$$

(b) Those isomorphisms describe the differential in both p and q direction. Show that Y_{pq} is a double complex, i.e. that the two differentials commute.

(c) Show that

$$X_{pq} \cong CS_p(CS_q(\Sigma^p M))_n \cong CS_q(CS_p(M(0)) \otimes M)_n.$$

(d) Those isomorphisms describe the differential in both p and q direction. Show that X_{pq} is a double complex, i.e. that the two differentials commute.

2.) We want to show that $d^1\colon \widehat{E}^1_{1,q} \to \widehat{E}^1_{0,q}$ is always zero. Find an isomorphism $\psi\colon \widehat{E}^0_{1,q} \to \widehat{E}^0_{0,q+1}$ that is a chain homotopy from the map of chain complexes $\widehat{E}^0_{1,*} \to \widehat{E}^0_{0,*}$ to the zero map.

For the remainder of the exercises, fix $r \in \mathbb{N}_0$, $d \in \mathbb{N}_0 \cup \{-\infty\}$, and $i \in \mathbb{N}_0$. Assume that

$$HS_q(N)_n \cong 0 \qquad \text{for all } n > \max(e+q, 2q+s)$$

if N is a $\mathsf{VIC}(\mathbb{Z})$–module with polynomial degree $\leq s$ in ranks $> e$ as long as $s < r$ or $q < i$. Let M be a $\mathsf{VIC}(\mathbb{Z})$–module with polynomial degree $\leq r$ in ranks $> d$.

3. We want to prove that

$$\widehat{E}^1_{pq} \longrightarrow E^1_{pq}$$

is surjective for $n > \max(d+p+q-1, p+2q+r-1)$ and injective for $n > \max(d+p+q, p+2q+r+1)$.

   (a) Prove that $\Sigma^p M$ has polynomial degree $\leq r$ in ranks $> d-p$.

   (b) For $p \geq 1$, prove that $\ker(M \to \Sigma^p M)_n \cong 0$ for all $n > d$ and $\operatorname{coker}(M \to \Sigma^p M)$ has polynomial degree $r-1$ in ranks $> d-1$. (Hint: Use Exercise 7.2a)

   (c) Use this information to show that

$$HS_q(M)_n \longrightarrow HS_q(\Sigma^p M)_n$$

   is surjective for $n > \max(d+q-1, 2q+r-1)$ and injective for $n > \max(d+q, 2q+r+1)$.

   (d) Finish the proof.

4. Let us finish the proof of Theorem 7.5:

   (a) Observe that $E^1_{0,i} \cong HS_i(M)_n$.

   (b) Use the previous exercises to show that $E^1_{0,i} = E^2_{0,i}$ for $n > \max(d+i, 2i+r)$.

   (c) Use the induction hypothesis to show that $E^2_{0,i} = E^\infty_{0,i}$ for $n > \max(d+i-1, 2i+r)$. [Hint: Consider E^1_{pq} for $q < i$, and $p+q = i+1$.]

   (d) Show that $E^\infty_{pq} \cong 0$ for $n > 2(p+q)$ to finish the proof.