Spectral sequences

Filtrations

Let (C, d) be a complex and $\dots \subset F_{p-1}C \subset F_pC \subset \dots$ a filtration of complexes. Define

$$A^r_{pq} = F_pC_{p+q} \cap d^{-1}F_{p-r}C_{p+q-1} = \{x \in F_pC_{p+q} \mid dx \in F_{p-r}C_{p+q-1}\}$$
$$E^r_{pq} = \frac{A^r_{pq}}{d(A^{r-1}_{p+r-1,q-r+2}) + A^{r-1}_{p-1,q+1}}$$
$$d^r \colon E^r_{pq} \longrightarrow E^r_{p-r,q+r-1}$$

where d^r is induced by d. (E^r, d^r) is a family of complexes and one checks that E^{r+1}_{pq} is the homology at the position E^r_{pq}.

Theorem 1 (Classical Convergence Theorem)**.** *If the filtration is bounded, i.e. $F_pC_n = C_n$ for large enough p and $F_pC_n = 0$ for small enough p (depending on n), then this spectral sequence converges to $H_*(C)$*

$$E^0_{pq} = F_pC_{p+q}/F_{p-1}C_{p+q} \implies H_{p+q}(C).$$

This means, that for fixed (p, q) every E^r_{pq} stabilizes to some E^∞_{pq} and

$$E^\infty_{pq} \cong F_pH_{p+q}(C)/F_{p-1}H_{p+q}(C),$$

where

$$F_pH_{p+q}(C) = \operatorname{im}(H_{p+q}(F_pC) \to H_{p+q}(C)) = \ker(H_{p+q}(C) \to H_{p+q}(C/F_pC)).$$

Double complexes

Let (C, d^v, d^h) be a double complex, i.e. $d^vd^h + d^hd^v = 0$, then the total complex $T = \operatorname{Tot}(C)$ is defined by $T_n = \bigoplus_{p+q=n} C_{pq}$ and $d = d^v + d^h$. There are two filtrations on T that can be used to get spectral sequences. The first is

$${}^IF_pT_n = \bigoplus_{\substack{a+b=n\\ a\le p}} C_{ab}.$$

This yields the spectral sequence ${}^IE^0_{pq} = C_{pq}$. ${}^IE^1_{pq} = H_q(C_{p*})$, $d^0 = d^v$, and $d^1 = d^h$. We get the second spectral sequence by flipping the role of p and q, i.e.

$${}^{II}F_qT_n = \bigoplus_{\substack{a+b=n\\ b\le q}} C_{ab}.$$

Then we get ${}^{II}E^0_{pq} = C_{qp}$. ${}^IIE^1_{pq} = H_q(C_{*p})$, $d^0 = d^h$, and $d^1 = d^v$.
Note that if the double complex is bounded, both spectral sequences converge to the homology of the total complex. (Although the E^∞_{pq} might very well be different.)