THE STABLE (CO)-HOMOLOGY OF $\mathrm{SL}(\mathbf{F}_p)$

Abstract. Some very rough notes and conjectures about the cohomology of $\mathrm{SL}(\mathbf{F}_p)$. Various statements are almost definitely wrong. Some of this comes from discussion with Will Sawin. These are notes associated from a talk at BIRS on Tuesday October 12, 2021.

0.1. **Guesses about the cohomology of $\mathrm{SL}(\mathbf{F}_p)$.** As the integer n increases, one can make sense of algebraic representations of SL_n compatibly in n, and label them by some data such as a pair of partitions $\Sigma = (\lambda; \mu)$ (e.g. [CF13]). These give representations of $\mathrm{SL}_n(\mathbf{Z})$ which admit lattices $\mathbb{L}_{\mathbf{Z},n,\Sigma}$. There is some ambiguity in the choice of these lattices, but if one defines them correctly then (van der Kalllen [vdK80] plus an automorphic computation for the last claim):

Theorem 0.1. *Fix m, Σ, and p. Then*

$$\lim H^m(\mathrm{SL}_n(\mathbf{Z}), \mathbb{L}_{\mathbf{Z},n,\Sigma})$$

stabilizes to a finitely generated abelian group. This group is finite unless Σ is trivial.

Now we imagine that we take m, Σ, p, and n in the following order:

1. Fix m and Σ.
2. Choose p sufficiently large with respect to m and Σ.
3. Choose n sufficiently large with respect to p. From now on, we drop n from the notation.

Let $\mathbb{L}_\Sigma = \mathbb{L}_{\mathbf{Z},\Sigma}/p$, which is a representation of $\mathrm{SL}(\mathbf{F}_p)$.

Problem 0.2. Compute $H^m(\mathrm{SL}(\mathbf{F}_p), \mathbb{L}_\Sigma)$.

Example 0.3. Let $\Gamma(p)$ denote the principal congruence subgroup. Then $\Gamma(p)/\Gamma(p^2)$ is the adjoint representation, and hence, assuming the congruence subgroup property,

$$M = H^1(\Gamma(p), \mathbf{F}_p)$$

is the dual of the adjoint representation. This is $\mathbb{L}_\Sigma$ for $\Sigma = (1; -1)$.

The ring of invariants of $R = \mathrm{Sym}^*(M)$ is given by a polynomial algebra $R^{\mathrm{SL}(\mathbf{F}_p)} = \mathbf{F}_p[x_2, x_4, \ldots]$. Let $S = R/(x_2, x_4, \ldots)$, considered as a graded algebra where the degrees are doubled, so that $S[k] = 0$ if k is odd.

Conjecture 0.4. We have (in the appropriate range)

$$H^*(\mathrm{SL}(\mathbf{F}_p), \mathbb{L}) = (\mathbb{L} \otimes S)^{\mathrm{SL}(\mathbf{F}_p)}.$$

0.2. **Guesses about the cohomology of $\Gamma(p) \subset \mathrm{SL}(\mathbf{Z})$ for regular primes.** Suppose that p is regular. Then in small degree, the mod-p cohomology of $\mathrm{SL}(\mathbf{Z})$ over $\mathbf{F}_p$ is given by $\wedge^*[x_5, x_9, \ldots]$. The mod p cohomology of $\mathrm{SL}(\mathbf{Z}_p)$ is given by $\wedge^*[x_3, x_5, x_7, \ldots]$. For a regular prime, the map on homotopy groups of the spaces $\mathrm{SK}(\mathbf{Z})$ and $\mathrm{SK}(\mathbf{Z}_p)$ sends x_{4n+1} to x_{4n+1}, so the cohomology of the homotopy fibre will be $\mathrm{Sym}^*[x_2, x_6, x_{10}, \ldots]$. These are the completed cohomology groups $\widetilde{H}^i(\mathbf{F}_p)$.

Remark 0.5. In a small range (up to m) one only needs to eliminate the finitely many primes which divide the torsion subgroup of $K_i(\mathbf{Z})$ for i in a finite range *together* with p such that the finitely many p-adic zeta values $\zeta_p(3)$, $\zeta_p(5)$, up to some point are p-adic units. The latter should eliminate $\log \log X$ primes $p < X$. (These primes are the primes for which either $K_n(\mathbf{Z})$ is either has extra torsion or such that the map $K_{4n+1}(\mathbf{Z}) \otimes \mathbf{F}_p \to K_{4n+1}(\mathbf{Z}_p; \mathbf{Z}_p) \otimes \mathbf{F}_p$ is not an isomorphism.)

Let $G(p)$ be the congruence subgroup of $\mathrm{SL}(\mathbf{Z}_p)$, so $H^1(G(p), \mathbf{F}_p) = M$, but $H^*(G(p), \mathbf{F}_p) = \wedge^*(M)$ by Lazard.

For a regular prime p, there is a spectral sequence

$$\wedge^* (M) \otimes \mathbf{F}_p[x_2, x_6, \ldots] = H^i(G(p), \widetilde{H}^j(\mathbf{F}_p)) \Rightarrow H^{i+j}(\Gamma(p), \mathbf{F}_p). \tag{1}$$

One also knows that $H^*(G(\mathbf{Z}_p), \mathbf{F}_p) = \mathbf{F}_p[y_3, y_5, y_7, \ldots]$ and (for these good p) one understands exactly the what the spectral sequence

$$H^i(G(\mathbf{Z}_p), \widetilde{H}_j(\mathbf{F}_p)) = \mathbf{F}_p[y_3, y_5, \ldots] \otimes \mathbf{F}_p[x_2, x_6, x_{10}, \ldots] \Rightarrow \mathbf{F}_p[y_5, y_9, \ldots] = H^*(\mathrm{SL}(\mathbf{Z}), \mathbf{F}_p)$$

is on every page. Explicitly — the y_{4n+1} survive, and otherwise the d^r maps are zero unless $r = 4n - 1$ and then $d^{4n-1}(x_{4n-2}) = y_{4n-1}$ (which one can see on homotopy groups).

Conjecture 0.6. The spectral sequence (1) degenerates on page 2. Moreover, the corresponding isomorphism respects the $\mathrm{SL}(\mathbf{F}_p)$-action, that is, there is an isomorphism of $\mathrm{SL}(\mathbf{F}_p)$-modules:

$$\wedge^*(M) \otimes \mathbf{F}_p[x_2, x_6, \ldots] = H^*(\Gamma(p), \mathbf{F}_p)$$

with the appropriate grading.

Proposition 0.7. This is true for H^1 (CSP) and H^2 (as proved in my paper [Cal15] for $p > 3$).

0.3. **Cohomology of $\Gamma(p)$ versus $\Gamma(p^k)$.** Since one also has $H^*(G(p^k), \mathbf{F}_p) = \wedge^* M$ by Lazard, the same formula for level p should apply equally for level $\Gamma(p^k)$. So One makes the same conjecture (and it's true in degrees 1 and 2). OTOH, the maps

$$H^*(G(p), \mathbf{F}_p) \to H^*(G(p^2), \mathbf{F}_p)$$

are clearly not isomorphisms because M in H^1 maps to 0. In my notes I seem to think that this might be able to make deductions about the splitting of the sequence for level p^k for big k, but no argument is given and I don't see any such argument now.

0.4. **Some global arguments.** We provably have:

$$H^0(\Gamma(p),\mathbb{L}) = \mathbb{L},$$

$$H^1(\Gamma(p),\mathbb{L}) = M \otimes \mathbb{L},$$

$$H^2(\Gamma(p),\mathbb{L}) = \mathbb{L} \oplus \wedge^2 M \otimes \mathbb{L},$$

Hence the spectral sequence $H^i(\mathrm{SL}(\mathbf{F}_p), H^j(\Gamma(p),\mathbb{L})) \Rightarrow H^{i+j}(\Gamma,\mathbb{L}) = 0$ is:

q				
2	$H^0(\mathrm{SL},\mathbb{L}) \oplus H^0(\mathrm{SL},\mathbb{L}\otimes\wedge^2 M)$	$H^1(\mathrm{SL},\mathbb{L}) \oplus H^1(\mathrm{SL},\mathbb{L}\otimes\wedge^2 M)$	$\ldots$	
1	$H^0(\mathrm{SL},\mathbb{L}\otimes M)$	$H^1(\mathrm{SL},\mathbb{L}\otimes M)$	$\ldots$	
0	$H^0(\mathrm{SL},\mathbb{L})$	$H^1(\mathrm{SL},\mathbb{L})$	$H^2(\mathrm{SL},\mathbb{L})$	$H^3(\mathrm{SL},\mathbb{L})$
	0	1	2	3

We immediately deduce, for parameters in the associated range:

(1) $H^1(\mathrm{SL}(\mathbf{F}_p),\mathbb{L}) = 0$ for any $\mathbb{L}$.
(2) There is an isomorphism $H^0(\mathrm{SL}(\mathbf{F}_p),\mathbb{L}\otimes M) \to H^2(\mathrm{SL}(\mathbf{F}_p),\mathbb{L})$. In particular, $H^2(\mathrm{SL}(\mathbf{F}_p),\mathbb{L}) = 0$ unless $\mathbb{L} = M^*$, in which case $H^2(\mathrm{SL}(\mathbf{F}_p), M^*) = \mathbf{F}_p$.

Note that $\mathrm{SL}(\mathbf{Z}/p^2\mathbf{Z})$ is a non-split extension of $\mathrm{SL}(\mathbf{F}_p)$ by M^*, and this gives the non-trivial class in $H^2(\mathrm{SL}(\mathbf{F}_p), M^*) = \mathbf{F}_p$.

0.5. **The K-theory of $\mathbf{Z}/p^2\mathbf{Z}$.** In [EF82], Evans and Friedlander compute some of the very small K-groups of $\mathbf{Z}/p^2\mathbf{Z}$ by explicitly exploiting some computations of $H^i(\mathrm{SL}(\mathbf{F}_p),\mathbb{L})$ for $i = 1$ and 2 and $\mathbb{L}$ related to M^* and $\wedge^2 M^*$ in order to compute the small K-groups. In [Cal15], I used some of these computations to prove the theorem for H^2 mentioned above, and then one bootstraps this argument to prove results about more general $\mathbb{L}$.

To give some indication of my ignorance, I surely imagine that the state of the art has improved since [EF82]. In particular, does one know what $K_i(\mathbf{Z}/p^2\mathbf{Z})$ is for all i? Or at least all $i < p$? This general question seems very related to this problem.

0.6. **Some heuristics.** There really should not be any reason to work globally (over $\mathbf{Z}$) to say something over $\mathbf{F}_p$. Instead, we should *start* with $\mathbf{Z}_p$. The groups $H^*(\mathrm{SL}(\mathbf{Z}_p),\mathbb{L})$ stabilize; what has to be true (by global considerations) but is not obvious is that (recall that $\mathbb{L}$ is irreducible over $\mathbf{F}_p$):

$$H^*(\mathrm{SL}(\mathbf{Z}_p),\mathbb{L}) = 0$$

for fixed $\mathbb{L}$ and sufficiently large p in small degree. We let $G = \mathrm{SL}(\mathbf{Z}_p)$ and $G(p^k)$ be the congruence subgroup.

To indicate the analogous groups we talked about in characteristic zero, we would have

$$\widetilde{H}^i = \lim H^*(G(p^k),\mathbf{F}_p) = \mathbf{F}_p$$

in degree 0. Hence the analogue of Hochschild–Serre is now the trivial

$$H^i(G(p), \mathbf{F}_p) = H^i(G(p), \widetilde{H}^j) \Rightarrow H^{i+j}(G(p), \mathbf{F}_p).$$

Also by Lazard we have

$$H^*(G(p), \mathbf{F}_p) = \wedge^* M,$$

and thus

$$H^*(G(p), \mathbb{L}) = \wedge^* M \otimes \mathbb{L},$$

so finally:

$$H^i(\mathrm{SL}(\mathbf{F}_p), \wedge^j M \otimes \mathbb{L}) \Rightarrow 0.$$

The remark (made by WS) is then that this looks like the Leray spectral sequence for a torus bundle where the total space has no cohomology, which suggests that

$$H^{2d}(\mathrm{SL}(\mathbf{F}_p), \mathbb{L}) = (\mathbb{L} \otimes \mathrm{Sym}^d M)^{\mathrm{SL}(\mathbf{F}_p)}.$$

All of this has ignored the fact, however, that this doesn't work for $\mathbb{L} = \mathbf{F}_p$, because of the two (related) reasons, namely:

(1) $H^*(\mathrm{SL}(\mathbf{Z}_p), \mathbf{F}_p) = \wedge^*[y_3, y_5, \ldots]$ is non-zero.
(2) $H^*(\mathrm{SL}(\mathbf{F}_p), \mathbf{F}_p) = \mathbf{F}_p$ is zero.

The simplest ansatz is just to excise those classes, and this leads to the original guess.

References

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[vdK80] Wilberd van der Kallen, *Homology stability for linear groups*, Invent. Math. **60** (1980), no. 3, 269–295. MR 586429