More open problems in representation stability and related areas

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1 Coherence/regularity for module categories

Recall that FI is the category of finite sets and injections. A module over a category C is a functor from C to abelian groups (or more generally to a category of R-modules). Church–Ellenberg [CE17] proved that any FI-module presented in finite degree has a resolution by free FI-modules generated in finite degree (see [CE17] for relevant definitions). Moreover, they gave quantitative bounds on the degrees of the syzygies. There are many natural generalizations of FI such as:

- OI, the category of finite ordered sets and injections.
- FS^{op}, the opposite category of the category of finite sets and surjections.
- FIM, the category of finite sets and injections with a choice of perfect paring of the complement of the image.
- $\mathsf{VI}(R)$, the category of free finitely generated R-modules and split linear injections.
- $\mathsf{VIC}(R)$, the category of free finitely generated R-modules and split linear injections with choice of splitting.
- $\mathsf{SI}(R)$, the category of free finitely generated R-modules with choice of symplectic form and split symplectic injections.

Question 1.1. *Let M be a C-module for $C = \mathsf{OI}$, FS^{op}, FIM, $\mathsf{VI}(R)$, $\mathsf{VIC}(R)$, $\mathsf{SI}(R)$, etc which is presented in finite degree. Is there a resolution of M by free C-modules generated in finite degree? Can the degrees of the syzygies be bounded in terms of the presentation degree of M?*

We want to point out that in particular answering this question positively for $\mathsf{VIC}(\mathbb{Z})$ and $\mathsf{SI}(\mathbb{Z})$ would imply (a variation of) a conjecture by Church–Farb [CF13] about the homology of the Torelli subgroups of the automorphism groups of free groups and of the mapping class groups of surfaces, respectively.

2 Ordered configuration spaces of graphs

Let X be a topological space. Recall that the ordered configuration space of X is

$$PConf_n(X) := Emb(\{1, \ldots, n\}, X).$$

This can be considered as functor from FI^{op} to topological spaces. For manifolds X of dimension at least 2, Church–Ellenberg–Farb [CEF15] proved that the cohomology (in fixed cohomological degree) of the ordered configuration spaces of X forms a finitely generated FI-module. The unordered configuration space of X is defined as

$$UConf_n(X) := PConf_n(X)/S_n.$$

The above result implies that $UConf_n(X)$ satisfies (rational) cohomological stability via a transfer argument.

When X is a graph (one-dimensional CW-complex), neither of these results generalize. In lieu of homological stability, An–Drummond-Cole–Knudsen [ADCK20] showed that the cohomology of the unordered configuration spaces of graphs are finitely generated modules over a polynomial ring.

Question 2.1. *Is there a category C such that the cohomology of ordered configuration spaces of graphs form a finitely generated C-module and so that a transfer argument implies that the cohomology of the unordered configuration spaces of graphs are finitely generated modules over a polynomial ring?*

3 Odd dimensional Madsen–Weiss theorem

Solving the Mumford conjecture, Madsen–Weiss [MW07] showed that, in a stable range, the homology of diffeomorphism groups of a surface agrees with the homology of an infinite loop space known as $\Omega^\infty MTSO(2)$. This was later generalized to high-dimensional manifolds of even dimension by Galatius–Randal-Williams [GRW18, GRW17] which allowed one to completely compute the stable rational homology of diffeomorphism groups of certain high dimensional manifolds. By work of Perlmutter [Per16a, Per16b], diffeomorphism groups of odd dimensional manifolds exhibit homological stability and it is natural to ask what the stable homology is.

Problem 3.1. *Compute $H^i(\mathrm{Diff}(\#_g(S^d \times S^{d+1}));\mathbb{Q})$ for $g \gg i$.*

See Hatcher [Hat] for the case of $d = 1$.

4 Dualizing module for $\mathrm{Aut}(F_n)$

Recall that a group G is a rational duality group of dimension d if there is a G-representation $\mathbb{D}$ (called the dualizing module) such that

$$H^{d-i}(G;\mathbb{Q}) \cong H_i(G;\mathbb{D}).$$

Special linear groups of number rings (Borel–Serre [BS73]), mapping class groups (Harer [Har86]), and automorphism groups of free groups (Bestvina–Feighn [BF00]) are all examples of rational duality groups. In the case of special linear groups of number rings, the dualizing module is the top homology of the Tits building and in the case of mapping class groups, the dualizing module is the top homology of the curve complex.

Question 4.1. *Is there a natural simplicial complex X_n with an action of $\mathrm{Aut}(F_n)$ which is spherical of dimension d_n such that $\widetilde{H}_{d_n}(X_n)$ is the dualizing module of $\mathrm{Aut}(F_n)$?*

If the answer to this question is yes, a natural follow up question is to use this description to produce generators and relations for the dualizing module.

5 Computing the top degree homology of congruence subgroups

Let $\Gamma_n(p)$ denote the kernel of the mod-p reduction map $\mathrm{SL}_n(\mathbb{Z}) \longrightarrow \mathrm{SL}_n(\mathbb{Z}/p)$. By work of Borel–Serre [BS73],

$$H^i(\Gamma_n(p);\mathbb{Q}) = 0 \text{ for } i > \binom{n}{2}.$$

Lee–Szczarba [LS76] proved that

$$H^{\binom{n}{2}}(\Gamma_n(3);\mathbb{Z}) \cong \mathbb{Z}^{3^{\binom{n}{2}}}$$

and M.–P.–Putman [MPP] gave a recursive formula for $H^{\binom{n}{2}}(\Gamma_n(5);\mathbb{Z})$ and directly computed $H^{\binom{n}{2}}(\Gamma_n(2);\mathbb{Q})$.

Problem 5.1. *Compute the group $H^{\binom{n}{2}}(\Gamma_n(p))$ for all primes p.*

The case that p is not a prime is also interesting but likely to be harder.

6 Top degree homology of special linear groups as a ring

Let $\mathcal{O}_{\mathbf{k}}$ be the ring of integers in a number field $\mathbf{k}$ with r_1 real embeddings and r_2 pairs of complex conjugate complex embeddings. By work of Borel–Serre [BS73], $\mathrm{SL}_n(\mathcal{O}_{\mathbf{k}})$ is a rational duality group of dimension

$$\nu_n := r_1\left(\frac{n^2+n-2}{2}\right) + r_2(n^2-1) - n + 1.$$

Let $T_n(\mathbf{k})$ denote the realization of the poset of proper nonempty subspaces of $\mathbf{k}^n$ and let

$$\mathrm{St}_n(\mathbf{k}) = \widetilde{H}_{n-2}(T_n(\mathbf{k});\mathbb{Q}).$$

By the work of Borel–Serre, $\mathrm{St}_n(\mathbf{k})$ is the dualizing module of $\mathrm{SL}_n(\mathcal{O}_{\mathbf{k}})$. In particular,

$$\mathrm{St}_n(\mathcal{O}_{\mathbf{k}})_{\mathrm{SL}_n(\mathcal{O}_{\mathbf{k}})} \cong H^{\nu_n}(\mathrm{SL}_n(\mathcal{O}_{\mathbf{k}});\mathbb{Q}).$$

The groups $H^{\nu_n}(\mathrm{SL}_n(\mathcal{O}_{\mathbf{k}});\mathbb{Q})$ do not stabilize in the classical sense; Church–Farb–Putman [CFP19] showed they have ranks that grow at least exponentially fast as n increases provided the class number of $\mathcal{O}_{\mathbf{k}}$ is at least 3. M.–Nagpal–P. [MNP20] (also see Galatius–Kupers–Randal-Williams [GKRW]) describe $\mathrm{GL}_n(\mathbf{k}) \times \mathrm{GL}_m(\mathbf{k})$-equivariant maps

$$\mathrm{St}_n(\mathbf{k}) \otimes \mathrm{St}_m(\mathbf{k}) \longrightarrow \mathrm{St}_{n+m}(\mathbf{k}).$$

After taking coinvariants and invoking Borel–Serre duality, these maps gives the group

$$H^{\nu}(\mathrm{SL}(\mathcal{O}_{\mathbf{k}});\mathbb{Q}) := \bigoplus_n H^{\nu_n}(\mathrm{SL}_n(\mathcal{O}_{\mathbf{k}});\mathbb{Q})$$

the structure of a graded ring.

Question 6.1. *Let $\mathcal{O}_{\mathbf{k}}$ be the ring of integers in a number field $\mathbf{k}$. Is $H^{\nu}(\mathrm{SL}(\mathcal{O}_{\mathbf{k}});\mathbb{Q})$ finitely generated as a ring?*

For $\mathcal{O}_{\mathbf{k}}$ Euclidean and $n > 1$, $\mathrm{St}_n(\mathbf{k})_{\mathrm{SL}_n(\mathcal{O}_{\mathbf{k}})} = 0$ hence $H^{\nu}(\mathrm{SL}(\mathcal{O}_{\mathbf{k}});\mathbb{Q})$ is finitely generated. When $\mathcal{O}_{\mathbf{k}}$ is not a PID, it might make sense to enlarge $H^{\nu}(\mathrm{SL}(\mathcal{O}_{\mathbf{k}});\mathbb{Q})$ to include automorphism groups of finitely generated projective modules that are not necessarily free.

7 Stability for the top degree homology of moduli space

Let $\mathcal{M}_g$ denote the moduli space of smooth complex curves of genus g. By work of Harer [Har86], Church–Farb–Putman [CFP12], and Morita–Sakasai–Suzuki [MSS13],

$$H_i(\mathcal{M}_g;\mathbb{Q}) \cong 0 \text{ for } i > 4g-6.$$

Chan–Galatius–Payne [CGP] showed that $H_{4g-6}(\mathcal{M}_g;\mathbb{Q})$ grows at least exponentially fast with g. In analogy with the theory of representation stability, we ask the following question.

Question 7.1. *Is there some natural categorical or algebraic object A that acts on the sequence $\{H_{4g-6}(\mathcal{M}_g;\mathbb{Q})\}_g$ that makes the sequence into a finitely generated A-module?*

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