The categorical graph minor theorem

(w/ Dane Miyata and Nick Proudfoot)

<u>Recall</u> The category of f.g FI-modules is abelian, i.e. FI-modules (over a noeth ring) satisfy the Noetherian property.

<u>Sam–Snowden Gröbner Perspective</u>

we can think of this result as a categorification of Higmann's Lemma.

$\downarrow$

Noetherianity for certain posets.

Two famous generalizations of Higman's Lemma

1.) Kruskal's tree theorem
"Trees are well-ordered" w.r.t the contraction

2.) graph minor theorem.
graphs are quasi-well-ordered by deletions/contractions
(i.e the minor order is a qwo)

If you take any infinite collection of graphs $\{G_i\}$ there is $i < j$ such that G_i is obtained from G_j by a ~~seqeu~~

of edge deletions and contractions

<u>Def</u>: $\mathcal{G}$ = graph minor category

obj - finite connected graphs

<u>mor</u> - <u>minor morphisms</u>

<u>Note</u> There exists a morphism

$\varphi: G \to G'$ iff

G' can be obtained from G by a sequence of edge deletions, contractions and graph automorphisms.

Thm (Proudfoot–Miyata, –)

The category of $\mathcal{G}^{op}$-mods satisfies a Noetherian property.

Cor: The following categories satisfy a Noetherian property in their reps

FI, FS^{op}, $VI_{\mathbb{F}_q}$,

$\mathcal{G}_g^{op}$ ← contraction category of graphs with first Betti # $= g$

For a graph G we can talk about its configuration space

$\mathcal{F}_n(G)$ = unordered conf space on G.

Fact: A minor morphism $\varphi : G \to G'$ induces
$$\varphi_* : H_i(\mathcal{F}_n(G')) \to H_i(\mathcal{F}_n(G))$$

Thm (Miyata, Proudfoot, -) the $\mathcal{G}^{op}$-module

$G \mapsto H_i(\mathcal{F}_n(G))$ is finitely generated for all i, n.

Moreover there is a version of this theorem where only ℓ is fixed.

<u>Cor</u>: The kinds of torsion which can appear in the homology of graph conf spaces only depends on the homological index.