# Homological Stability for Spaces of Representations

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$G$ – compact, ctd Lie gp

$\mathrm{Hom}(\mathbb{Z}^n, G) \subseteq G^n$ – space of commuting $n$-tuples

$\ni (g_1, \dots, g_n)$

$[g_i, g_j] = 1$

Studied by Witten due to connections with gauge theory.

<u>Note</u>: If $G$ is the cplx'n of a cpt gp $K$, e.g. $G = GL_n\mathbb{C}$, $K = U(n)$ then $\mathrm{Hom}(\mathbb{Z}^n, G) \simeq \mathrm{Hom}(\mathbb{Z}^n, K)$
(Pettet–Souto, Bergeron)

<u>Goal</u>: Understand the <u>rational</u> homology of $\mathrm{Hom}(\mathbb{Z}^n, G)$ and related spaces, e.g. $\mathrm{Hom}(\mathbb{Z}^n, G)/G$, where $G$ acts by conj'n in each coord.

Poincaré series can be described via Lie theoretic data about $G$:

<u>Stafa</u>: $P_t(\mathrm{Hom}(\mathbb{Z}^n, G)_1 / G) = \frac{1}{|W|} \sum_{w \in W} \det(1 + tw)^n$

identity component

$W$ – Weyl gp of $G$, acting on $\mathrm{Lie}(T)$

max'l torus

<u>R.–Stafa</u>:
Similar formula for $\mathrm{Hom}(\mathbb{Z}^n, G)_1$

Computer calc'n suggested that $\mathrm{rank}\left(H_k \mathrm{Hom}(\mathbb{Z}^n, G)_1\right) = $ coeff of $t^k$ in $P_t$ stabilizes as $G$ varies through the classical sequences $U(r)$, $Sp(r)$, ...

## Main Results:

- $H_k(\mathrm{Hom}(\mathbb{Z}^n, G_r)_1/G_r) \to H_k(\mathrm{Hom}(\mathbb{Z}^n, G_{r+1})_1/G_{r+1})$ is an isom. once $r \geq k$
- $H_k(\mathrm{Hom}(\mathbb{Z}^n, G_r)_1) \to H_k(\mathrm{Hom}(\mathbb{Z}^n, G_{r+1})_1)$ is an isom. once $r - \lfloor \sqrt{r} \rfloor \geq k$

Note: Comp'n suggests actual stable range is closer to $r \geq k/2$.

Methods: • Representation Stability (Church-Ellenberg-Farb, Wilson).

• Rational models for these spaces (Baird)

Key Observation: To prove that

$$\cdots \to H_k(X_r) \to H_k(X_{r+1}) \to \cdots$$

stabilizes, realize $X_r$ as

$$X_r = \tilde{X}_r / \Sigma_r \leftarrow \text{symmetric gp}$$

If $r \mapsto H_k(\tilde{X}_r)^{\circlearrowleft \Sigma_r}$ is rep'n stable, then

$$H_k(X_r) = H_k(\tilde{X}_r)^{\Sigma_r} = \begin{array}{l}\text{isotypic comp}\\ \text{of triv. rep.}\end{array}$$

stabilizes as well, w/ same stable range

**Baird:** Conjugating elements from a max'l torus $T \le G$ induces rational equivalences

$$(\star)\quad (G/T \times T^n)/W \longrightarrow \mathrm{Hom}(\mathbb{Z}^n, G)_1$$
$$(g, t_1, \dots, t_n) \longmapsto (g t_1 g^{-1}, \dots, g t_n g^{-1})$$

and

$$(BT \times T^n)/W \simeq \left(\mathrm{Hom}(\mathbb{Z}^n, G)_1\right)_{hG}$$

htpy orbit sp

$(\star)$ reduces the question of hom. stability to rep. stability for $G_r/T_r$ & $T_r$.

**Wilson:** $H_k(G_r/T_r)$ has structure of a fin. gen. $FI_W$-module.

For $T_r \cong (S^1)^r$, we apply:

> **Exercise:** If $H_* X$ is fin dim'l $\forall *$, then $r \mapsto H_k(X^r) \circlearrowleft \Sigma_r$ is a fin. gen. FI-module w/ gen's in $H_k(X^k)$ (Hint: Künneth Thm)

In fact, $r \mapsto H_k(X^r) \circlearrowleft \Sigma_r$ is an FI#-mod, so $H_k(X^r)^{\Sigma_r} = H_k(X^r/\Sigma_r)$ stabilizes at $r = k$.

General theory then implies

$$H_k(\mathrm{Hom}(\mathbb{Z}^n, G_r)_1) = H_k(G_r/T_r \times T_r^n)^{W_r}$$

stabilizes, but does not yield a stable range.

(Wilson's result uses Noetherian property of FI-modules.)

To bound the stable range:

Baird's model

$$(BT \times T^n)/W \simeq \left(\mathrm{Hom}(\mathbb{Z}^n, G)_1\right)_{hG}$$

yields

$$H_k^G\left(\mathrm{Hom}(\mathbb{Z}^n, G)_1\right) = H_k(BT \times T^n)^W.$$

Now $r \longmapsto H_k(BT_r) = H_k((BS^1)^r)$

& $r \longmapsto H_k(T_r{}^n) \cong H_k(((S^1)^n)^r)$

are both FI$_W$#-modules, gen'd by the $r=k$-terms.

<u>Conclusion:</u> $\mathrm{Hom}(\mathbb{Z}^n, G_r)_1$ satisfies $G_r$-equiv. homological stab. w/ range $r \geq k$.

<u>Main Thm:</u>

$$H_k\left(\mathrm{Hom}(\mathbb{Z}^n, G_r)_1\right) \to H_k\left(\mathrm{Hom}(\mathbb{Z}^n, G_{r+1})_1\right)$$

is an isom. once $r - \lfloor \sqrt{r} \rfloor \geq k$

<u>Pf:</u> Analyze the Eilenberg-Moore spectral sequence for

$$\begin{array}{ccc} \mathrm{Hom}(\mathbb{Z}^n, G_r)_1 & \longrightarrow & \left(\mathrm{Hom}(\mathbb{Z}^n, G_r)_1\right)_{hG_r} \\ & & \downarrow \\ & & BG_r \end{array}$$

as $r$ varies. $\square$

## Further results:

**Thm (R-Stafa):** $n \mapsto H_k(\mathrm{Hom}(\mathbb{Z}^n, G)_1)^{\mathbb{Q}\Sigma_n}$ is a rep. stable for $n \geq 2k$, and

$$n \mapsto H_k(\mathrm{Hom}(\mathbb{Z}^n, G)_1/\Sigma_n)$$

is stable for $n \geq k$.

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**Thm** (Á. Cruz, Master's Thesis '19)

Stability holds for subspaces of abelian configurations $(g_1, \ldots, g_n)$ s.t. $g_i \neq g_j$ & $[g_i, g_j] = 1$.

Method: Compare to $\mathrm{Hom}(\mathbb{Z}^n, G)$ via the Totaro spectral sequence.

## Questions:

- Can stable ranges be improved? In particular, what is the stable range for $\{Gr/Tr\}_{r=2}^{\infty}$ in Wilson's theorem?

- What happens for the char. var's

$$\mathcal{X}_n(G) = \mathrm{Hom}(F_n, G)/G\,?$$

When $G = SU(2)$, work of Baird implies

$$k \mapsto H_k(\mathcal{X}_n(G))^{\mathbb{Q}\Sigma_n}$$

is rep. stable.

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