Two tales from equivariant commutative algebra

Andrew Snowden

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University of Michigan

§1. Background

Equivariant commutative algebra

Suppose that a group G acts on a ring R.

Equivariant commutative algebra studies the commutative algebra of R taking into account the G action.

Motivation in representation stability:

- $G = \mathfrak{S}_\infty$ acting on $R = \mathbf{C}[x_i]_{i \geq 1}$.
- $G = \mathbf{GL}_\infty$ acting on $\mathrm{Sym}(V)$, where V is a polynomial representation (twisted commutative algberas).

Equivariant definitions

There is a long list of important concepts in commutative algebra: noetherian ring, prime ideal, radical of an ideal, etc.

How to import concepts to the equivariant world

1. Write down the usual definition using ideals (no elements!).
2. Change "ideal" to "G-ideal" ($=G$-stable ideal).

Prime ideals: definitions

Fact

An ideal $\mathfrak{p}$ of R is prime if and only $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$ implies $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$, for ideals $\mathfrak{a}$ and $\mathfrak{b}$.

Definition (G-prime)

A G-ideal $\mathfrak{p}$ is *G-prime* if $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$ implies $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$, for G-ideals $\mathfrak{a}$ and $\mathfrak{b}$.

Proposition (Elemental criterion for G-primality)

Let $\mathfrak{p}$ be a G-ideal. Then $\mathfrak{p}$ is G-prime if and only if $f \cdot (\sigma g) \in \mathfrak{p}$ for all $\sigma \in G$ implies $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$.

Prime ideals: example 1

Let $R = \mathbf{C}[x, y]$ and let $G = \mathfrak{S}_2$ act on R by transposing the variables.

Proposition

The ideal $\mathfrak{p} = (xy)$ is G-prime.

Proof

- Suppose $f \cdot (\sigma g) \in \mathfrak{p}$ for all σ, but $f \notin \mathfrak{p}$.
- Assume (WLOG) f is not identically zero on $y = 0$.
- Since $f(x, y)g(x, y)$ and $f(x, y)g(y, x)$ both vanish on $y = 0$, it follows that $g(x, y)$ and $g(y, x)$ both vanish on $y = 0$.
- Thus g vanishes on $x = 0$ and $y = 0$, and so $g \in \mathfrak{p}$.

Prime ideals: finite groups

Suppose for the moment that the group G is finite.

Proposition

- *If $\mathfrak{p}$ is a G-prime of R then $\mathfrak{p} = \bigcap_{\sigma \in G} \sigma\mathfrak{q}$ for some ordinary prime ideal $\mathfrak{q}$.*
- *Have bijection $\{G\text{-primes}\} \leftrightarrow \{G\text{-orbits of primes}\}$.*

Corollary

Every G-prime is radical.

Prime ideals: example 2

Let $R = \mathbf{C}[x_i]_{i\geq 1}$ and let $G = \mathfrak{S}_\infty$ act by permuting variables.

Proposition

The ideal $\mathfrak{p} = (x_1^2, x_2^2, \ldots)$ is G-prime.

Proof

- Suppose $f \cdot (\sigma g) \in \mathfrak{p}$ for all σ.
- Pick σ so that f and σg have no variable in common.
- Every monomial in $f \cdot (\sigma g)$ has a square. Since f and σg have no common variable, the same must be true for either f or g.
- Thus $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$.

Radicals: definition

Fact

Let $\mathfrak{a}$ be an ideal of R. The radical of $\mathfrak{a}$ is the sum of all ideals $\mathfrak{c}$ such that $\mathfrak{c}^n \subset \mathfrak{a}$ for some n.

Definition (G-radical)

Let $\mathfrak{a}$ be a G-ideal. The *G-radical* of $\mathfrak{a}$, denoted $\mathrm{rad}_G(\mathfrak{a})$, is the sum of all G-ideals $\mathfrak{c}$ such that $\mathfrak{c}^n \subset \mathfrak{a}$ for some n.

Proposition (Elemental formulation of G-radical)

Let $\mathfrak{a}$ be a G-ideal. Then $x \in \mathrm{rad}_G(\mathfrak{a})$ if and only if there exists $n \geq 1$ such that $(\sigma_1 x) \cdots (\sigma_n x) \in \mathfrak{a}$ for all $\sigma_1, \ldots, \sigma_n \in G$.

The connection between radicals and primes

Proposition

Let $\mathfrak{a}$ be a G-ideal of R. Then $\operatorname{rad}_G(\mathfrak{a})$ is the intersection of all G-primes containing $\mathfrak{a}$.

Proof

The usual proof applies with minimal changes.

Spectra

Definition

The *G-spectrum* of R, denoted $\mathrm{Spec}_G(R)$, is the set of all G-primes of R, equipped with the usual Zariski topology.

Remark

The closed subsets of $\mathrm{Spec}_G(R)$ are in 1–1 correspondence with the G-radical ideals of R, by the previous slide.

The main problem

The main problem

Describe $\mathrm{Spec}_G(R)$ in cases of interest.

§2. Symmetric ideals

The set-up

Let $R = \mathbf{C}[x_i]_{i \geq 1}$ and $G = \mathfrak{S}_\infty$.

Goal of this section

Classify the G-primes of R.

Everything in this part is joint work with Rohit Nagpal. The papers should appear in the near future.

The general strategy

It is not hard to see that if $\mathfrak{p}$ is G-prime then $\mathrm{rad}(\mathfrak{p})$ is also a G-prime. Note that here we are taking the ordinary radical.

We can therefore consider the map

$$\{G\text{-primes}\} \xrightarrow{\mathrm{rad}} \{\text{radical } G\text{-primes}\}.$$

The general strategy

First classify the radical G-primes. Then analyze the fibers of the above map.

Radical G-primes via geometry

Let $\mathfrak{X} = \mathrm{Spec}(R)$, which we identify with $\mathbf{C}^\infty$, the set of all sequences $(a_1, a_2, \ldots)$ of complex numbers.

The construction $\mathfrak{a} \mapsto V(\mathfrak{a})$ provides a bijection between ideals of R and closed subsets of $\mathfrak{X}$.

Observation

This induces a bijection between radical G-primes of R and G-irreducible closed subsets of $\mathfrak{X}$.

(Note: G-irreducible means the set cannot be written as a union of two proper closed G-stable subsets.)

It therefore suffices to study G-irreducible closed subsets of $\mathfrak{X}$.

How to build loci in $\mathfrak{X}$

There are three things one can do to build G-stable closed loci:

- Bound the number of distinct values in the sequence (a_i).
- Bound the multiplicities of these values.
- Impose algebraic relations between the values.

We'll look at examples to illustrate what these points mean, and then state a precise theorem.

Example: bounding number of distinct values

- Let $Z \subset \mathfrak{X}$ be the set of all sequences (a_i) that assume at most two distinct values, i.e., such that $\{a_i \mid i \geq 1\} \subset \mathbf{C}$ has cardinality at most two.
- An element of Z looks something like $(a, b, b, a, a, b, \ldots)$ with $a, b \in \mathbf{C}$.
- The set Z is Zariski closed: it is the common vanishing locus of the discriminants $\Delta_{i,j,k} = (x_i - x_j)(x_i - x_k)(x_j - x_k)$.

Example: bounding multiplicities of values

- Let $Z' \subset Z$ consist of those two-valued sequences where one value occurs at most one time.
- An element of Z' looks something like $(a, b, a, a, a, a, \ldots)$, where there is one b and the remaining entries are a.
- The set Z' is Zariski closed: it is cut out (inside of Z) by the equations $(x_i - x_j)(x_k - x_\ell) = 0$ (all indices distinct).

Example: algebraic relations between values

- Fix some polynomial $f \in \mathbf{C}[x, y]$.
- Let $Z'' \subset Z'$ consist of those sequences (a_i) such that $f(a, b) = 0$, where a is the value occurring infinitely often and b is the value occurring at most once.
- The set Z'' is Zariski closed: it is cut out (inside of Z') by the equations $(x_i - x_j)(x_i - x_k)f(x_j, x_i) = 0$ (all indices distinct).

The general construction

- Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a non-increasing sequence with $1 \leq \lambda_i \leq \infty$ and $\lambda_1 = \infty$ (a "partition of infinity").
- Define $\mathfrak{X}_\lambda$ to be the set of sequences that have exactly r values with multiplicities given by λ (i.e., the 3rd most common value occurs exactly λ_3 times).
- Let $H \subset \mathfrak{S}_r$ be the stabilizer of λ. We have a well-defined map $\mathfrak{X}_\lambda \to \mathbf{C}^r/H$ taking a sequence to the tuple of its values ordered by multiplicity.
- Given an irreducible closed subset W of $\mathbf{C}^r/H$, let $\mathfrak{X}_\lambda(W)$ be its inverse image in $\mathfrak{X}_\lambda$.

Classification of radical G-primes

Theorem (Nagpal–Snowden)

We have a bijection

$$\begin{aligned} \{\textit{pairs } (\lambda, W)\} &\to \{\textit{proper } G\textit{-irreducible closed subsets of } \mathfrak{X}\} \\ (\lambda, W) &\mapsto \textit{closure of } \mathfrak{X}_\lambda(W). \end{aligned}$$

On to general G-primes

The previous theorem gives a complete description of the target of the map

$$\{G\text{-primes}\} \xrightarrow{\text{rad}} \{\text{radical } G\text{-primes}\}.$$

We must now understand the fibers of this map.

Let $\mathfrak{q}$ be the ideal of R generated by $x_i - x_j$ for $i, j \geq 1$, which is a radical G-prime.

Via a localization argument, we reduce the problem of analyzing the general fibers to analyzing the fiber above $\mathfrak{q}$.

The fiber over $\mathfrak{q}$

Let $\mathfrak{q}_n$ be the ideal of R generated by $(x_i - x_j)^n$ for $i, j \geq 1$.
Clearly, $\mathrm{rad}(\mathfrak{q}_n) = \mathfrak{q}$ for all n.

Theorem (Nagpal–Snowden)

- *The ideal $\mathfrak{q}_n$ is G-prime if and only if n is odd.*
- *Any G-prime with radical $\mathfrak{q}$ has the form $\mathfrak{q}_n$ with n odd.*

The key result

We now state an elementary result that is a key piece in our proof of the previous theorem.

Let $A = \mathbf{C}[x_1, \ldots, x_N]$ and $\mathfrak{a} = \langle (x_i - x_j)^{2n-1} \rangle_{1 \leq i,j \leq N}$.

Let $B = A[t]$ and $\mathfrak{b} = \langle (x_i - t)^n \rangle_{1 \leq i \leq N}$.

Proposition

The contraction of $\mathfrak{b}$ to A is $\mathfrak{a}$.

The containment $\mathfrak{a} \subset \mathfrak{b}^c$ is clear: indeed,

$$(x_i - x_j)^{2n-1} = ((x_i - t) - (x_j - t))^{2n-1} = \sum_{k=0}^{2n-1} \binom{2n-1}{k} (x_i - t)^k (x_j - t)^{2n-1-k}$$

and every term on the right has a nth power in it. Showing that $\mathfrak{a} = \mathfrak{b}^c$ is rather involved.

Further remarks

- Rohit and I have many additional results, such as:
  - description of $\mathrm{Spec}_G(R)$ as a topological space
  - primary decomposition for G-ideals
  - results on the structure of G-equivariant modules.
- Some of these results work in positive characteristic and some do not.
- It would be interesting to generalize these results to the ring $\mathbf{C}[x_{i,j}]_{1\leq i\leq r, 1\leq j}$ with r sets of infinite variables. In joint work with Rohit and Vignesh Jagathese we have described the radical G-primes.

§3. Twisted commutative algebras

Twisted commutative algebras

Work over $\mathbf{C}$. A *polynomial representation* of $G = \mathbf{GL}_\infty$ is one that decomposes as a sum of Schur functors.

A *twisted commutative algebra* (tca) is a $\mathbf{C}$-algebra equipped with an action of G under which it forms a polynomial representation.

Goal of this section

Understand G-primes in tca's.

Example 1

Let $A = \mathrm{Sym}(\mathbf{C}^r \otimes \mathbf{C}^\infty) = \mathbf{C}[x_{i,j}]_{1 \le i \le r, 1 \le j}$.

Proposition (Sam–Snowden)

The G-primes of A are exactly the G-stable prime ideals of A.

Proof

See [SS, §9.3].

Remark

Every tca generated in degree 1 is a quotient of A (for some r), and so the proposition holds for these tca's too. More generally, it holds for bounded tca's.

Example 2

Let $A = \bigoplus_{n \geq 0} \bigwedge^{2n}(\mathbf{C}^\infty)$. Every positive degree element of A is nilpotent. Thus $\mathrm{Spec}(A)$ is a single point. However:

Proposition

The ideal (0) *is* G*-prime, that is,* A *is a* G*-domain.*

Proof

- Each graded piece of A is an irreducible representation.
- Thus the non-zero G-ideals of A are those of the form $\mathfrak{a}_r = \bigoplus_{n \geq r} \bigwedge^{2n}(\mathbf{C}^\infty)$.
- Since $\mathfrak{a}_r \mathfrak{a}_s = \mathfrak{a}_{r+s}$, the result follows.

We thus see that $\mathrm{Spec}_G(A) = \{(0), \mathfrak{a}_1\}$.

Example 3

Let $A = \mathrm{Sym}(\mathrm{Sym}^2(\mathbf{C}^\infty)) = \mathbf{C}[x_{i,j}]_{1 \le i \le j \le \infty}$.

We have the well-known decomposition $A = \bigoplus_{\lambda \text{ even}} \mathbf{S}_\lambda(\mathbf{C}^\infty)$, where "even" means all parts are even.

Let $\mathfrak{p}_{r,s}$ be the ideal generated by the $(2s+2) \times (r+1)$ rectangle.

It turns out that these are exactly the non-zero G-primes of A, and that $\mathfrak{p}_{r,s}$ is radical if and only if $s = 0$.

We find $\mathrm{Spec}_G(A) = \mathbf{N}^2 \cup \{\infty\}$ with the order topology.

Geometric interpretation: part 1

Before moving on to the general case, we should try to understand Example 3 more conceptually. This is best done geometrically.

$\mathrm{Spec}(A)$ is the space of symmetric bilinear forms on $\mathbf{C}^{\infty}$.

The ideals $\mathfrak{p}_{r,0}$ are well-known classical objects, called determinantal ideals. $V(\mathfrak{p}_{r,0})$ is the locus of forms of rank $\leq r$.

Key insight (Sam–Snowden)

This picture suggests that $\mathfrak{p}_{r,s}$ should be related to ranks of super bilinear forms.

Interlude: super mathematics

- A *super vector space* is a $\mathbf{Z}/2\mathbf{Z}$-graded vector space: $V = V_0 \oplus V_1[1]$. The $[1]$ indicates that V_1 is in odd degree.
- Notation: $\mathbf{C}^{p|q} = \mathbf{C}^p \oplus \mathbf{C}^q[1]$.
- Tensor products are formed as usual.
- Sign rule: the symmetry isomorphism $V \otimes W \to W \otimes V$ is given by $x \otimes y \mapsto (\pm 1) \cdot y \otimes x$, where we use -1 if both x and y have odd degree, and $+1$ otherwise.
- The sign rule is entirely responsible for the interesting aspects of the theory.
- Much of algebra can be generalized to superalgebra: for example, there are supergroups, superschemes, and so on.

Interlude: super mathematics (continued)

- A symmetric bilinear form on a super vector space V is a map $\omega\colon \mathrm{Sym}^2(V) \to \mathbf{C}$.
- We have

$$\mathrm{Sym}^2(V) = \mathrm{Sym}^2(V_0) \oplus \mathrm{Sym}^2(V_1[1]) \oplus (V_0 \otimes V_1)[1]$$

- $(V_0 \otimes V_1)[1]$ must map to 0 (degree reasons).
- Sign rule: $\mathrm{Sym}^2(V_1[1]) = \bigwedge^2(V_1)$.
- Thus ω corresponds to a pair (ω_0, ω_1) where
  - ω_0 is a symmetric bilinear form on V_0
  - ω_1 is an alternating bilinear form on V_1.

Geometric interpretation: part 2

The super scheme $\mathrm{Spec}(A(\mathbf{C}^{\infty|\infty}))$ is the space of symmetric bilinear forms on $\mathbf{C}^{\infty|\infty}$.

Note: the reduced subscheme is the ordinary scheme $\mathrm{Spec}(A) \times \mathrm{Spec}(B)$ where $B = \mathrm{Sym}(\bigwedge^2(\mathbf{C}^{\infty}))$.

$V(\mathfrak{p}_{r,s}(\mathbf{C}^{\infty|\infty}))$ is exactly the locus of forms ω where ω_0 has rank $\leq r$ and ω_1 has rank $\leq 2s$.

Thus the geometry of the super scheme can "see" the non-radical G-primes!

The main theorem

Theorem (Snowden, [Sn])

Let A be a tca and let $\mathfrak{a}$ and $\mathfrak{b}$ be G-ideals.

- $\mathrm{rad}_G(\mathfrak{a}) = \mathrm{rad}_G(\mathfrak{b}) \iff V(\mathfrak{a}(\mathbf{C}^{\infty|\infty})) = V(\mathfrak{b}(\mathbf{C}^{\infty|\infty}))$.
- $\mathrm{rad}_G(\mathfrak{a})$ *is G-prime* $\iff V(\mathfrak{a}(\mathbf{C}^{\infty|\infty}))$ *is irreducible.*

Remark

Compare the theorem with the statements:

- $\mathrm{rad}(\mathfrak{a}) = \mathrm{rad}(\mathfrak{b}) \iff V(\mathfrak{a}) = V(\mathfrak{b})$.
- $\mathrm{rad}(\mathfrak{a})$ is prime $\iff V(\mathfrak{a})$ is irreducible.

Upshot: $\mathrm{Spec}(A(\mathbf{C}^{\infty|\infty}))$ is rich enough to "see" $\mathrm{Spec}_G(A)$.

Further remarks

- Consequence of main theorem + Draisma's theorem: if A is a finitely generated tca then $\mathrm{Spec}_G(A)$ is a noetherian topological space.
- The main theorem is reminiscent of a theorem of Deligne, which asserts that, under certain natural conditions, a tensor category admits a fiber functor to super vector spaces.
- All of this breaks in positive characteristic. It would be very interesting to extend the results there!

References

[Sn] A. Snowden. The spectrum of a twisted commutative algebra. arXiv:2002.01152

[SS] S. Sam, A. Snowden. Introduction to twisted commutative algebras. arXiv:1209.5122