I. Nonstable cohomology of arithmetic groups

$SO(p,q;\mathbb{Z}) = \{ g \in SL_{p+q}(\mathbb{Z}) : g^t I_{p,q} g = I_{p,q} \}$ $\quad I_{p,q} = \begin{pmatrix} Id_p & \\ & -I_q \end{pmatrix}$

<u>Main Thm</u> (T, 2017) $1 \le p \le q$, $p+q \geq 3$, p odd

$\forall N \geq 1$ $\exists$ finite index $\Gamma < SO(p,q;\mathbb{Z})$ s.t.

$$\dim H^p(\Gamma;\mathbb{Q}) \geq N.$$

<u>Rmks</u> / Context

- Given ℓ prime can take Γ congruence subgroup

$$\Gamma(\ell^n) = SO(p,q;\mathbb{Z}) \cap \operatorname{Ker}\left[SL_{p+q}(\mathbb{Z}) \longrightarrow SL_{p+q}(\mathbb{Z}/\ell^n\mathbb{Z}) \right]$$

for $n \gg 0$.

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- Millson-Raghunathan (1980) similar thm for cocompact lattices in $SO(p,q)$ p even

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- Borel stability. say $p = q$.

not trivial.

$H^\cdot(\Gamma;\mathbb{Q}) \cong H^\cdot(SO(p,q;\mathbb{Z});\mathbb{Q})$ poly ring generators in deg $4k$

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Rmks / Context

- classes produced in $H^p(\Gamma;\mathbb{Q})$ have an interpretation as characteristic classes, in terms of obstruction theory

Applications

① $W_g^{4k} = \#_g (S^{2k} \times S^{2k})$

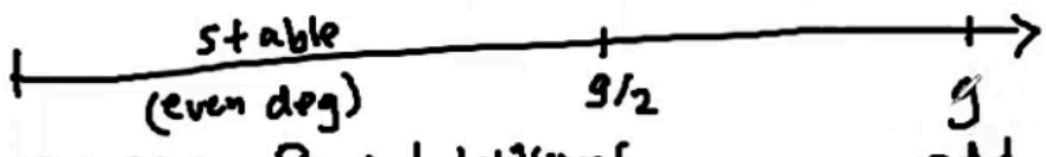
$$\mathrm{Diff}(W_g) \xrightarrow{\alpha} \mathrm{Aut}(H_{2k}(W_g), \text{int form}) \doteq SO(g,g;\mathbb{Z})$$

$$\begin{array}{ccc} \cup & & \cup \ \text{f.i.} \\ \mathrm{Diff}^{\Gamma}(W_g) = \alpha^{-1}(\Gamma) & \longrightarrow & \Gamma \end{array}$$

Thm (Berglund-Madsen, Krannich-T) $k \geq 2$

$H^i(B\Gamma;\mathbb{Q}) \longrightarrow H^i(B\,\mathrm{Diff}^{\Gamma}(W_g);\mathbb{Q})$ injective for $i \leq 2k$.

$g \leq 2k$



$H^*(B\,\mathrm{Diff}^{\Gamma}(W_g);\mathbb{Q})$

Galatius - Randal-Williams

Thm ⇒ dim $\geq N$ not in algebra gen by stable classes

② M^4 K3 surface $\cong \{x^4+y^4+z^4+w^4=0\} \subset \mathbb{CP}^3$

$$\mathrm{Diff}(M) \longrightarrow \mathrm{Aut}(H_2(M), \text{int form}) \doteq O(3,19;\mathbb{Z})$$

<u>Thm</u> (Giansiracusa, 2009) $\Gamma < SO(3,19;\mathbb{Z})$ finite index

$$H^i(B\Gamma;\mathbb{Q}) \longrightarrow H^i(B\pi_0\mathrm{Diff}(M);\mathbb{Q}) \quad \text{injective} \quad \forall i$$

Main thm $\Rightarrow$ $\exists$ nonzero $z \in H^3(B\pi_0\mathrm{Diff}(M);\mathbb{Q})$

<u>Rmk</u> if z lifts to $H^3(B\mathrm{Diff}(M);\mathbb{Q})$ then $\exists$ $\begin{array}{c} M^4 \to E \\ \downarrow \\ B^3 \end{array}$

w/ no fiberwise Einstein metric

(Donaldson) for $\begin{array}{c} M^4 \to E \\ \downarrow \\ S^1 \end{array}$ always has fib. Ein metric.

II. Characteristic classes for vector bundles w/ lattice

<u>Plan</u> for certain $\Gamma < SL_n\mathbb{Z}$ define char class $c \in H^{n-1}(B\Gamma)$ for v.b. $\mathbb{R}^n \to E \to B$ w/ str. gp Γ.

(similar construction for $SO(p,q;\mathbb{Z})$)

- Fix decomposition $\mathbb{R}^n = P \oplus L$ $\dim L = 1$, defined over $\mathbb{Q}$.

- Given

$$\begin{tikzcd}
\mathbb{Z}^n \arrow[d] & \mathbb{R}^n \arrow[d] \\
\Lambda \arrow[r, hook] \arrow[dr] & W, q \arrow[d] \\
& B
\end{tikzcd}$$

fiberwise inner prod

v.b. str group

$SL_n(\mathbb{Z}) < SL_n(\mathbb{R}) \sim SO(n)$

<u>Defn</u> (P,L) is <u>q-orthogonal at $b \in B$</u> if $\exists$ iso $\phi: (\mathbb{R}^n, \mathbb{Z}^n) \to (W_b, \Lambda_b)$ s.t. $W_b = \phi(P) \oplus \phi(L)$ orthog. wrt q_b. If (P,L) not q-orth. at any $b \in B$ say (P,L) <u>nowhere q-orthogonal</u>.

__Ex__ $B = pt$

$$\begin{tikzcd} X = SO(n) \backslash SL_n(\mathbb{R}) \arrow[r, "\cong"] & \left\{ \begin{matrix} \text{inner prod} \\ \text{on } \mathbb{R}^n \end{matrix} \right\} \\ SO(n) \cdot g \arrow[u, hook, "\substack{\text{codim} \\ n-1}"'] \arrow[r, mapsto] & g^t g \end{tikzcd}$$

$H = \{ \text{inn. prod. s.t. } \mathbb{R}^n = P \oplus L \text{ orthogonal} \}.$

For inner prod q on $\underset{x}{\underset{\downarrow}{W}} \cong \mathbb{R}^n$, (P, L) is nowhere q-orthogonal

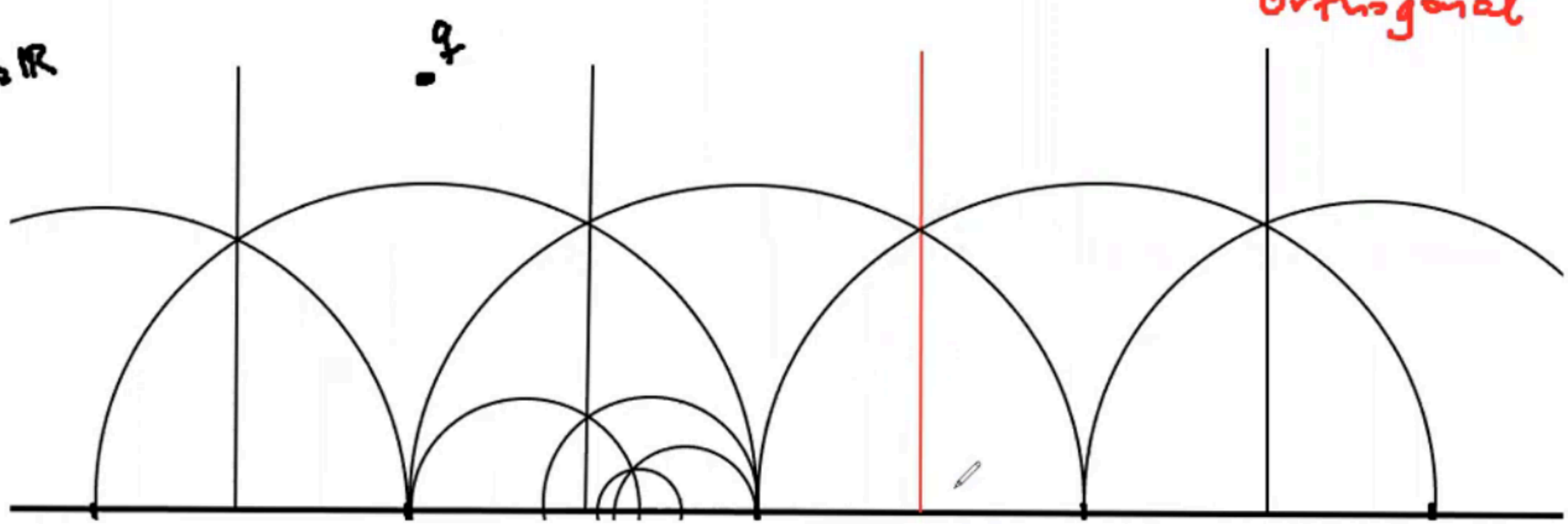
$\iff \quad q \in X \backslash (H \cdot SL_n(\mathbb{Z}))$

Fact $\exists$ f.i. torsion free $\Gamma < SL_n(\mathbb{Z})$ st Γ orbit of H is embedded in X.

H = inner prod st. $R^2 = \langle e_1 \rangle \oplus \langle \frac{e_1}{2} + e_2 \rangle$ orthogonal

$n = 2$

$\mathbb{H}^2 = SL_2\mathbb{R} / SO(2)$



$\underline{\text{Ex}}$ $B = pt$

$$X = SO(n)\backslash SL_n(\mathbb{R}) \xrightarrow{\cong} \left\{ \begin{array}{c} \text{inner prod} \\ \text{on } \mathbb{R}^n \end{array} \right\}$$

$$\text{codim } n-1 \uparrow \quad SO(n)\cdot g \longmapsto g^t g$$

$H = \{$ inn. prod. s.t. $\mathbb{R}^n = P \oplus L$ orthogonal $\}$.

For inner prod q on $W \cong \mathbb{R}^n$, (P, L) is nowhere q-orthogonal
$\downarrow$
$*$

$$\Leftrightarrow \quad q \in X \backslash (H \cdot SL_n(\mathbb{Z}))$$

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for $W \to B$ str gp in $\Gamma < SL_n(\mathbb{Z})$

if $\exists\, q$ s.t. (P, L) nowhere q-orthogonal

then $\dfrac{\tilde{B} \times (X \backslash H \cdot \Gamma)}{\pi_1(B)} \to B$ has a section

obstruction theory $\rightsquigarrow$ $c(W) \in H^{n-1}(B)$

A: yes $SL_n\mathbb{Z}$
(Avramidi - Phan)
and $SO(p,q;\mathbb{Z})$
unknown for $Sp_{2g}(\mathbb{Z})$ p odd
Prob: Study
$\mathrm{Mod}(S_g) \to Sp_{2g}(\mathbb{Z})$
on $H^*(-;\mathbb{Q})$
outside stable range

$\underline{Q}$ Is $c \neq 0$?
$\uparrow$
$H^{n-1}(B\Gamma;\mathbb{Q})$