Second order terms in arithmetic statistics

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Statistics of number fields

Let $n > 0$, and $A < B$. Define

$$N_n(A, B) := \#\{ K/\mathbb{Q} \mid \deg(K/\mathbb{Q}) = n,\ A < \Delta_K < B \}$$

__Conjecture__ (Linnik?) For each n, $N_n(A, B)$ is asymptotically linear in A and B.

__$n=2$__: Classical, related to counting squarefree integers: $N_2(-X, X) \approx X/\zeta(2)$.

__$n=3$__: Davenport - Heilbronn   __$n=4,5$__: Bhargava

__Variant__: If $G \hookrightarrow S_n$ is a transitive subgroup, set

$$N_{G,n}(A, B) := \#\left\{ K/\mathbb{Q} \;\middle|\; \begin{array}{l} \deg(K/\mathbb{Q}) = n, \quad A < \Delta_K < B \\ \mathrm{Gal}(K/\mathbb{Q}) \cong G \end{array} \right\}$$

(isomorphism also encodes action of Gal on $\mathrm{Emb}(K, \bar{K})$).

A conjecture of Malle predicts the asymptotics of $N_{G,n}(-X, X)$. Ellenberg-Tran-W. proved the upper bound in Malle's conjecture if $\mathbb{Q}$ goes $\mathbb{F}_q(t)$, with $(\#G, q) = 1$ and $q > Q(G)$.

Second order term for $n = 3$: (Taniguchi-Thorne, Bhargava-Shankar-Tsimerman)

$$N_3(0, X) = \frac{1}{12\zeta(3)} X + \frac{4\zeta(1/3)}{5\Gamma(2/3)^3 \zeta(5/3)} X^{5/6} + O_\varepsilon\left(X^{5/6 - 1/48 + \varepsilon}\right)$$

Goal: investigate second order terms in the $\mathbb{F}_q(t)$-setting.

Method: 1. identify the set of $K/\mathbb{F}_q(t)$ w/ specified discriminant as the $\mathbb{F}_q$ points of a Hurwitz moduli space:

$$\mathrm{Hur}^c_{G,n}(K) := \left\{ \pi : \Sigma \to \mathbb{A}^1_K \;\middle|\; \begin{array}{l} \mathrm{Gal}(\pi) = G, \ \exists\, n \text{ branch pts} \\ \text{over } \overline{K}, \text{ monodromy in } c \end{array} \right\}$$

2. Bound $\mathrm{rk}\left(H_j\left(\mathrm{Hur}^c_{G,n}(\mathbb{C}), \mathbb{Q}_\ell\right)\right)$

3. Asymptotically bound point count using the Weil conjectures

<u>For Malle's conjecture</u>: we bounded $\mathrm{rk}(H_j(\mathrm{Hur}^c_{G,n}))$ polynomially in n and exponentially in j.

For <u>second order terms</u>: improve these bounds.

<u>Topology of</u> $\mathrm{Hur}^c_{G,n}$: $\exists$ covering space $\mathrm{Hur}^c_{G,n} \to \mathrm{Conf}_n(\mathbb{C})$ which records the branch locus of a cover. The fibre is $c^{\times n}$, recording the monodromy at the branch locus. So:

$$H_j(\mathrm{Hur}^c_{G,n}, k) \cong H_j(\mathrm{Conf}_n(\mathbb{C}), k[c^{\times n}]) \cong H_j(B_n, V^{\otimes n})$$

where $V := k[c]$ is a <u>braided vector space</u> with $\sigma: V \otimes V \to V \otimes V$ given by $\sigma(g \otimes h) = h \otimes g^h$

<u>Thm</u> (ETW) $\exists$ ring isomorphism

$$H_j(B_n, V^{\otimes n}) \cong \mathrm{Ext}^{n-j,n}_{A(V_\epsilon)}(k,k)$$

where $A(V_\epsilon)$ is the <u>quantum shuffle algebra</u> on $V_\epsilon = V \otimes \mathrm{sgn}$: $A(W) \cong T^{c}(W)$ as a braided Hopf algebra.

The cohomology of the quantum shuffle algebra

$A(W) = T^{c}(W)$ contains W; let

$B(W) := \langle W \rangle$ = subalgebra gen. by $W \subseteq A(W)$

be the Nichols algebra.

ex: if W is a vector space in char. 0 w/ symmetric braiding, $A(W) \cong \mathrm{Sym}(\mathrm{Lie}(W))$, and $B(W) \cong \mathrm{Sym}(W)$.

Filter $A(W)$ by powers of the augmentation ideal:

Thm: $A(W)^{gr} \cong B(W) \otimes^{\alpha} \mathcal{E}$, where $\mathcal{E}$ is generated in degrees ≥ 2. Hence $\exists$ spectral sequences

$$\mathrm{Ext}_{B(W)}(k,k) \otimes \mathrm{Ext}_{\mathcal{E}}(k,k) \underset{CESS}{\Rightarrow} \mathrm{Ext}_{A(W)^{gr}}(k,k) \underset{May}{\Rightarrow} \mathrm{Ext}_{A(W)}(k,k)$$

Further, upon identifying $\mathrm{Ext}^{n-j,n}_{A(V_\varepsilon)}(k,k) \cong H_j(B_n, V^{\otimes n})$, $\mathrm{Ext}_{\mathcal{E}}(k,k)$ contributes to classes with $j \geq n/2 - 1$.

slogan: $\mathrm{Ext}_{B(V_\varepsilon)}(k,k)$ should contribute the "dominant terms" in the cohomology.

The Nichols algebra's contribution to the point count

Thm (Kapranov–Schechtman) There is an isomorphism

$$\mathrm{Ext}^{n-j,n}_{B(V_\epsilon)}(k,k) \cong H_j(\mathrm{Sym}^n \mathbb{A}^1, \iota_!(V^{\otimes n}))$$

where $\iota_!$ denotes perverse pushforward of sheaves along $\iota : \mathrm{Conf}_n \mathbb{A}^1 \hookrightarrow \mathrm{Sym}^n \mathbb{A}^1$.

Corollary: $\mathrm{Ext}_{B(V_\epsilon)}(\mathbb{Q}_\ell, \mathbb{Q}_\ell)$ supports a rep. of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ consistent w/ the map

$$\begin{array}{ccc} \mathrm{Ext}_{A(V_\epsilon)}(\mathbb{Q}_\ell, \mathbb{Q}_\ell) & \to & \mathrm{Ext}_{B(V_\epsilon)}(\mathbb{Q}_\ell, \mathbb{Q}_\ell), \\ \wr\wr & & \wr\wr \\ H_j(\mathrm{Conf}_n, V^{\otimes n}) & & H_j(\mathrm{Sym}_n, \iota_! V^{\otimes n}) \end{array}$$

Thm (BMTW)

$$\# \mathrm{Hur}^c_{G,n}(\mathbb{F}_q) = \left(\Lambda(\mathrm{Frob}_q \mid \mathrm{Ext}^{n-*,n}_{B(V_\epsilon)}(k,k)) \right) (1 + O(q^{3n/4}))$$

Computing $\mathrm{Ext}_{B(W)}(k,k)$

Notice that $\exists$ surj $T(W) \to B(W) = T(W)/I$; the <u>quadratic cover</u> of $B(W)$ is $\widehat{B(W)} := T(W)/(I_2)$.

The <u>quadratic dual</u> is $R := \widehat{B(W)}^! : T(W^*)/(I_2^\perp)$

Degree zero:

There is an iso

$$H_0(B_n, V^{\otimes n}) = \mathrm{Ext}^{n,n}_{A(V_\epsilon)}(k,k) \cong \mathrm{Ext}^{n,n}_{B(V_\epsilon)} = \left(\widehat{B(V_\epsilon)}^{!}\right)_n = R_n$$

Action of Frobenius is well under-stood on R_n:

$$\mathrm{tr}(\mathrm{Frob}_q \mid R_n) = \# \pi_0(\mathrm{Hur}^c_{G,n}(\mathbb{F}_q))\, q^n$$

Case study: $G = S_3$, $c =$ transpositions

Thm (Ştefan-Vay) If $V = k[c]$, char $k = 0$, $\exists$ an iso

$$\mathrm{Ext}_{B(V_\epsilon)}(k,k) \cong R[X]\ ; \quad X \in \mathrm{Ext}^{4,6} \rightsquigarrow H_2(\mathrm{Sym}^6 \mathbb{A}^1, \mathcal{L}, V^{\otimes 6})$$

$$\cong H^{10}_c(\mathrm{Sym}^6 \mathbb{A}^1, V^{\otimes 6})$$

Conj. (Purity) $\mathrm{Frob}_q(X) = q^5 X$

Conj: (Multiplicativity): The E_2-alg. structure on $\mathrm{Ext}_{B(V_\epsilon)}(k,k)^G$ is Galois invariant

If true, then

$$\Lambda\left(\mathrm{Ext}^{n-*,n}_{B(V_\epsilon)}(k,k)\right) = \sum_{k=0}^{[n/6]} \# \pi_0(\mathrm{Hur}^c_{G,n-6k})\, q^{5k} \approx \frac{q}{q-1}\left(q^n - q^{(5/6)n}\right)$$

What about other groups?

Let K^* be the dual Koszul complex associated to $B(V_\epsilon)$:

$$K^* := (B(V_\epsilon)^* \otimes R, d) \quad ; \quad d = \sum_{i=1}^{n} \partial_{v_i} \otimes v_i^*$$

where $\{v_i\}$ is a basis of V_ϵ.

Thm (ETW)

$\mathrm{Ext}_{B(V_\epsilon)}(k,k)$ is a subquotient of $R \otimes T^{co}(H_* K^*)$

Upshot: · $H_* K^*$ controls the growth rate of $\mathrm{Ext}_{B(V_\epsilon)}(k,k)$

· If conjectures hold, it controls point count up to $X^{3/4}$

ex: $(G,c) = (S_3, \text{transpositions})$: $H_* K^* = k\{X\}$ in bidegree $(4,6) \rightsquigarrow X^{5/6}$ term

ex: (Berglund): $(G,c) = (S_4, \text{transpositions}) \rightarrow H_* K^*$ has classes in bidegrees $(4,6)$ and $(4,8)$. Should yield an $X^{5/6}$ term and an $X^{3/4}$ term.

ex: (Michel): $(G,c) = (D_{2p}, \text{reflections}) \rightsquigarrow H_* K^* = k\{X\}$ in bidegree $(4, 2p) \rightsquigarrow X^{1/2 + 1/p}$ term.