AMS Special session Spring

The high-degree cohomology of the special linear group.

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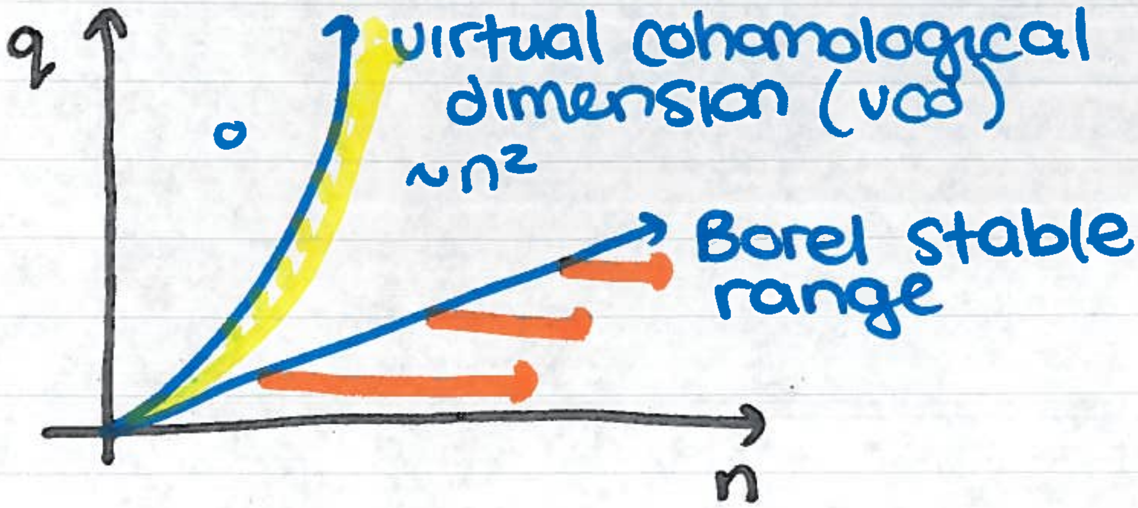
Notation K - number field
$\mathcal{O}_K$ - its ring of integer

Eg $\mathcal{O}_K = \mathbb{Z}, \mathbb{Z}[i], \mathbb{Z}[\frac{1}{2}(1+$
....

Goal: Understand $H^q(SL_n$

Borel: For $q << n$,
$SL_n \mathcal{O}_K \longrightarrow SL_{n+1} \mathcal{O}_K$
$A \longmapsto [A \ 0]$

induce isos on cohomology.



<u>Our goal</u> Understand $H^q(SL_n \mathcal{O}_K ; \mathbb{Q})$ for $q >> n$.

Virtual Bieri-Eckmann Duality.

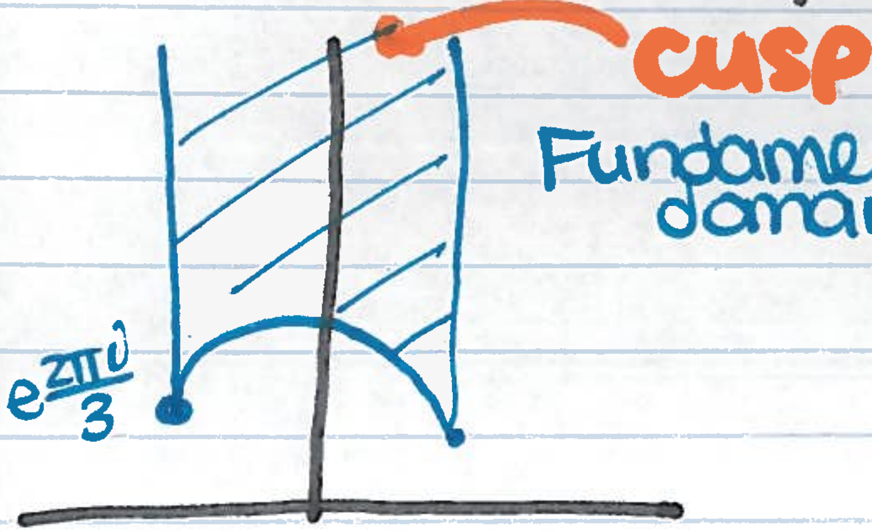
Embed
$SL_n \mathcal{O}_K \hookrightarrow G$
$G = (SL_n \mathbb{R})^{r_1} \times (SL_n \mathbb{C})^{r_2}$
as a lattice

Key: Study action on
$X := G / \text{max compact}$ } symmetric space

contractible!

Warm-Up :

$SL_2\mathbb{Z} \circlearrowright SL_2\mathbb{R}/SO(2) \cong \mathbb{H}$



First disappointment:
this action is <u>not</u> free.

<u>but</u> has finite stabilizers.

- suffices to compute rational cohomology.

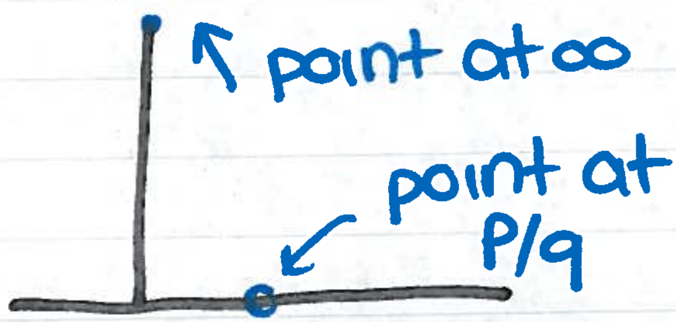
$H^q(SL_2\mathbb{Z};\mathbb{Q})$

$= H^q(SL_2\mathbb{Z}\backslash SL_2\mathbb{R}/SO(2)$

Second disappointment:
Action is not cocompact.

but we can compactify the quotient.

$\rightsquigarrow$ Add a point for each line $\mathbb{Z}\left[\begin{smallmatrix}p\\q\end{smallmatrix}\right]$ in $\mathbb{Z}^2$.



Borel-Serre bordification of X
- htpy equivalent to X
- quotient by $SL_n\mathcal{O}_K$ is cpt.
- boundary is the Bruhat-Tits bldg $T_n(K)$.

Thm (Solomon-Tits)
$T_n(K) \simeq \bigvee S^{n-2}$

By a formal argument

$$H^i(SL_n \mathcal{O}_K; \mathbb{Q})$$
$$\cong H_{vcd-i}(SL_n \mathcal{O}_K; \tilde{H}_{n-2}(T_n) \otimes \mathbb{Q})$$

<u>upshot</u>: $SL_n \mathcal{O}_K$ is a (rational) virtual duality gp with (rational) dualizing module

$$St_n(K) := \tilde{H}_{n-2}(T_n(K))$$

called the <u>Steinberg module.</u>

<u>Eg</u> $H^{vcd}(SL_n \mathcal{O}_K; \mathbb{Q})$
$$= (St_n(K) \otimes \mathbb{Q})_{SL_n \mathcal{O}_K}$$

<u>Goal becomes</u>:

- understand generators / relations for $St_n(K)$
- compute a (partial) resolution of $St_n(K)$ by $SL_n \mathcal{O}_K$-modules with understandable action.

- understand topology of $\Pi_n(k)$.

$\Pi_n(k)$ = geometric realization of the poset of nonzero proper subspaces of k^n under inclusion

ie, vertices $\longleftrightarrow$ subspaces V_i

edges $\longleftrightarrow$ nested subspaces $V_i \subseteq V_{i+1}$

p-simplex $\longleftrightarrow$ chain $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_p$

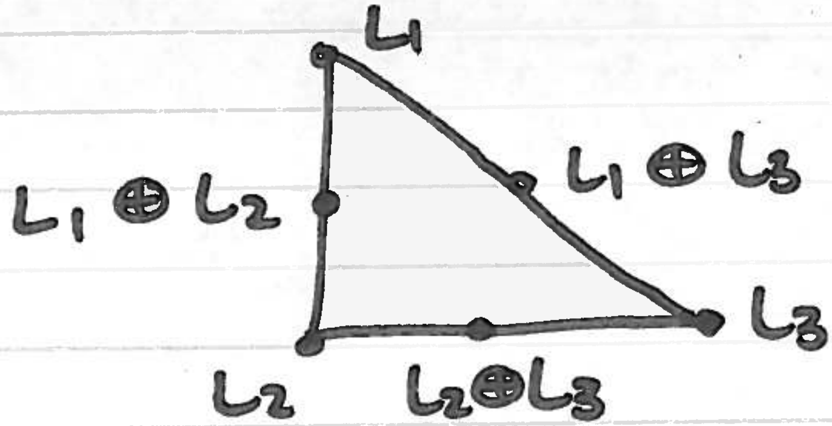
<u>Generaters for $St_n(k)$.</u>

<u>Thm</u> (Solomon-Tits) $St_n(k)$ is generated by <u>apartment classes</u>

indexed by

$$\left\{ (L_1, \dots, L_n) \;\middle|\; \begin{array}{l} L_i \text{ line in } K^n \\ K^n = L_1 \oplus \cdots \oplus L_n \end{array} \right\}$$

<u>Eg</u> $\{L_1, L_2, L_3\} \longleftrightarrow$



<u>Def^n</u> An apartment class $\{L_1, \dots, L_n\}$ is <u>integral</u> if

$$(L_1 \cap \mathcal{O}_K^n) \oplus \cdots \oplus (L_n \cap \mathcal{O}_K^n) = \mathcal{O}_K^n$$

<u>Eg</u> $\mathcal{O}_K = \mathbb{Z}$

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ integral

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ <u>not</u> integral

$\mathbb{Z}$-span is index 2 in $\mathbb{Z}^2$

<u>Thm</u> (Ash-Rudolph)
$\mathcal{O}_K$ Euclidean domain

Then $St_n(K)$ is generated by integral apartments.

<u>Eg</u> $\mathbb{Z}$, $\mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{-2}]$

Consequence:

$$H^{vcd}(SL_n \mathcal{O}_K; \mathbb{Q}) = 0$$

Why?

Given $\{L_1, \dots, L_n\}$ integral
$\exists g \in SL_n \mathcal{O}_K$ with
$g(L_1) = L_2$, $g(L_2) = L_1$
$g(L_i) = L_i$. otherwise

$\Rightarrow$ g negates apartment class

<u>Q</u> When is $St_n(K)$ gen. by integral apt. classes?

Thm (Church-Farb-Putman)
$\mathcal{O}_K$ is **not** a PID

$St_n(K)$ is **not** generated by integral apartment classes.

$H^{ucd}(SL_n \mathcal{O}_K; \mathbb{Q}) \neq 0$

Thm (MPaWY) $K = \mathbb{Q}(\sqrt{d})$
$d = -19, -43, -67, -163$
(PID's, not Euclidean)

then $St_n(K)$ is **not** gen. by integral apt classes.

$$\dim_{\mathbb{Q}} H^{ucd}(SL_{2n} \mathcal{O}_K; \mathbb{Q}) \geq \begin{cases} 1, & d=-43 \\ 2^n, & d=-67 \\ 6^n, & d=-163 \end{cases}$$

(Weinberger) Assuming GRH, these **all** $\mathcal{O}_K$ that are PID's, but not Euclidean.

Relations ← understanding $H^{vcd-1}(SL_n \mathcal{O}_K; \mathbb{Q})$

(Church-Putman) $\mathcal{O}_K = \mathbb{Z}$, $n \geq 3$

- compute a flat (partial) resolution of $St_n(\mathbb{Q}) \otimes \mathbb{Q}$ by $\mathbb{Q}[SL_n \mathbb{Z}]$-modules

$$I_1 \to I_0 \to St_n(\mathbb{Q})^{\mathbb{Q}} \to 0$$

Thm (CP)

$$H^{vcd-1}(SL_n \mathbb{Z}; \mathbb{Q}) = 0$$

$\forall n \geq 3$.

Work in progress (K M Pa W)

Do the same for $\mathcal{O}_K$ the Gaussian or Eisenstein integers.

<u>But</u> the same presentation fails (eg) for $\mathcal{O}_K = \mathbb{Z}[\sqrt{7}]$.

<u>Higher syzygies</u>

Conjecture (Church-Farb-Putman)

$H^{vcd-i}(SL_n\mathbb{Z};\mathbb{Q}) = 0$

$\forall n \geq i+2$.

(work-in-progress)

(KMNPaPuW)

- compute 3-step res. of $St_n(\mathbb{Q}) \otimes \mathbb{Q}$

$\rightsquigarrow H^{vcd-2}(SL_n\mathbb{Z};\mathbb{Q}) = 0$

$\forall n \geq 4$.