Twist coefficients for the periodic orbit of period 2 of the billiard map

1. Derivatives of the billiard map $(s, u) \to (s_1, u_1) \to (s_2, u_2)$

Functions for the map from point A to point B

```
Clear[rad0, rad1, tau, s1, u1];
(*set up the radius function*)
rad0[0, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] := rad0[0, t, ra, ra2, ra4, rb, rb2, rb4] = ra;
Derivative[1, 0, 0, 0, 0, 0, 0, 0][rad0][0, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] := 0;
Derivative[2, 0, 0, 0, 0, 0, 0, 0][rad0][0, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] := ra2;
Derivative[3, 0, 0, 0, 0, 0, 0, 0][rad0][0, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] := 0;
Derivative[4, 0, 0, 0, 0, 0, 0, 0][rad0][0, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] := ra4;
rad1[0, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] := rad1[0, t, ra, ra2, ra4, rb, rb2, rb4] = rb;
Derivative[1, 0, 0, 0, 0, 0, 0, 0][rad1][0, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] := 0;
Derivative[2, 0, 0, 0, 0, 0, 0, 0][rad1][0, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] := rb2;
Derivative[3, 0, 0, 0, 0, 0, 0, 0][rad1][0, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] := 0;
Derivative[4, 0, 0, 0, 0, 0, 0, 0][rad1][0, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] := rb4;
(*set up the orbit length function and the billiard map*)
tau[0, 0, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] := tau[0, 0, t, ra, ra2, ra4, rb, rb2, rb4] = t;
u1[0, 0, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] := u1[0, 0, t, ra, ra2, ra4, rb, rb2, rb4] = 0;
s1[0, 0, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] := s1[0, 0, t, ra, ra2, ra4, rb, rb2, rb4] = 0;
Derivative[1, 0, 0, 0, 0, 0, 0, 0, 0][s1][s_, u_, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] :=
  tau[s, u, t, ra, ra2, ra4, rb, rb2, rb4] / (rad0[s, t, ra, ra2, ra4, rb, rb2, rb4] *
      Sqrt[1 - u1[s, u, t, ra, ra2, ra4, rb, rb2, rb4]^2]) -
   Sqrt[1 - u^2] / (Sqrt[1 - u1[s, u, t, ra, ra2, ra4, rb, rb2, rb4]^2] * rad0[s, t, ra, ra2, ra4, rb, rb2, rb4]);
Derivative[0, 1, 0, 0, 0, 0, 0, 0, 0][s1][s_, u_, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] :=
  tau[s, u, t, ra, ra2, ra4, rb, rb2, rb4] / (Sqrt[1 - u^2] * Sqrt[1 - u1[s, u, t, ra, ra2, ra4, rb, rb2, rb4]^2]);
Derivative[1, 0, 0, 0, 0, 0, 0, 0, 0][u1][s_, u_, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] :=
  tau[s, u, t, ra, ra2, ra4, rb, rb2, rb4] / (rad0[s, t, ra, ra2, ra4, rb, rb2, rb4] *
      rad1[s1[s, u, t, ra, ra2, ra4, rb, rb2, rb4], t, ra, ra2, ra4, rb, rb2, rb4]) -
   Sqrt[1 - u^2] / rad1[s1[s, u, t, ra, ra2, ra4, rb, rb2, rb4], t, ra, ra2, ra4, rb, rb2, rb4] -
   Sqrt[1 - u1[s, u, t, ra, ra2, ra4, rb, rb2, rb4]^2] / rad0[s, t, ra, ra2, ra4, rb, rb2, rb4];
Derivative[0, 1, 0, 0, 0, 0, 0, 0, 0][u1][s_, u_, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] :=
  tau[s, u, t, ra, ra2, ra4, rb, rb2, rb4] /
    (rad1[s1[s, u, t, ra, ra2, ra4, rb, rb2, rb4], t, ra, ra2, ra4, rb, rb2, rb4] *
      Sqrt[1 - u^2]) - Sqrt[1 - u1[s, u, t, ra, ra2, ra4, rb, rb2, rb4]^2] / Sqrt[1 - u^2];
Derivative[1, 0, 0, 0, 0, 0, 0, 0, 0][tau][s_, u_, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] :=
  u - u1[s, u, t, ra, ra2, ra4, rb, rb2, rb4] *
    (tau[s, u, t, ra, ra2, ra4, rb, rb2, rb4] / (rad0[s, t, ra, ra2, ra4, rb, rb2, rb4] *
        Sqrt[1 - u1[s, u, t, ra, ra2, ra4, rb, rb2, rb4]^2]) -
     Sqrt[1 - u^2] / (Sqrt[1 - u1[s, u, t, ra, ra2, ra4, rb, rb2, rb4]^2] * rad0[s, t, ra, ra2, ra4, rb, rb2, rb4]));
Derivative[0, 1, 0, 0, 0, 0, 0, 0, 0][tau][s_, u_, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] :=
  -u1[s, u, t, ra, ra2, ra4, rb, rb2, rb4] *
   tau[s, u, t, ra, ra2, ra4, rb, rb2, rb4] / (Sqrt[1 - u^2] * Sqrt[1 - u1[s, u, t, ra, ra2, ra4, rb, rb2, rb4]^2]);
```

Functions for the map from point B to point A

In[ ]:=
```
Clear[tau1, s2, u2];
tau1[0, 0, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] :=
  tau1[0, 0, t, ra, ra2, ra4, rb, rb2, rb4] = t;
u2[0, 0, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] := u2[0, 0, t, ra, ra2, ra4, rb, rb2, rb4] = 0;
s2[0, 0, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] := s2[0, 0, t, ra, ra2, ra4, rb, rb2, rb4] = 0;
Derivative[1, 0, 0, 0, 0, 0, 0, 0, 0][s2][s_, u_, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] :=
  tau1[s, u, t, ra, ra2, ra4, rb, rb2, rb4] / (rad1[s, t, ra, ra2, ra4, rb, rb2, rb4] *
      Sqrt[1 - u2[s, u, t, ra, ra2, ra4, rb, rb2, rb4]^2]) -
   Sqrt[1 - u^2] / Sqrt[1 - u2[s, u, t, ra, ra2, ra4, rb, rb2, rb4]^2];
Derivative[0, 1, 0, 0, 0, 0, 0, 0, 0][s2][s_, u_, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] :=
  tau1[s, u, t, ra, ra2, ra4, rb, rb2, rb4] / (Sqrt[1 - u^2] * Sqrt[1 - u2[s, u, t, ra, ra2, ra4, rb, rb2, rb4]^2]);
Derivative[1, 0, 0, 0, 0, 0, 0, 0, 0][u2][s_, u_, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] :=
  tau1[s, u, t, ra, ra2, ra4, rb, rb2, rb4] / (rad1[s, t, ra, ra2, ra4, rb, rb2, rb4] *
      rad0[s2[s, u, t, ra, ra2, ra4, rb, rb2, rb4], t, ra, ra2, ra4, rb, rb2, rb4]) -
   Sqrt[1 - u^2] / rad0[s2[s, u, t, ra, ra2, ra4, rb, rb2, rb4], t, ra, ra2, ra4, rb, rb2, rb4] -
   Sqrt[1 - u2[s, u, t, ra, ra2, ra4, rb, rb2, rb4]^2] / rad1[s, t, ra, ra2, ra4, rb, rb2, rb4];
Derivative[0, 1, 0, 0, 0, 0, 0, 0, 0][u2][s_, u_, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] :=
  tau1[s, u, t, ra, ra2, ra4, rb, rb2, rb4] /
    (rad0[s2[s, u, t, ra, ra2, ra4, rb, rb2, rb4], t, ra, ra2, ra4, rb, rb2, rb4] *
      Sqrt[1 - u^2]) - Sqrt[1 - u2[s, u, t, ra, ra2, ra4, rb, rb2, rb4]^2] / Sqrt[1 - u^2];
Derivative[1, 0, 0, 0, 0, 0, 0, 0, 0][tau1][s_, u_, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] :=
  u - u2[s, u, t, ra, ra2, ra4, rb, rb2, rb4] *
    (tau1[s, u, t, ra, ra2, ra4, rb, rb2, rb4] / (rad1[s, t, ra, ra2, ra4, rb, rb2, rb4] *
        Sqrt[1 - u2[s, u, t, ra, ra2, ra4, rb, rb2, rb4]^2]) -
     Sqrt[1 - u^2] / Sqrt[1 - u2[s, u, t, ra, ra2, ra4, rb, rb2, rb4]^2]);
Derivative[0, 1, 0, 0, 0, 0, 0, 0, 0][tau1][s_, u_, t_, ra_, ra2_, ra4_, rb_, rb2_, rb4_] :=
  -u2[s, u, t, ra, ra2, ra4, rb, rb2, rb4] *
   tau1[s, u, t, ra, ra2, ra4, rb, rb2, rb4] / (Sqrt[1 - u^2] * Sqrt[1 - u2[s, u, t, ra, ra2, ra4, rb, rb2, rb4]^2]);
```

Evaluate along the orbit from A to B and then back to A

In[ ]:=
```
(*v={t,ra,ra2,ra4,rb,rb2,rb4}*)
Clear[t, ra, ra2, ra4, rb, rb2, rb4];
t[v__] := t[v] = v[[1]];
ra[v__] := ra[v] = v[[2]];
ra2[v__] := ra2[v] = v[[3]];
ra4[v__] := ra4[v] = v[[4]];
rb[v__] := rb[v] = v[[5]];
rb2[v__] := rb2[v] = v[[6]];
rb4[v__] := rb4[v] = v[[7]];

Clear[scomp, ucomp, eta, lambda];
(*j-th derivative of s and k-th derivative of u at the periodic point O*)
scomp[j_, k_, v__] := scomp[j, k, v] =
    (D[s2[s1[svariable, uvariable, t[v], ra[v], ra2[v], ra4[v], rb[v], rb2[v], rb4[v]],
         u1[svariable, uvariable, t[v], ra[v], ra2[v], ra4[v], rb[v], rb2[v], rb4[v]],
         t[v], ra[v], ra2[v], ra4[v], rb[v], rb2[v], rb4[v]],
        {svariable, j}, {uvariable, k}] /. svariable → 0) /. uvariable → 0;
ucomp[j_, k_, v__] := ucomp[j, k, v] =
    (D[u2[s1[svariable, uvariable, t[v], ra[v], ra2[v], ra4[v], rb[v], rb2[v], rb4[v]],
         u1[svariable, uvariable, t[v], ra[v], ra2[v], ra4[v], rb[v], rb2[v], rb4[v]],
         t[v], ra[v], ra2[v], ra4[v], rb[v], rb2[v], rb4[v]],
        {svariable, j}, {uvariable, k}] /. svariable → 0) /. uvariable → 0;
eta[v__] := eta[v] = (-scomp[0, 1, v] / ucomp[1, 0, v]) ^ (1 / 4);
(*the preferred eigenvalue for the preferred eigenvector*)
lambda[v__] := lambda[v] = scomp[1, 0, v] + I * Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]];
(*only consider the case t<ra and t<rb*)
(*the general case: If[scomp[0,1,v]<0,scomp[1,0,v]+I*Sqrt[-scomp[0,1,v]*ucomp[1,0,v]],
   scomp[1,0,v]-I*Sqrt[-scomp[0,1,v]*ucomp[1,0,v]]]*)
```

2. Taylor expansion of the billiard map in the coordinate system (x,y)=(s/η, ηu)

Coefficients of $x_1 = \sum a_{jk}\, x^j\, y^k$, $y_1 = \sum b_{jk}\, x^j\, y^k$

```
Clear[a10, a01, b10, b01, a30, a21, a12, a03, b30, b21,
  b12, b03, a50, a41, a32, a23, a14, a05, b50, b41, b32, b23, b14, b05];
a10[v__] := a10[v] = scomp[1, 0, v];
a01[v__] := a01[v] = eta[v]^(-2) * scomp[0, 1, v];
a30[v__] := a30[v] = eta[v]^2 * scomp[3, 0, v] / 6;
a21[v__] := a21[v] = scomp[2, 1, v] / 2;
a12[v__] := a12[v] = eta[v]^(-2) * scomp[1, 2, v] / 2;
a03[v__] := a03[v] = eta[v]^(-4) * scomp[0, 3, v] / 6;
a50[v__] := a50[v] = eta[v]^4 * scomp[5, 0, v] / 5!;
a41[v__] := a41[v] = eta[v]^2 * scomp[4, 1, v] / 24;
a32[v__] := a32[v] = scomp[3, 2, v] / 12;
a23[v__] := a23[v] = eta[v]^(-2) * scomp[2, 3, v] / 12;
a14[v__] := a14[v] = eta[v]^(-4) * scomp[1, 4, v] / 24;
a05[v__] := a05[v] = eta[v]^(-6) * scomp[0, 5, v] / 5!;
b10[v__] := b10[v] = eta[v]^(2) * ucomp[1, 0, v];
b01[v__] := b01[v] = ucomp[0, 1, v];
b30[v__] := b30[v] = eta[v]^4 * ucomp[3, 0, v] / 6;
b21[v__] := b21[v] = eta[v]^2 * ucomp[2, 1, v] / 2;
b12[v__] := b12[v] = ucomp[1, 2, v] / 2;
b03[v__] := b03[v] = eta[v]^(-2) * ucomp[0, 3, v] / 6;
b50[v__] := b50[v] = eta[v]^6 * ucomp[5, 0, v] / 5!;
b41[v__] := b41[v] = eta[v]^4 * ucomp[4, 1, v] / 24;
b32[v__] := b32[v] = eta[v]^2 * ucomp[3, 2, v] / 12;
b23[v__] := b23[v] = ucomp[2, 3, v] / 12;
b14[v__] := b14[v] = eta[v]^(-2) * ucomp[1, 4, v] / 24;
b05[v__] := b05[v] = eta[v]^(-4) * ucomp[0, 5, v] / 5!;
```

3. Coefficients of the coordinate transform $(x, y) \to (X, Y) = \left(\sum p_{jk}\, x^j\, y^k, \sum q_{jk}\, x^j\, y^k\right)$

```
In[•]:= Clear[p30, p21, p12, p03, q30, q21, q12, q03,
  p50, p41, p32, p23, p14, p05, q50, q41, q32, q23, q14, q05];
p30[v__] := p30[v] = -(2 scomp[1, 0, v] Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]] (a30[v] +
        b21[v] - a12[v] - b03[v]) - (scomp[1, 0, v]^2 + scomp[0, 1, v] * ucomp[1, 0, v])
      (b30[v] - a21[v] - b12[v] + a03[v])) / (8 Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]]) +
   (Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]] (a30[v] - b21[v] - a12[v] + b03[v]) +
      scomp[1, 0, v] (b30[v] + a21[v] - b12[v] - a03[v])) /
    (32 scomp[1, 0, v] Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]]);
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p21[v__] := p21[v] = 3 (Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]] (b30[v] + a21[v] -
            b12[v] - a03[v]) - scomp[1, 0, v] (a30[v] - b21[v] - a12[v] + b03[v])) /
      (32 scomp[1, 0, v] Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]]);
p12[v__] := p12[v] = -3 (2 scomp[1, 0, v] Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]] (a30[v] +
             b21[v] - a12[v] - b03[v]) - (scomp[1, 0, v]^2 + scomp[0, 1, v] * ucomp[1, 0, v])
           (b30[v] - a21[v] - b12[v] + a03[v])) / (8 Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]]) -
     3 (Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]] (a30[v] - b21[v] - a12[v] + b03[v]) +
         scomp[1, 0, v] (b30[v] + a21[v] - b12[v] - a03[v])) /
       (32 scomp[1, 0, v] Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]]);
p03[v__] := p03[v] = (2 scomp[1, 0, v] Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]] (b30[v] -
             a21[v] - b12[v] + a03[v]) + (scomp[1, 0, v]^2 + scomp[0, 1, v] * ucomp[1, 0, v])
           (a30[v] + b21[v] - a12[v] - b03[v])) / (4 Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]]) -
     (Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]] (b30[v] + a21[v] - b12[v] - a03[v]) -
        scomp[1, 0, v] (a30[v] - b21[v] - a12[v] + b03[v])) /
      (32 scomp[1, 0, v] Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]]);
q30[v__] := q30[v] = (2 scomp[1, 0, v] Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]] (b30[v] -
             a21[v] - b12[v] + a03[v]) + (scomp[1, 0, v]^2 + scomp[0, 1, v] * ucomp[1, 0, v])
           (a30[v] + b21[v] - a12[v] - b03[v])) / (4 Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]]) +
     (Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]] (b30[v] + a21[v] - b12[v] - a03[v]) -
        scomp[1, 0, v] (a30[v] - b21[v] - a12[v] + b03[v])) /
      (32 scomp[1, 0, v] Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]]);
q21[v__] := q21[v] = 3 (2 scomp[1, 0, v] Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]] (a30[v] +
             b21[v] - a12[v] - b03[v]) - (scomp[1, 0, v]^2 + scomp[0, 1, v] * ucomp[1, 0, v])
           (b30[v] - a21[v] - b12[v] + a03[v])) / (8 Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]]) -
     3 (Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]] (a30[v] - b21[v] - a12[v] + b03[v]) +
         scomp[1, 0, v] (b30[v] + a21[v] - b12[v] - a03[v])) /
       (32 scomp[1, 0, v] Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]]);
q12[v__] := q12[v] = -3 (Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]] (b30[v] + a21[v] -
             b12[v] - a03[v]) - scomp[1, 0, v] (a30[v] - b21[v] - a12[v] + b03[v])) /
      (32 scomp[1, 0, v] Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]]);
q03[v__] := q03[v] = (2 scomp[1, 0, v] Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]] (a30[v] +
             b21[v] - a12[v] - b03[v]) - (scomp[1, 0, v]^2 + scomp[0, 1, v] * ucomp[1, 0, v])
           (b30[v] - a21[v] - b12[v] + a03[v])) / (8 Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]]) +
     (Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]] (a30[v] - b21[v] - a12[v] + b03[v]) +
        scomp[1, 0, v] (b30[v] + a21[v] - b12[v] - a03[v])) /
      (32 scomp[1, 0, v] Sqrt[-scomp[0, 1, v] * ucomp[1, 0, v]]);
p50[v__] := p50[v] = p21[v] * q30[v];
p41[v__] := p41[v] = p21[v] * q21[v] + 2 p12[v] * q30[v];
p32[v__] := p32[v] = p21[v] * q12[v] + 2 p12[v] * q21[v] + 3 p03[v] * q30[v];
p23[v__] := p23[v] = p21[v] * q03[v] + 2 p12[v] * q12[v] + 3 p03[v] * q21[v];
p14[v__] := p14[v] = 2 p12[v] * q03[v] + 3 p03[v] * q12[v];
p05[v__] := p05[v] = 3 p03[v] * q03[v]; q50[v__] := q50[v] = q21[v] * q30[v];
q41[v__] := q41[v] = q21[v] * q21[v] + 2 q12[v] * q30[v];
q32[v__] := q32[v] = q21[v] * q12[v] + 2 q12[v] * q21[v] + 3 q03[v] * q30[v];
q23[v__] := q23[v] = q21[v] * q03[v] + 2 q12[v] * q12[v] + 3 q03[v] * q21[v];
q14[v__] := q14[v] = 2 q12[v] * q03[v] + 3 q03[v] * q12[v];
q05[v__] := q05[v] = 3 q03[v] * q03[v];
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4. Taylor expansion of the billiard map F^2 in terms of new coordinate system (X,Y):

$$(X, Y) \to (X_1, Y_1) = \left(\sum A_{jk} X^j Y^k, \sum B_{jk} X^j Y^k\right)$$

```
In[•]:= Clear[aa30, aa21, aa12, aa03, bb30, bb21, bb12, bb03];
aa30[v__] := aa30[v] = a30[v] + b10[v]^3 p03[v] + a10[v] b10[v]^2 p12[v] +
    a10[v]^2 b10[v] * p21[v] + a10[v]^3 p30[v] - (a10[v] * p30[v] + a01[v] * q30[v]);
aa21[v__] := aa21[v] = a21[v] + b01[v]
    (3 b10[v]^2 p03[v] + 2 a10[v] * b10[v] * p12[v] + a10[v]^2 p21[v]) +
   a01[v] (b10[v]^2 p12[v] + 2 a10[v] * b10[v] * p21[v] + 3 a10[v]^2 p30[v]) -
   (a10[v] * p21[v] + a01[v] * q21[v]);
aa12[v__] := aa12[v] = a12[v] + b01[v]^2 (3 b10[v] * p03[v] + a10[v] * p12[v]) +
   2 a01[v] * b01[v] (b10[v] * p12[v] + a10[v] * p21[v]) +
   a01[v]^2 (b10[v] * p21[v] + 3 a10[v] * p30[v]) - (a10[v] * p12[v] + a01[v] * q12[v]);
aa03[v__] := aa03[v] = a03[v] + b01[v]^3 p03[v] + a01[v] b01[v]^2 p12[v] +
   a01[v]^2 b01[v] * p21[v] + a01[v]^3 p30[v] - (a10[v] * p03[v] + a01[v] * q03[v]);
bb30[v__] := bb30[v] = b30[v] + b10[v]^3 q03[v] + a10[v] b10[v]^2 q12[v] +
    a10[v]^2 b10[v] * q21[v] + a10[v]^3 q30[v] - (b10[v] * p30[v] + b01[v] * q30[v]);
bb21[v__] := bb21[v] = b21[v] + b01[v]
    (3 b10[v]^2 q03[v] + 2 a10[v] * b10[v] * q12[v] + a10[v]^2 q21[v]) +
   a01[v] (b10[v]^2 q12[v] + 2 a10[v] * b10[v] * q21[v] + 3 a10[v]^2 q30[v]) -
   (b10[v] * p21[v] + b01[v] * q21[v]);
bb12[v__] := bb12[v] = b12[v] + b01[v]^2 (3 b10[v] * q03[v] + a10[v] * q12[v]) +
   2 a01[v] * b01[v] (b10[v] * q12[v] + a10[v] * q21[v]) +
   a01[v]^2 (b10[v] * q21[v] + 3 a10[v] * q30[v]) - (b10[v] * p12[v] + b01[v] * q12[v]);
bb03[v__] := bb03[v] = b03[v] + b01[v]^3 q03[v] + a01[v] b01[v]^2 q12[v] +
   a01[v]^2 b01[v] * q21[v] + a01[v]^3 q30[v] - (b10[v] * p03[v] + b01[v] * q03[v]);
```

```
In[•]:= Clear[aa50, aa41, aa32, aa23, aa14, aa05];
aa50[v__] := aa50[v] = a50[v] + 3 b10[v]^2 b30[v] × p03[v] + b10[v]^5 p05[v] +
    (a30[v] b10[v]^2 + 2 a10[v] × b10[v] × b30[v]) p12[v] + a10[v] b10[v]^4 p14[v] +
    (2 a10[v] × a30[v] × b10[v] + a10[v]^2 b30[v]) p21[v] + a10[v]^2 b10[v]^3 p23[v] +
    3 a10[v]^2 a30[v] × p30[v] + a10[v]^3 b10[v]^2 p32[v] + a10[v]^4 b10[v] × p41[v] +
    a10[v]^5 p50[v] - (3 aa30[v] × p30[v] + a10[v] × p50[v] + aa21[v] × q30[v] + a01[v] × q50[v]);
aa41[v__] := aa41[v] = a41[v] + b10[v]^4 (5 b01[v] × p05[v] + a01[v] × p14[v]) +
    2 a10[v] b10[v]^3 (2 b01[v] × p14[v] + a01[v] × p23[v]) + b10[v]^2
     (3 b21[v] × p03[v] + a21[v] × p12[v] + 3 a10[v]^2 (b01[v] × p23[v] + a01[v] × p32[v])) +
    2 b10[v] (a10[v] × b21[v] × p12[v] + a01[v] × b30[v] × p12[v] + a10[v] × a21[v] × p21[v] +
       a01[v] × a30[v] × p21[v] + b01[v] (3 b30[v] × p03[v] + a30[v] × p12[v] + a10[v]^3 p32[v]) +
       2 a01[v] a10[v]^3 p41[v]) + a10[v] (a10[v] × b21[v] × p21[v] + 2 a01[v] × b30[v] × p21[v] +
       3 a10[v] × a21[v] × p30[v] + 6 a01[v] × a30[v] × p30[v] + b01[v]
        (2 b30[v] × p12[v] + 2 a30[v] × p21[v] + a10[v]^3 p41[v]) + 5 a01[v] a10[v]^3 p50[v]) -
    (3 aa30[v] × p21[v] + 2 aa21[v] × p30[v] + a10[v] × p41[v] + aa21[v] × q21[v] +
      2 aa12[v] × q30[v] + a01[v] × q41[v]);
```

```
aa32[v__] := aa32[v] = a32[v] + 3 b01[v]^2 b30[v] × p03[v] + a30[v] b01[v]^2 p12[v] +
    2 a10[v] × b01[v] × b21[v] × p12[v] + 2 a01[v] × b01[v] × b30[v] × p12[v] +
    2 a10[v] × a21[v] × b01[v] × p21[v] + 2 a01[v] × a30[v] × b01[v] × p21[v] +
    a10[v]^2 b12[v] × p21[v] + 2 a01[v] × a10[v] × b21[v] × p21[v] + a01[v]^2 b30[v] × p21[v] +
    b10[v]^3 (10 b01[v]^2 p05[v] + 4 a01[v] × b01[v] × p14[v] + a01[v]^2 p23[v]) +
    3 a10[v]^2 a12[v] × p30[v] + 6 a01[v] × a10[v] × a21[v] × p30[v] +
    3 a01[v]^2 a30[v] × p30[v] + a10[v]^3 b01[v]^2 p32[v] +
    b10[v]^2 (3 b12[v] × p03[v] + a12[v] × p12[v] + 6 a10[v] b01[v]^2 p14[v] +
       6 a01[v] × a10[v] × b01[v] × p23[v] + 3 a01[v]^2 a10[v] × p32[v]) +
    4 a01[v] a10[v]^3 b01[v] × p41[v] + b10[v] (3 a10[v]^2 b01[v]^2 p23[v] +
       2 b01[v] (3 b21[v] × p03[v] + a21[v] × p12[v] + 3 a01[v] a10[v]^2 p32[v]) +
       2 (a10[v] × b12[v] × p12[v] + a01[v] × b21[v] × p12[v] + a10[v] × a12[v] × p21[v] +
          a01[v] × a21[v] × p21[v] + 3 a01[v]^2 a10[v]^2 p41[v])) + 10 a01[v]^2 a10[v]^3 p50[v] -
    (3 aa30[v] × p12[v] + 2 aa21[v] × p21[v] + aa12[v] × p30[v] + a10[v] × p32[v] +
      aa21[v] × q12[v] + 2 aa12[v] × q21[v] + 3 aa03[v] × q30[v] + a01[v] × q32[v]);
aa23[v__] := aa23[v] = a23[v] + 6 b01[v] × b10[v] × b12[v] × p03[v] + 3 b01[v]^2 b21[v] × p03[v] +
    10 b01[v]^3 b10[v]^2 p05[v] + a21[v] b01[v]^2 p12[v] + 2 a12[v] × b01[v] × b10[v] × p12[v] +
    a03[v] b10[v]^2 p12[v] + 2 a10[v] × b01[v] × b12[v] × p12[v] +
    2 a01[v] × b10[v] × b12[v] × p12[v] + 2 a01[v] × b01[v] × b21[v] × p12[v] +
    4 a10[v] b01[v]^3 b10[v] × p14[v] + 6 a01[v] b01[v]^2 b10[v]^2 p14[v] +
    2 a10[v] × a12[v] × b01[v] × p21[v] + 2 a01[v] × a21[v] × b01[v] × p21[v] +
    2 a03[v] × a10[v] × b10[v] × p21[v] + 2 a01[v] × a12[v] × b10[v] × p21[v] +
    2 a01[v] × a10[v] × b12[v] × p21[v] + a01[v]^2 b21[v] × p21[v] +
    b03[v] (3 b10[v]^2 p03[v] + 2 a10[v] × b10[v] × p12[v] + a10[v]^2 p21[v]) +
    a10[v]^2 b01[v]^3 p23[v] + 6 a01[v] × a10[v] b01[v]^2 b10[v] × p23[v] +
    3 a01[v]^2 b01[v] b10[v]^2 p23[v] + 3 a03[v] a10[v]^2 p30[v] +
    6 a01[v] × a10[v] × a12[v] × p30[v] + 3 a01[v]^2 a21[v] × p30[v] +
    3 a01[v] a10[v]^2 b01[v]^2 p32[v] + 6 a01[v]^2 a10[v] × b01[v] × b10[v] × p32[v] +
    a01[v]^3 b10[v]^2 p32[v] + 6 a01[v]^2 a10[v]^2 b01[v] × p41[v] +
    4 a01[v]^3 a10[v] × b10[v] × p41[v] + 10 a01[v]^3 a10[v]^2 p50[v] -
    (3 aa30[v] × p03[v] + 2 aa21[v] × p12[v] + aa12[v] × p21[v] + a10[v] × p23[v] +
      aa21[v] × q03[v] + 2 aa12[v] × q12[v] + 3 aa03[v] × q21[v] + a01[v] × q23[v]);
aa14[v__] := aa14[v] = a14[v] + b01[v]^4 (5 b10[v] × p05[v] + a10[v] × p14[v]) +
    2 a01[v] b01[v]^3 (2 b10[v] × p14[v] + a10[v] × p23[v]) + b01[v]^2
     (3 b12[v] × p03[v] + a12[v] × p12[v] + 3 a01[v]^2 (b10[v] × p23[v] + a10[v] × p32[v])) +
    2 b01[v] (3 b03[v] × b10[v] × p03[v] + a10[v] × b03[v] × p12[v] +
       a03[v] × b10[v] × p12[v] + a01[v] × b12[v] × p12[v] + a03[v] × a10[v] × p21[v] +
       a01[v] × a12[v] × p21[v] + a01[v]^3 b10[v] × p32[v] + 2 a01[v]^3 a10[v] × p41[v]) +
    a01[v] (2 a03[v] × b10[v] × p21[v] + a01[v] × b12[v] × p21[v] +
       2 b03[v] (b10[v] × p12[v] + a10[v] × p21[v]) + 6 a03[v] × a10[v] × p30[v] +
       3 a01[v] × a12[v] × p30[v] + a01[v]^3 b10[v] × p41[v] + 5 a01[v]^3 a10[v] × p50[v]) -
    (2 aa21[v] × p03[v] + aa12[v] × p12[v] + a10[v] × p14[v] + 2 aa12[v] × q03[v] +
      3 aa03[v] × q12[v] + a01[v] × q14[v]);
aa05[v__] := aa05[v] = a05[v] + b01[v]^5 p05[v] + a01[v] b01[v]^4 p14[v] +
    a01[v]^2 b01[v]^3 p23[v] + b01[v]^2 (3 b03[v] × p03[v] + a03[v] × p12[v] + a01[v]^3 p32[v]) +
    a01[v] × b01[v] (2 b03[v] × p12[v] + 2 a03[v] × p21[v] + a01[v]^3 p41[v]) +
    a01[v]^2 (b03[v] × p21[v] + 3 a03[v] × p30[v] + a01[v]^3 p50[v]) -
    (aa12[v] × p03[v] + a10[v] × p05[v] + 3 aa03[v] × q03[v] + a01[v] × q05[v]);
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In[ ]:=
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Clear[bb50, bb41, bb32, bb23, bb14, bb05];
bb50[v__] := bb50[v] = b50[v] + 3 b10[v]^2 b30[v] × q03[v] + b10[v]^5 q05[v] +
    (a30[v] b10[v]^2 + 2 a10[v] × b10[v] × b30[v]) q12[v] + a10[v] b10[v]^4 q14[v] +
    (2 a10[v] × a30[v] × b10[v] + a10[v]^2 b30[v]) q21[v] + a10[v]^2 b10[v]^3 q23[v] +
    3 a10[v]^2 a30[v] × q30[v] + a10[v]^3 b10[v]^2 q32[v] + a10[v]^4 b10[v] × q41[v] +
    a10[v]^5 q50[v] - (3 bb30[v] × p30[v] + b10[v] × p50[v] + bb21[v] × q30[v] + b01[v] × q50[v]);
bb41[v__] := bb41[v] = b41[v] + b10[v]^4 (5 b01[v] × q05[v] + a01[v] × q14[v]) +
    2 a10[v] b10[v]^3 (2 b01[v] × q14[v] + a01[v] × q23[v]) + b10[v]^2
     (3 b21[v] × q03[v] + a21[v] × q12[v] + 3 a10[v]^2 (b01[v] × q23[v] + a01[v] × q32[v])) +
    2 b10[v] (a10[v] × b21[v] × q12[v] + a01[v] × b30[v] × q12[v] + a10[v] × a21[v] × q21[v] +
       a01[v] × a30[v] × q21[v] + b01[v] (3 b30[v] × q03[v] + a30[v] × q12[v] + a10[v]^3 q32[v]) +
       2 a01[v] a10[v]^3 q41[v]) + a10[v] (a10[v] × b21[v] × q21[v] + 2 a01[v] × b30[v] × q21[v] +
       3 a10[v] × a21[v] × q30[v] + 6 a01[v] × a30[v] × q30[v] + b01[v]
        (2 b30[v] × q12[v] + 2 a30[v] × q21[v] + a10[v]^3 q41[v]) + 5 a01[v] a10[v]^3 q50[v]) -
    (3 bb30[v] × p21[v] + 2 bb21[v] × p30[v] + b10[v] × p41[v] + bb21[v] × q21[v] +
      2 bb12[v] × q30[v] + b01[v] × q41[v]);
bb32[v__] := bb32[v] = b32[v] + 3 b01[v]^2 b30[v] × q03[v] + a30[v] b01[v]^2 q12[v] +
    2 a10[v] × b01[v] × b21[v] × q12[v] + 2 a01[v] × b01[v] × b30[v] × q12[v] +
    2 a10[v] × a21[v] × b01[v] × q21[v] + 2 a01[v] × a30[v] × b01[v] × q21[v] +
    a10[v]^2 b12[v] × q21[v] + 2 a01[v] × a10[v] × b21[v] × q21[v] + a01[v]^2 b30[v] × q21[v] +
    b10[v]^3 (10 b01[v]^2 q05[v] + 4 a01[v] × b01[v] × q14[v] + a01[v]^2 q23[v]) +
    3 a10[v]^2 a12[v] × q30[v] + 6 a01[v] × a10[v] × a21[v] × q30[v] +
    3 a01[v]^2 a30[v] × q30[v] + a10[v]^3 b01[v]^2 q32[v] +
    b10[v]^2 (3 b12[v] × q03[v] + a12[v] × q12[v] + 6 a10[v] b01[v]^2 q14[v] +
       6 a01[v] × a10[v] × b01[v] × q23[v] + 3 a01[v]^2 a10[v] × q32[v]) +
    4 a01[v] a10[v]^3 b01[v] × q41[v] + b10[v] (3 a10[v]^2 b01[v]^2 q23[v] +
       2 b01[v] (3 b21[v] × q03[v] + a21[v] × q12[v] + 3 a01[v] a10[v]^2 q32[v]) +
       2 (a10[v] × b12[v] × q12[v] + a01[v] × b21[v] × q12[v] + a10[v] × a12[v] × q21[v] +
          a01[v] × a21[v] × q21[v] + 3 a01[v]^2 a10[v]^2 q41[v])) + 10 a01[v]^2 a10[v]^3 q50[v] -
    (3 bb30[v] × p12[v] + 2 bb21[v] × p21[v] + bb12[v] × p30[v] + b10[v] × p32[v] +
      bb21[v] × q12[v] + 2 bb12[v] × q21[v] + 3 bb03[v] × q30[v] + b01[v] × q32[v]);
bb23[v__] := bb23[v] = b23[v] + 6 b01[v] × b10[v] × b12[v] × q03[v] + 3 b01[v]^2 b21[v] × q03[v] +
    10 b01[v]^3 b10[v]^2 q05[v] + a21[v] b01[v]^2 q12[v] + 2 a12[v] × b01[v] × b10[v] × q12[v] +
    a03[v] b10[v]^2 q12[v] + 2 a10[v] × b01[v] × b12[v] × q12[v] +
    2 a01[v] × b10[v] × b12[v] × q12[v] + 2 a01[v] × b01[v] × b21[v] × q12[v] +
    4 a10[v] b01[v]^3 b10[v] × q14[v] + 6 a01[v] b01[v]^2 b10[v]^2 q14[v] +
    2 a10[v] × a12[v] × b01[v] × q21[v] + 2 a01[v] × a21[v] × b01[v] × q21[v] +
    2 a03[v] × a10[v] × b10[v] × q21[v] + 2 a01[v] × a12[v] × b10[v] × q21[v] +
    2 a01[v] × a10[v] × b12[v] × q21[v] + a01[v]^2 b21[v] × q21[v] +
    b03[v] (3 b10[v]^2 q03[v] + 2 a10[v] × b10[v] × q12[v] + a10[v]^2 q21[v]) +
    a10[v]^2 b01[v]^3 q23[v] + 6 a01[v] × a10[v] b01[v]^2 b10[v] × q23[v] +
    3 a01[v]^2 b01[v] b10[v]^2 q23[v] + 3 a03[v] a10[v]^2 q30[v] +
    6 a01[v] × a10[v] × a12[v] × q30[v] + 3 a01[v]^2 a21[v] × q30[v] +
    3 a01[v] a10[v]^2 b01[v]^2 q32[v] + 6 a01[v]^2 a10[v] × b01[v] × b10[v] × q32[v] +
    a01[v]^3 b10[v]^2 q32[v] + 6 a01[v]^2 a10[v]^2 b01[v] × q41[v] +
    4 a01[v]^3 a10[v] × b10[v] × q41[v] + 10 a01[v]^3 a10[v]^2 q50[v] -
    (3 bb30[v] × p03[v] + 2 bb21[v] × p12[v] + bb12[v] × p21[v] + b10[v] × p23[v] +
      bb21[v] × q03[v] + 2 bb12[v] × q12[v] + 3 bb03[v] × q21[v] + b01[v] × q23[v]);
bb14[v__] := bb14[v] = b14[v] + b01[v]^4 (5 b10[v] × q05[v] + a10[v] × q14[v]) +
```

```
    2 a01[v] b01[v]^3 (2 b10[v] × q14[v] + a10[v] × q23[v]) + b01[v]^2
     (3 b12[v] × q03[v] + a12[v] × q12[v] + 3 a01[v]^2 (b10[v] × q23[v] + a10[v] × q32[v])) +
    2 b01[v] (3 b03[v] × b10[v] × q03[v] + a10[v] × b03[v] × q12[v] +
       a03[v] × b10[v] × q12[v] + a01[v] × b12[v] × q12[v] + a03[v] × a10[v] × q21[v] +
       a01[v] × a12[v] × q21[v] + a01[v]^3 b10[v] × q32[v] + 2 a01[v]^3 a10[v] × q41[v]) +
    a01[v] (2 a03[v] × b10[v] × q21[v] + a01[v] × b12[v] × q21[v] +
       2 b03[v] (b10[v] × q12[v] + a10[v] × q21[v]) + 6 a03[v] × a10[v] × q30[v] +
       3 a01[v] × a12[v] × q30[v] + a01[v]^3 b10[v] × q41[v] + 5 a01[v]^3 a10[v] × q50[v]) -
    (2 bb21[v] × p03[v] + bb12[v] × p12[v] + b10[v] × p14[v] + 2 bb12[v] × q03[v] +
      3 bb03[v] × q12[v] + b01[v] × q14[v]);
bb05[v__] := bb05[v] = b05[v] + 3 b01[v]^2 b03[v] × q03[v] + b01[v]^5 q05[v] +
    (a03[v] b01[v]^2 + 2 a01[v] × b01[v] × b03[v]) q12[v] + a01[v] b01[v]^4 q14[v] +
    (2 a01[v] × a03[v] × b01[v] + a01[v]^2 b03[v]) q21[v] + a01[v]^2 b01[v]^3 q23[v] +
    3 a01[v]^2 a03[v] × q30[v] + a01[v]^3 b01[v]^2 q32[v] + a01[v]^4 b01[v] × q41[v] +
    a01[v]^5 q50[v] - (bb12[v] × p03[v] + b10[v] × p05[v] + 3 bb03[v] × q03[v] + b01[v] × q05[v]);
```

5. Twist Coefficients

```
(*to find τ1 and τ2, use the input of the form v={L,ra,ra2,ra4,rb,rb2,rb4},
where L is the the length, ra and rb are the radii of the two arcs at s=0, ,
ra2 and rb2 are the second-derivatives, and ra4 and rb4 are the four-derivatives*)

Clear[twist1, twist2];
twist1[v__] := twist1[v] = (a10[v] (3 b30[v] - a21[v] + b12[v] - 3 a03[v]) -
      Sqrt[-a01[v] * b10[v]] (3 a30[v] + b21[v] + a12[v] + 3 b03[v])) / 8;
twist2[v__] := twist2[v] = (a10[v] (-2 aa41[v] - 2 aa23[v] - 10 aa05[v] +
         10 bb50[v] + 2 bb32[v] + 2 bb14[v]) - Sqrt[-a01[v] * b10[v]]
       (10 aa50[v] + 2 aa32[v] + 2 aa14[v] + 2 bb41[v] + 2 bb23[v] + 10 bb05[v])) / 32;
```