1. Using the results from twistAsymmetry to obtain the explicit expressions of the coefficients

$$x_1 = \sum a_{jk}\, x^j\, y^k,\ \ y_1 = \sum b_{jk}\, x^j\, y^k$$

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In[ ]:= Clear[va10, va01, vb10, vb01, va30, va21, va12, va03, vb30, vb21, vb12,
    vb03, va50, va41, va32, va23, va14, va05, vb50, vb41, vb32, vb23, vb14, vb05];
  (*Taylor expansion in terms of (x,y)=(s/η, ηu)*)
  va10[L_, R_, S_] := va10[L, R, S] = R^-1 S^-1 ((R - L) (S - L) - L (R + S - L));
  va01[L_, R_, S_] := va01[L, R, S] = -2 R^-1 S^-1 L^1/2 (R - L)^1/2 (S - L)^1/2 (R + S - L)^1/2;
  vb10[L_, R_, S_] := vb10[L, R, S] = 2 R^-1 S^-1 L^1/2 (R - L)^1/2 (S - L)^1/2 (R + S - L)^1/2;
  vb01[L_, R_, S_] := vb01[L, R, S] = R^-1 S^-1 ((R - L) (S - L) - L (R + S - L));
  (*third order derivatives *)
  va30[L_, R_, R2_, S_, S2_] :=
   va30[L, R, R2, S, S2] = R * L^1/2 (S - L)^1/2 (R - L)^-1/2 (R + S - L)^-1/2 (1/(R^5 S^3) 2
       (4 L^6 + 3 R^3 S (R + S)^2 - 12 L^5 (2 R + S) + L^2 R (68 R S^2 + 12 S^3 + R^2 S (99 - R2 S) + R^3 (39 - 3 S S2)) +
         L^4 (60 R S + 12 S^2 + R^2 (57 - S S2)) + L R^2 (-36 R^2 S - 11 S^3 + R S^2 (-38 + R2 S) +
           R^3 (-9 + S S2)) + L^3 (-114 R^2 S - 48 R S^2 - 4 S^3 + R^3 (-67 + 3 S S2)))) / 6;
  va21[L_, R_, R2_, S_, S2_] := va21[L, R, R2, S, S2] = (1/(R^4 S^3) 2 (4 L^6 + R^3 S^2 (R + S) -
       4 L^5 (5 R + 3 S) - L R^2 S (8 R^2 + 16 R S + 7 S^2) + L^2 R (44 R^2 S + 44 R S^2 + 10 S^3 + R^3 (9 - S S2)) +
       L^4 (50 R S + 12 S^2 + R^2 (37 - S S2)) + 2 L^3 (-37 R^2 S - 20 R S^2 - 2 S^3 + R^3 (-15 + S S2)))) / 2;
  va12[L_, R_, R2_, S_, S2_] := va12[L, R, R2, S, S2] = R^-1 L^-1/2 (S - L)^-1/2 (R - L)^1/2
    (R + S - L)^1/2 * (1/(R^3 S^3) 2 L (4 L^5 - 4 R^2 S^2 (R + S) - 4 L^4 (4 R + 3 S) + L R S (13 R^2 + 25 R S + 8 S^2) +
       L^3 (40 R S + 12 S^2 + R^2 (21 - S S2)) + L^2 (-42 R^2 S - 32 R S^2 - 4 S^3 + R^3 (-9 + S S2)))) / 2;
  va03[L_, R_, R2_, S_, S2_] := va03[L, R, R2, S, S2] = R^-2 L^-1 (S - L)^-1 (R - L)
    (R + S - L) * (1/(R^2 S^3) 2 L (4 L^5 - 3 R^2 S^3 - 12 L^4 (R + S) + 6 L R S^2 (2 R + S) -
       2 L^2 S (9 R^2 + 12 R S + 2 S^2) + L^3 (30 R S + 12 S^2 + R^2 (9 - S S2)))) / 6;
  vb30[L_, R_, R2_, S_, S2_] := vb30[L, R, R2, S, S2] = R^2 L (S - L) (R - L)^-1 (R + S - L)^-1 *
    (-1/(R^5 S^3) 2 (4 L^6 R2 - 12 L^5 R2 (R + S) + R^2 (6 R^2 S + S^3 + R S^2 (4 + R2 S) + R^3 (3 + S S2)) -
       L R (S^3 + 4 R S^2 (2 + R2 S) + 2 R^2 S (9 + 2 R2 S) + 4 R^3 (3 + S S2)) +
       2 L^2 R (S^2 (2 + 3 R2 S) + R S (9 + 8 R2 S) + 3 R^2 (3 + R2 S + S S2)) -
       2 L^3 (2 R^3 R2 + 2 R2 S^3 + 3 R S (1 + 4 R2 S) + 2 R^2 (3 + 6 R2 S + S S2)) +
       L^4 (12 R^2 R2 + 12 R2 S^2 + R (3 + 30 R2 S + S S2)))) / 6;
  vb21[L_, R_, R2_, S_, S2_] := vb21[L, R, R2, S, S2] = R L^1/2 (S - L)^1/2 (R - L)^-1/2 (R + S - L)^-1/2 *
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    (-1/(R^4 S^3) 2 (4 L^6 R2 + R^2 S (R + S)^2 - 4 L^5 R2 (2 R + 3 S) + L^2 R (4 S^2 (1 + R2 S) + R S (13 + 5 R2 S) +
            3 R^2 (3 + S S2)) - L R (8 R^2 S + S^3 + R S^2 (6 + R2 S) + R^3 (3 + S S2)) + L^4 (4 R^2 R2 + 12 R2 S^2 +
            R (3 + 20 R2 S + S S2)) - L^3 (4 R2 S^3 + 2 R S (3 + 8 R2 S) + R^2 (9 + 8 R2 S + 3 S S2)))) / 2;

vb12[L_, R_, R2_, S_, S2_] := vb12[L, R, R2, S, S2] = (-1/(R^3 S^3) 2 (4 L^6 R2 + R^2 S^2 (R + S) -
        4 L^5 R2 (R + 3 S) - L R S (2 R^2 + 4 R S + S^2) + L^2 R (8 R S + 2 S^2 (2 + R2 S) + R^2 (3 + S S2)) -
        2 L^3 (2 R2 S^3 + R S (3 + 4 R2 S) + R^2 (3 + S S2)) + L^4 (12 R2 S^2 + R (3 + 10 R2 S + S S2)))) / 2;

vb03[L_, R_, R2_, S_, S2_] := vb03[L, R, R2, S, S2] = R^-1 L^-1/2 (S - L)^-1/2
    (R - L)^1/2 (R + S - L)^1/2 * (-1/(R^2 S^3) 2 L^2 (4 L^4 R2 - 12 L^3 R2 S + 3 R S (R + S) +
        L^2 (12 R2 S^2 + R (3 + S S2)) - L (6 R S + 4 R2 S^3 + R^2 (3 + S S2)))) / 6;

(*fifth order derivatives*)
va50[L_, R_, R2_, R4_, S_, S2_, S4_] :=
 va50[L, R, R2, R4, S, S2, S4] = R^2 L (S - L) (R - L)^-1 (R + S - L)^-1 *
    (1/(R^9 S^5) 2 (-16 L^10 (-9 + 4 R R2) + 80 L^9 (5 R^2 R2 - 9 S + 2 R (-9 + 2 R2 S)) -
        15 R^5 S (R + S) (-22 R^2 S - 7 S^3 + R S^2 (-22 + R2 S) + R^3 (-7 + S S2)) + 80 L^7 (18 R^4 R2 -
            18 S^3 + 8 R S^2 (-18 + R2 S) + 4 R^3 (-51 + 13 R2 S + S S2) + R^2 S (-318 + 40 R2 S + S S2)) -
        40 L^8 (26 R^3 R2 - 36 S^2 + 2 R S (-81 + 8 R2 S) + R^2 (-159 + 45 R2 S + S S2)) + L R^4
          (-601 S^5 + 2 R S^4 (-1643 + 55 R2 S) - 6 R^2 S^3 (1100 - 41 R2 S + R2^2 S^2) + 30 R^4 S (-83 + 9 S S2) +
            R^3 S^2 (-6060 + 120 R2 S + R4 S^3 + 160 S S2) + R^5 (-345 + 94 S S2 - 6 S^2 S2^2 + S^3 S4)) +
        L^2 R^3 (1320 S^5 + R S^4 (9331 - 240 R2 S) + 20 R^2 S^3 (1191 - 46 R2 S) + 6 R^3 S^2 (4615 -
              150 R2 S + R2^2 S^2 - 85 S S2) - R^4 S (-14 670 + R4 S^3 + 1170 S S2 - 10 R2 S (-21 + S S2)) -
            5 R^5 (-561 + 118 S S2 - 6 S^2 S2^2 + S^3 S4)) + 10 L^3 R^2 (-140 S^5 + 28 R S^4 (-48 + R2 S) +
            12 R^2 S^3 (-369 + 14 R2 S) + 2 R^3 S^2 (-3235 + 137 R2 S + 38 S S2) - 2 R^4 S
              (2130 - 118 S S2 + R2 S (-69 + S S2)) + R^5 (-1017 + 16 R2 S + 158 S S2 - 6 S^2 S2^2 + S^3 S4)) -
        5 L^4 R (16 R^6 R2 - 144 S^5 + 8 R S^4 (-264 + 5 R2 S) + 8 R^2 S^3 (-1185 + 47 R2 S) +
            R^3 S^2 (-17 847 + 960 R2 S + 119 S S2) + R^4 S (-14 685 + 523 S S2 - 2 R2 S (-417 + S S2)) +
            2 R^5 (-2157 + 116 R2 S + 234 S S2 - 6 S^2 S2^2 + S^3 S4)) -
        L^6 (1120 R^5 R2 - 720 S^4 + 160 R S^3 (-63 + 2 R2 S) + 40 R^2 S^2 (-989 + 70 R2 S + S S2) + 40 R^3
            S (-1428 + 163 R2 S + 14 S S2) + R^4 (-26 865 + 5040 R2 S + 1094 S S2 - 6 S^2 S2^2 + S^3 S4)) +
        L^5 (464 R^6 R2 - 144 S^5 + 32 R S^4 (-135 + 2 R2 S) + 120 R^2 S^3 (-247 + 10 R2 S) +
            40 R^3 S^2 (-1902 + 125 R2 S + 6 S S2) + 30 R^4 S (-2685 + 232 R2 S + 55 S S2) +
            R^5 (-29 469 + 3360 R2 S + 2070 S S2 - 30 S^2 S2^2 + 5 S^3 S4))) / 120;

va41[L_, R_, R2_, R4_, S_, S2_, S4_] := va41[L, R, R2, R4, S, S2, S4] =
  R L^1/2 (S - L)^1/2 (R - L)^-1/2 (R + S - L)^-1/2 *
    (-1/(R^8 S^5) 2 (16 L^10 (-9 + 4 R R2) - 16 L^9 (21 R^2 R2 - 45 S + R (-81 + 20 R2 S)) + R^5 S^2 (-132 R^2 S -
            41 S^3 + R S^2 (-128 + R2 S) + R^3 (-45 + S S2)) + L R^4 S (329 S^4 + 2 R^2 S^2 (1118 - 15 R2 S) +
            2 R S^3 (727 - 14 R2 S) + R^4 (306 - 38 S S2) - 8 R^3 S (-177 + 5 S S2)) + 8 L^8 (88 R^3 R2 -
            180 S^2 + R S (-729 + 80 R2 S) + R^2 (-633 + 189 R2 S + 5 S S2)) - 8 L^7 (92 R^4 R2 - 180 S^3 + 16 R
              S^2 (-81 + 5 R2 S) + 2 R^2 S (-1266 + 168 R2 S + 5 S S2) + R^3 (-1407 + 352 R2 S + 35 S S2)) +
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L^2 R^3 (-896 S^5 + 8 R^2 S^3 (-1379 + 32 R2 S) + R S^4 (-5323 + 100 R2 S) + 12 R^4 S (-297 + 29 S S2) +
    4 R^3 S^2 (-2457 + 32 R2 S + 54 S S2) + R^5 (-345 + 94 S S2 - 6 S^2 S2^2 + S^3 S4)) -
4 L^3 R^2 (-278 S^5 + 2 R S^4 (-1153 + 22 R2 S) + 2 R^2 S^3 (-3203 + 96 R2 S) +
    3 R^3 S^2 (-2535 + 70 R2 S + 37 S S2) + R^4 S (-3822 + 266 S S2 + R2 S (55 - S S2)) +
    R^5 (-615 + 124 S S2 - 6 S^2 S2^2 + S^3 S4)) - 2 L^5 (40 R^6 R2 - 72 S^5 + 8 R S^4 (-243 + 4 R2 S) +
    12 R^2 S^3 (-983 + 42 R2 S) + 8 R^3 S^2 (-3280 + 209 R2 S + 13 S S2) + R^4 S
     (-23427 + 1756 R2 S + 609 S S2) + 2 R^5 (-3465 + 288 R2 S + 314 S S2 - 6 S^2 S2^2 + S^3 S4)) +
L^6 (384 R^5 R2 - 720 S^4 + 16 R S^3 (-567 + 20 R2 S) + 8 R^2 S^2 (-3937 + 294 R2 S + 5 S S2) +
    8 R^3 S (-4926 + 550 R2 S + 61 S S2) + R^4 (-15609 + 2576 R2 S + 814 S S2 - 6 S^2 S2^2 + S^3 S4)) +
L^4 R (-648 S^5 + 168 R S^4 (-50 + R2 S) + 16 R^2 S^3 (-2041 + 77 R2 S) +
    5 R^3 S^2 (-10375 + 468 R2 S + 87 S S2) - 4 R^4 S (8655 - 391 S S2 + R2 S (-345 + S S2)) +
    2 R^5 (-3855 + 100 R2 S + 542 S S2 - 18 S^2 S2^2 + 3 S^3 S4)))) / 24;

va32[L_, R_, R2_, R4_, S_, S2_, S4_] := va32[L, R, R2, R4, S, S2, S4] = (1/(R^7 S^5)) 2 (-16 L^10
    (-9 + 4 R R2) + 12 R^5 S^3 (R + S)^2 + 16 L^9 (17 R^2 R2 - 45 S + 4 R (-18 + 5 R2 S)) + L^2 R^3 S (580 S^4 +
    R^2 S^2 (4394 - 41 R2 S) + R S^3 (2783 - 32 R2 S) + R^4 (507 - 61 S S2) + 2 R^3 S (1341 - 34 S S2)) +
  2 L R^4 S^2 (-282 R^2 S - 82 S^3 + R S^2 (-271 + 2 R2 S) + R^3 (-93 + 2 S S2)) + 16 L^7 (19 R^4 R2 - 90 S^3 +
    8 R S^2 (-72 + 5 R2 S) + 3 R^3 (-153 + 36 R2 S + 5 S S2) + R^2 S (-978 + 136 R2 S + 5 S S2)) -
  8 L^8 (54 R^3 R2 - 180 S^2 + 8 R S (-81 + 10 R2 S) + R^2 (-489 + 153 R2 S + 5 S S2)) +
  L^3 R^2 (-860 S^5 + 4 R S^4 (-1504 + 25 R2 S) + 2 R^2 S^3 (-6789 + 140 R2 S) + 4 R^4 S (-1059 + 95 S S2) +
    2 R^3 S^2 (-6168 + 77 R2 S + 114 S S2) + R^5 (-345 + 94 S S2 - 6 S^2 S2^2 + S^3 S4)) + L^4 R (576 S^5 +
    4 R^2 S^3 (5333 - 183 R2 S) + 4 R S^4 (1623 - 34 R2 S) - 4 R^3 S^2 (-6891 + 230 R2 S + 75 S S2) +
    R^4 S (14055 - 833 S S2 + R2 S (-277 + S S2)) - 3 R^5 (-705 + 134 S S2 - 6 S^2 S2^2 + S^3 S4)) -
  L^6 (80 R^5 R2 - 720 S^4 + 64 R S^3 (-126 + 5 R2 S) + 4 R^2 S^2 (-6083 + 476 R2 S + 10 S S2) +
    4 R^3 S (-6432 + 673 R2 S + 104 S S2) + R^4 (-8265 + 1064 R2 S + 574 S S2 - 6 S^2 S2^2 + S^3 S4)) +
  L^5 (-144 S^5 + 64 R S^4 (-54 + R2 S) + 12 R^2 S^3 (-1519 + 68 R2 S) +
    4 R^3 S^2 (-8570 + 507 R2 S + 44 S S2) + 2 R^4 S (-12435 + 716 R2 S + 425 S S2) +
    R^5 (-5595 + 240 R2 S + 682 S S2 - 18 S^2 S2^2 + 3 S^3 S4))))) / 12;

va23[L_, R_, R2_, R4_, S_, S2_, S4_] := va23[L, R, R2, R4, S, S2, S4] = R^-1 L^-1/2 (S - L)^-1/2
  (R - L)^1/2 (R + S - L)^1/2 * (-(1/(R^6 S^5)) 2 (16 L^10 (-9 + 4 R R2) - 3 R^5 S^4 (R + S) + 3 L R^4 S^3 (30 R^2 + 58 R S +
    27 S^2) - 16 L^9 (13 R^2 R2 - 45 S + R (-63 + 20 R2 S)) + 3 L^2 R^3 S^2 (-480 R^2 S - 122 S^3 +
    R S^2 (-449 + 2 R2 S) + 2 R^3 (-75 + 2 S S2)) + 8 L^8 (28 R^3 R2 - 180 S^2 + R S (-567 + 80 R2 S) +
    R^2 (-363 + 117 R2 S + 5 S S2)) + 4 L^3 R^2 S (162 S^4 + 6 R S^3 (155 - 2 R2 S) - 2 R^2 S^2
     (-807 + 8 R2 S) - 6 R^3 S (-167 + 4 S S2) - 3 R^4 (-59 + 7 S S2)) - 8 L^7 (10 R^4 R2 - 180 S^3 +
    16 R S^2 (-63 + 5 R2 S) + 2 R^2 S (-726 + 104 R2 S + 5 S S2) + R^3 (-555 + 112 R2 S + 25 S S2)) +
  L^4 R (-504 S^5 + 8 R S^4 (-606 + 13 R2 S) + 2 R^2 S^3 (-6519 + 184 R2 S) + 6 R^4 S (-751 + 61 S S2) +
    R^3 S^2 (-12939 + 230 R2 S + 191 S S2) + R^5 (-345 + 94 S S2 - 6 S^2 S2^2 + S^3 S4)) -
  2 L^5 (-72 S^5 + 8 R S^4 (-189 + 4 R2 S) + 12 R^2 S^3 (-565 + 26 R2 S) + 2 R^3 S^2 (-5205 + 260 R2 S +
      36 S S2) + 3 R^4 S (-1929 + 62 R2 S + 91 S S2) + R^5 (-885 + 154 S S2 - 6 S^2 S2^2 + S^3 S4)) +
  L^6 (-720 S^4 + 16 R S^3 (-441 + 20 R2 S) + 8 R^2 S^2 (-2259 + 182 R2 S + 5 S S2) + 8 R^3 S
     (-1947 + 174 R2 S + 43 S S2) + R^4 (-3825 + 280 R2 S + 374 S S2 - 6 S^2 S2^2 + S^3 S4)))) / 12;

va14[L_, R_, R2_, R4_, S_, S2_, S4_] := va14[L, R, R2, R4, S, S2, S4] =
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$$R^{-2}\,L^{-1}\,(S-L)^{-1}\,(R-L)\,(R+S-L) * \left(-\frac{1}{R^5\,S^5}\,2\,L\,\left(16\,L^9\,\left(-9+4\,R\,R2\right)+48\,R^4\,S^4\,\left(R+S\right)-\right.\right.$$
$$12\,L\,R^3\,S^3\,\left(29\,R^2+53\,R\,S+20\,S^2\right)-16\,L^8\,\left(9\,R^2\,R2-45\,S+R\,\left(-54+20\,R2\,S\right)\right)-16\,L^6$$
$$\left(-90\,S^3+8\,R\,S^2\,\left(-54+5\,R2\,S\right)+10\,R^3\,\left(-15+2\,R2\,S+S\,S2\right)+R^2\,S\,\left(-510+72\,R2\,S+5\,S\,S2\right)\right)+$$
$$8\,L^7\,\left(10\,R^3\,R2-180\,S^2+2\,R\,S\,\left(-243+40\,R2\,S\right)+R^2\,\left(-255+81\,R2\,S+5\,S\,S2\right)\right)-$$
$$4\,L^2\,R^2\,S^2\,\left(-675\,R^2\,S-120\,S^3+4\,R\,S^2\,\left(-138+R2\,S\right)+R^3\,\left(-216+7\,S\,S2\right)\right)+$$
$$L^3\,R\,S\,\left(-432\,S^4+24\,R\,S^3\,\left(-145+3\,R2\,S\right)+16\,R^2\,S^2\,\left(-459+8\,R2\,S\right)+R^4\,\left(-909+107\,S\,S2\right)+\right.$$
$$R^3\,S\,\left(-5061+109\,S\,S2\right)\Big)+L^4\,\left(144\,S^5-32\,R\,S^4\,\left(-81+2\,R2\,S\right)-48\,R^2\,S^3\,\left(-200+9\,R2\,S\right)+$$
$$18\,R^4\,S\,\left(243-17\,S\,S2\right)-16\,R^3\,S^2\,\left(-714+23\,R2\,S+7\,S\,S2\right)+R^5\,\left(345-94\,S\,S2+6\,S^2\,S2^2-\right.$$
$$\left.\left.S^3\,S4\right)\right)+L^5\,\left(-720\,S^4+32\,R\,S^3\,\left(-189+10\,R2\,S\right)+8\,R^2\,S^2\,\left(-1590+126\,R2\,S+5\,S\,S2\right)+\right.$$
$$\left.\left.\left.16\,R^3\,S\,\left(-528+31\,R2\,S+17\,S\,S2\right)+R^4\,\left(-1425+214\,S\,S2-6\,S^2\,S2^2+S^3\,S4\right)\right)\right)\right)\Big/\,24;$$

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va05[L_, R_, R2_, R4_, S_, S2_, S4_] := va05[L, R, R2, R4, S, S2, S4] =
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$$R^{-3}\,L^{-3/2}\,(S-L)^{-3/2}\,(R-L)^{3/2}\,(R+S-L)^{3/2} * \left(-\frac{1}{R^4\,S^5}\,2\,L\,\left(16\,L^9\,\left(-9+4\,R\,R2\right)+45\,R^4\,S^5-\right.\right.$$
$$45\,L\,R^3\,S^4\,\left(7\,R+4\,S\right)+120\,L^2\,R^2\,S^3\,\left(8\,R^2+11\,R\,S+3\,S^2\right)-80\,L^8\,\left(R^2\,R2-9\,S+R\,\left(-9+4\,R2\,S\right)\right)-$$
$$40\,L^6\,\left(-36\,S^3+16\,R\,S^2\,\left(-9+R2\,S\right)+3\,R^3\,\left(-9+S\,S2\right)+2\,R^2\,S\,\left(-66+8\,R2\,S+S\,S2\right)\right)+$$
$$40\,L^7\,\left(-36\,S^2+R\,S\,\left(-81+16\,R2\,S\right)+R^2\,\left(-33+9\,R2\,S+S\,S2\right)\right)+$$
$$5\,L^3\,R\,S^2\,\left(-756\,R^2\,S-72\,S^3+8\,R\,S^2\,\left(-60+R2\,S\right)+R^3\,\left(-291+11\,S\,S2\right)\right)-2\,L^4\,S\,\left(-72\,S^4+60\,R^2\right.$$
$$\left.S^2\,\left(-53+2\,R2\,S\right)+8\,R\,S^3\,\left(-135+4\,R2\,S\right)+20\,R^3\,S\,\left(-135+2\,S\,S2\right)+R^4\,\left(-555+65\,S\,S2\right)\right)+$$
$$L^5\,\left(-720\,S^4+80\,R\,S^3\,\left(-63+4\,R2\,S\right)+40\,R^2\,S^2\,\left(-207+14\,R2\,S+S\,S2\right)+\right.$$
$$\left.\left.\left.40\,R^3\,S\,\left(-96+5\,S\,S2\right)+R^4\,\left(-345+94\,S\,S2-6\,S^2\,S2^2+S^3\,S4\right)\right)\right)\right)\Big/\,120;$$

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vb50[L_, R_, R2_, R4_, S_, S2_, S4_] := vb50[L, R, R2, R4, S, S2, S4] =
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$$R^3\,L^{3/2}\,(S-L)^{3/2}\,(R-L)^{-3/2}\,(R+S-L)^{-3/2} * \left(-\frac{1}{R^9\,S^5}\,2\,\left(16\,L^{10}\,\left(4\,R2-6\,R\,R2^2+R^2\,R4\right)-80\,L^9\right.\right.$$
$$\left(R^3\,R4+4\,R2\,S+R\,R2\,\left(7-6\,R2\,S\right)+R^2\,\left(-6\,R2^2+R4\,S\right)\right)-40\,L^7\,\left(4\,R^5\,R4+16\,R2\,S^3-8\,R\,R2\,S^2\right.$$
$$\left(-14+3\,R2\,S\right)-8\,R^4\,\left(3\,R2^2-2\,R4\,S\right)+R^3\,\left(101\,R2-96\,R2^2\,S+16\,R4\,S^2-5\,R2\,S\,S2\right)+2\,R^2\,S$$
$$\left(102\,R2-48\,R2^2\,S+2\,R4\,S^2-R2\,S\,S2\right)\Big)+40\,L^8\,\left(4\,R^4\,R4+16\,R2\,S^2+3\,R\,R2\,S\,\left(21-8\,R2\,S\right)+\right.$$
$$R^3\,\left(-24\,R2^2+9\,R4\,S\right)+R^2\,\left(51\,R2-54\,R2^2\,S+4\,R4\,S^2-R2\,S\,S2\right)\Big)+L^2\,R^3\,\left(S^4\,\left(29+320\,R2\,S\right)-$$
$$40\,R\,S^3\,\left(-9-38\,R2\,S+4\,R2^2\,S^2\right)+2\,R^3\,S\,\left(450-60\,R2^2\,S^2+23\,R4\,S^3-150\,S\,S2+\right.$$
$$\left.10\,R2\,S\,\left(63-5\,S\,S2\right)\right)+2\,R^2\,S^2\,\left(1120\,R2\,S-163\,R2^2\,S^2+5\,\left(99+2\,R4\,S^3-3\,S\,S2\right)\right)-5\,R^4$$
$$\left(-45-4\,R4\,S^3+78\,S\,S2+18\,S^2\,S2^2+8\,R2\,S\,\left(-6+S\,S2\right)-3\,S^3\,S4\right)\Big)+20\,L^3\,R^2\,\left(-22\,R2\,S^5+$$
$$2\,R\,S^3\,\left(-3-81\,R2\,S+7\,R2^2\,S^2\right)+R^2\,S^2\,\left(-33-366\,R2\,S+56\,R2^2\,S^2-2\,R4\,S^3+S\,S2\right)-2\,R^5$$
$$\left(R4\,S^2+R2\,\left(5-S\,S2\right)\right)+R^3\,S\,\left(-45+52\,R2^2\,S^2-8\,R4\,S^3+15\,S\,S2+R2\,S\,\left(-325+11\,S\,S2\right)\right)+$$
$$R^4\,\left(-15+12\,R2^2\,S^2-8\,R4\,S^3+26\,S\,S2+6\,S^2\,S2^2+4\,R2\,S\,\left(-28+3\,S\,S2\right)-S^3\,S4\right)\Big)+R^4\,\left(-S^5+$$
$$R\,S^4\,\left(29+5\,R2\,S\right)+6\,R^2\,S^3\,\left(20+R2\,S-R2^2\,S^2\right)+R^3\,S^2\,\left(165+R4\,S^3-5\,S\,S2\right)-30\,R^4\,S\,\left(-3+\right.$$
$$\left.S\,S2\right)+R^5\,\left(15-26\,S\,S2-6\,S^2\,S2^2+S^3\,S4\right)\Big)-L\,R^3\,\left(-S^5+2\,R\,S^4\,\left(29+50\,R2\,S\right)+2\,R^2\,S^3$$
$$\left(180+143\,R2\,S-23\,R2^2\,S^2\right)+2\,R^3\,S^2\,\left(330+120\,R2\,S-18\,R2^2\,S^2+3\,R4\,S^3-10\,S\,S2\right)+2\,R^4$$
$$S\,\left(3\,\left(75+R4\,S^3-25\,S\,S2\right)-10\,R2\,S\,\left(-3+S\,S2\right)\right)+6\,R^5\,\left(15-26\,S\,S2-6\,S^2\,S2^2+S^3\,S4\right)\Big)+$$
$$L^6\,\left(80\,R^6\,R4+320\,R2\,S^4-80\,R\,R2\,S^3\,\left(-49+6\,R2\,S\right)-80\,R^5\,\left(6\,R2^2-7\,R4\,S\right)+40\,R^2\,S^2\right.$$
$$\left(-84\,R2^2\,S+2\,R4\,S^2+R2\,\left(317-S\,S2\right)\right)-40\,R^4\,\left(84\,R2^2\,S-25\,R4\,S^2+2\,R2\,\left(-59+5\,S\,S2\right)\right)+$$
$$R^3\,\left(15-6040\,R2^2\,S^2+560\,R4\,S^3-26\,S\,S2-6\,S^2\,S2^2-40\,R2\,S\,\left(-352+9\,S\,S2\right)+S^3\,S4\right)\Big)-$$
$$2\,L^5\,\left(8\,R^7\,R4+32\,R2\,S^5+24\,R\,R2\,S^4\,\left(35-2\,R2\,S\right)-24\,R^6\,\left(2\,R2^2-5\,R4\,S\right)+\right.$$
$$R^2\,\left(4740\,R2\,S^3-720\,R2^2\,S^4+8\,R4\,S^5\right)+5\,R^3\,S\,\left(9-468\,R2^2\,S^2+24\,R4\,S^3-3\,S\,S2-\right.$$
$$\left.8\,R2\,S\,\left(-233+2\,S\,S2\right)\right)-4\,R^5\,\left(180\,R2^2\,S-95\,R4\,S^2+R2\,\left(-408+50\,S\,S2\right)\right)+$$
$$R^4\,\left(45-2320\,R2^2\,S^2+380\,R4\,S^3-78\,S\,S2-18\,S^2\,S2^2-80\,R2\,S\,\left(-87+4\,S\,S2\right)+3\,S^3\,S4\right)+$$

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5 L^4 R (8 R^6 R4 S + 56 R2 S^5 - 48 R R2 S^4 (-14 + R2 S) + R^2 S^2 (33 + 2304 R2 S - 360 R2^2 S^2 +
      8 R4 S^3 - S S2) + 2 R^3 S (45 - 320 R2^2 S^2 + 28 R4 S^3 - 15 S S2 + 2 R2 S (755 - 13 S S2)) -
   8 R^5 (6 R2^2 S - 7 R4 S^2 + R2 (-31 + 5 S S2)) + R^4 (45 - 344 R2^2 S^2 + 100 R4 S^3 -
      78 S S2 - 18 S^2 S2^2 - 8 R2 S (-195 + 14 S S2) + 3 S^3 S4))) / 120;

vb41[L_, R_, R2_, R4_, S_, S2_, S4_] := vb41[L, R, R2, R4, S, S2, S4] =
 R^2 L (S - L) (R - L)^-1 (R + S - L)^-1 * (1/(R^8 S^5) 2 (16 L^10 (-4 R2 + 6 R R2^2 - R^2 R4) + 16 L^9
     (4 R^3 R4 + 20 R2 S + R R2 (31 - 30 R2 S) + R^2 (-24 R2^2 + 5 R4 S)) + R^4 S (R + S) (-30 R^2 S + S^3 +
      R S^2 (-14 + R2 S) + R^3 (-15 + S S2)) - 8 L^8 (12 R^4 R4 + 80 R2 S^2 - 3 R R2 S (-93 + 40 R2 S) +
      R^3 (-72 R2^2 + 36 R4 S) + R^2 (193 R2 - 216 R2^2 S + 20 R4 S^2 - 5 R2 S S2)) + 16 L^7 (4 R^5 R4 +
      40 R2 S^3 + 4 R R2 S^2 (62 - 15 R2 S) - 24 R^4 (R2^2 - R4 S) + R^2 S (386 R2 - 192 R2^2 S + 10 R4 S^2 -
        5 R2 S S2) - 2 R^3 (-78 R2 + 72 R2^2 S - 16 R4 S^2 + 5 R2 S S2)) - L^2 R^3 (S^4 (29 + 184 R2 S) +
      4 R S^3 (71 + 162 R2 S - 14 R2^2 S^2) + 2 R^2 S^2 (315 + 322 R2 S - 27 R2^2 S^2 + 4 R4 S^3 - 9 S S2) +
      3 R^3 S (150 + 62 R2 S + 3 R4 S^3 - 42 S S2 - 6 R2 S^2 S2) + 5 R^4 (15 - 26 S S2 - 6 S^2 S2^2 + S^3 S4)) +
   L R^3 (-S^5 + 2 R S^4 (21 + 19 R2 S) + 2 R^2 S^3 (104 + 35 R2 S - 3 R2^2 S^2) + 2 R^4 S (75 - 17 S S2) +
      R^3 S^2 (300 + 32 R2 S + R4 S^3 - 8 S S2) + R^5 (15 - 26 S S2 - 6 S^2 S2^2 + S^3 S4)) -
   L^6 (16 R^6 R4 + 320 R2 S^4 + 16 R R2 S^3 (217 - 30 R2 S) - 32 R^5 (3 R2^2 - 7 R4 S) + 8 R^2 S^2 (-336 R2^2
        S + 10 R4 S^2 + R2 (1199 - 5 S S2)) - 8 R^4 (168 R2^2 S - 75 R4 S^2 + R2 (-278 + 30 S S2)) +
      R^3 (15 - 3616 R2^2 S^2 + 448 R4 S^3 - 26 S S2 - 6 S^2 S2^2 - 48 R2 S (-181 + 6 S S2) + S^3 S4)) +
   L^5 (48 R^6 R4 S + 64 R2 S^5 - 48 R R2 S^4 (-31 + 2 R2 S) + 8 R^2 S^3 (895 R2 - 144 R2^2 S + 2 R4 S^2) +
      2 R^3 S (45 - 1392 R2^2 S^2 + 96 R4 S^3 - 15 S S2 + 32 R2 S (179 - 2 S S2)) -
      16 R^5 (18 R2^2 S - 19 R4 S^2 + 5 R2 (-13 + 2 S S2)) +
      R^4 (75 - 1840 R2^2 S^2 + 456 R4 S^3 - 130 S S2 - 30 S^2 S2^2 - 384 R2 S (-17 + S S2) + 5 S^3 S4)) +
   2 L^3 R^2 (164 R2 S^5 + 4 R S^3 (15 + 242 R2 S - 20 R2^2 S^2) + 2 R^2 S^2 (135 + 825 R2 S - 104 R2^2 S^2 + 6 R4
        S^3 - 4 S S2) + 2 R^3 S (-48 R2^2 S^2 + 2 (75 + 8 R4 S^3 - 23 S S2) - 7 R2 S (-71 + 3 S S2)) +
      R^4 (75 + 16 R4 S^3 - 130 S S2 - 30 S^2 S2^2 - 8 R2 S (-22 + 3 S S2) + 5 S^3 S4)) +
   L^4 R (-248 R2 S^5 + 32 R R2 S^4 (-79 + 6 R2 S) + R^2 S^2 (-165 - 7016 R2 S + 1056 R2^2 S^2 -
        32 R4 S^3 + 5 S S2) + 8 R^5 (-7 R4 S^2 + 5 R2 (-5 + S S2)) + R^3 S
      (-375 + 1240 R2^2 S^2 - 168 R4 S^3 + 121 S S2 + 2 R2 S (-3497 + 77 S S2)) + 2 R^4 (168 R2^2 S^2 +
        4 R2 S (-307 + 28 S S2) - 5 (15 + 20 R4 S^3 - 26 S S2 - 6 S^2 S2^2 + S^3 S4)))) / 24;

vb32[L_, R_, R2_, R4_, S_, S2_, S4_] := vb32[L, R, R2, R4, S, S2, S4] =
 R L^(1/2) (S - L)^(1/2) (R - L)^-(1/2) (R + S - L)^-(1/2) * (-1/(R^7 S^5) 2 (16 L^10 (4 R2 - 6 R R2^2 + R^2 R4) - 16 L^9 (3 R^3 R4 +
     20 R2 S + 3 R R2 (9 - 10 R2 S) + R^2 (-18 R2^2 + 5 R4 S)) + L R^3 S (S^4 - 2 R^2 S^2 (47 + 3 R2 S) -
      2 R S^3 (13 + 4 R2 S) + 4 R^3 S (-24 + S S2) + 2 R^4 (-15 + S S2)) - R^4 S^2 (-6 R^2 S + S^3 +
      R S^2 (-2 + R2 S) + R^3 (-3 + S S2)) - 8 L^7 (2 R^5 R4 + 80 R2 S^3 - 24 R R2 S^2 (-18 + 5 R2 S) -
      12 R^4 (R2^2 - 2 R4 S) + R^3 (173 R2 - 144 R2^2 S + 48 R4 S^2 - 15 R2 S S2) + 2 R^2 S (278 R2 -
        144 R2^2 S + 10 R4 S^2 - 5 R2 S S2)) + 8 L^8 (6 R^4 R4 + 80 R2 S^2 - 3 R R2 S (-81 + 40 R2 S) -
      9 R^3 (4 R2^2 - 3 R4 S) + R^2 (139 R2 - 162 R2^2 S + 20 R4 S^2 - 5 R2 S S2)) + 2 L^5 (-32 R2 S^5 +
      24 R R2 S^4 (-27 + 2 R2 S) - 4 R^2 S^3 (643 R2 - 108 R2^2 S + 2 R4 S^2) + 2 R^5 (-19 R4 S^2 + 10 R2
        (-5 + S S2)) + 3 R^3 S (-15 + 230 R2^2 S^2 - 24 R4 S^3 + 5 S S2 + 2 R2 S (-525 + 8 S S2)) +
      2 R^4 (-15 + 114 R2^2 S^2 - 57 R4 S^3 + 26 S S2 + 6 S^2 S2^2 + 12 R2 S (-51 + 4 S S2) - S^3 S4)) +
   L^2 R^3 (S^4 (29 + 90 R2 S) - 4 R S^3 (-52 - 50 R2 S + 3 R2^2 S^2) + 2 R^2 S^2 (174 + 50 R2 S +
        R4 S^3 - 5 S S2) - 36 R^3 S (-5 + S S2) + R^4 (15 - 26 S S2 - 6 S^2 S2^2 + S^3 S4)) - 4 L^3 R^2
    (58 R2 S^5 + R S^3 (30 + 254 R2 S - 19 R2^2 S^2) + 3 R^2 S^2 (35 + 96 R2 S - 8 R2^2 S^2 + R4 S^3 - S S2) +
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        R^3 S (90 + 4 R4 S^3 - 24 S S2 + R2 S (89 - 5 S S2)) + R^4 (15 - 26 S S2 - 6 S^2 S2^2 + S^3 S4)) +
      L^6 (56 R^5 R4 S + 320 R2 S^4 - 48 R R2 S^3 (-63 + 10 R2 S) + 8 R^2 S^2 (-252 R2^2 S +
          10 R4 S^2 + R2 (863 - 5 S S2)) - 12 R^4 (28 R2^2 S - 25 R4 S^2 + 10 R2 (-7 + S S2)) +
        R^3 (15 - 1804 R2^2 S^2 + 336 R4 S^3 - 26 S S2 - 6 S^2 S2^2 - 8 R2 S (-601 + 27 S S2) + S^3 S4)) +
      L^4 R (216 R2 S^5 - 16 R R2 S^4 (-113 + 9 R2 S) + R^2 S^2 (165 + 3802 R2 S - 516 R2^2 S^2 +
          24 R4 S^3 - 5 S S2) + 4 R^3 S (75 - 75 R2^2 S^2 + 21 R4 S^3 - 23 S S2 + R2 S (642 - 19 S S2)) +
        R^4 (90 + 50 R4 S^3 - 156 S S2 - 36 S^2 S2^2 - 8 R2 S (-58 + 7 S S2) + 6 S^3 S4))) / 12;

vb23[L_, R_, R2_, R4_, S_, S2_, S4_] := vb23[L, R, R2, R4, S, S2, S4] =
  (-1/(R^6 S^5)) 2 L (16 L^9 (4 R2 - 6 R R2^2 + R^2 R4) - 18 R^4 S^2 (R + S)^2 - 16 L^8 (2 R^3 R4 + 20 R2 S +
        R R2 (23 - 30 R2 S) + R^2 (-12 R2^2 + 5 R4 S)) + 3 L R^3 S (54 R^2 S + R S^2 (50 + 11 R2 S) +
        S^3 (11 + 12 R2 S) + R^3 (15 - S S2)) - 16 L^6 (4 R^4 R4 S + 40 R2 S^3 + 4 R R2 S^2 (46 - 15 R2 S) +
        R^2 S (186 R2 - 96 R2^2 S + 10 R4 S^2 - 5 R2 S S2) + R^3 (40 R2 - 24 R2^2 S + 16 R4 S^2 - 5 R2 S S2)) +
      8 L^7 (2 R^4 R4 + 80 R2 S^2 - 3 R R2 S (-69 + 40 R2 S) - 6 R^3 (2 R2^2 - 3 R4 S) + R^2 (93 R2 - 108 R2^2 S +
          20 R4 S^2 - 5 R2 S S2)) - L^2 R^2 (156 R2 S^5 + 2 R S^3 (63 + 220 R2 S - 12 R2^2 S^2) + 2 R^2 S^2 (156 +
          121 R2 S + 2 R4 S^3 - 2 S S2) - 36 R^3 S (-5 + S S2) + R^4 (15 - 26 S S2 - 6 S^2 S2^2 + S^3 S4)) +
      L^3 R (184 R2 S^5 + 12 R R2 S^4 (101 - 8 R2 S) + 4 R^2 S^2 (42 + 427 R2 S - 42 R2^2 S^2 + 4 R4 S^3 - S S2) +
        R^3 S (225 + 28 R4 S^3 - 63 S S2 + 5 R2 S (117 - 5 S S2)) + 3 R^4 (15 - 26 S S2 - 6 S^2 S2^2 + S^3 S4)) +
      L^5 (320 R2 S^4 + 16 R R2 S^3 (161 - 30 R2 S) + 20 R^4 (5 R4 S^2 - 2 R2 (-5 + S S2)) +
        4 R^2 S^2 (-336 R2^2 S + 20 R4 S^2 - 5 R2 (-231 + 2 S S2)) +
        R^3 (15 - 600 R2^2 S^2 + 224 R4 S^3 - 26 S S2 - 6 S^2 S2^2 - 8 R2 S (-277 + 18 S S2) + S^3 S4)) -
      L^4 (64 R2 S^5 - 48 R R2 S^4 (-23 + 2 R2 S) + 4 R^2 S^3 (861 R2 - 144 R2^2 S + 4 R4 S^2) +
        2 R^3 S (45 - 228 R2^2 S^2 + 48 R4 S^3 - 15 S S2 - 32 R2 S (-45 + S S2)) +
        R^4 (45 + 76 R4 S^3 - 78 S S2 - 18 S^2 S2^2 - 64 R2 S (-9 + S S2) + 3 S^3 S4))) / 12;

vb14[L_, R_, R2_, R4_, S_, S2_, S4_] := vb14[L, R, R2, R4, S, S2, S4] = R^-1 L^-1/2
  (S - L)^-1/2 (R - L)^1/2 (R + S - L)^1/2 * (-1/(R^5 S^5)) 2 (16 L^10 (4 R2 - 6 R R2^2 + R^2 R4) + 3 R^4 S^4 (R + S) -
      3 L R^3 S^3 (4 R^2 + 6 R S + S^2) - 16 L^9 (R^3 R4 + 20 R2 S + R R2 (19 - 30 R2 S) + R^2 (-6 R2^2 + 5 R4 S)) +
      3 L^2 R^3 S^2 (36 R S + S^2 (15 + 4 R2 S) + 2 R^2 (9 + S S2)) +
      4 L^3 R^2 S (-26 R2 S^4 - 2 R S^2 (18 + 17 R2 S) + R^3 (-15 + S S2) - 2 R^2 S (27 + S S2)) + 8 L^8
       (9 R^3 R4 S + 80 R2 S^2 - 3 R R2 S (-57 + 40 R2 S) + R^2 (55 R2 - 54 R2^2 S + 20 R4 S^2 - 5 R2 S S2)) +
      8 L^7 (-80 R2 S^3 + 8 R R2 S^2 (-38 + 15 R2 S) + 2 R^2 S (48 R2^2 S - 10 R4 S^2 + 5 R2 (-22 + S S2)) +
        R^3 (-16 R4 S^2 + 5 R2 (-5 + S S2))) + L^4 R (152 R2 S^5 + 16 R R2 S^4 (47 - 3 R2 S) + 2 R^3 S (75 -
          17 S S2) + R^2 S^2 (177 + 524 R2 S + 8 R4 S^3 - S S2) + R^4 (15 - 26 S S2 - 6 S^2 S2^2 + S^3 S4)) -
      2 L^5 (32 R2 S^5 + 24 R R2 S^4 (19 - 2 R2 S) + 4 R^2 S^3 (259 R2 - 36 R2^2 S + 2 R4 S^2) + R^3 S
          (3 (15 + 8 R4 S^3 - 5 S S2) + 4 R2 S (111 - 4 S S2)) + R^4 (15 - 26 S S2 - 6 S^2 S2^2 + S^3 S4)) +
      L^6 (320 R2 S^4 + 16 R R2 S^3 (133 - 30 R2 S) + 8 R^2 S^2 (-84 R2^2 S + 10 R4 S^2 + R2 (343 - 5 S S2)) +
        R^3 (15 + 112 R4 S^3 - 26 S S2 - 6 S^2 S2^2 - 8 R2 S (-86 + 9 S S2) + S^3 S4))) / 24;

vb05[L_, R_, R2_, R4_, S_, S2_, S4_] := vb05[L, R, R2, R4, S, S2, S4] = R^-2 L^-1 (S - L)^-1 (R - L)
  (R + S - L) * (-1/(R^4 S^5)) 2 L^2 (16 L^8 (4 R2 - 6 R R2^2 + R^2 R4) + 60 R^3 S^3 (R + S) - 80 L^7 (4 R2 S + R^2 R4 S +
      3 R R2 (1 - 2 R2 S)) - 20 L R^2 S^2 (9 R S + 4 R2 S^3 + R^2 (6 + S S2)) + 5 L^2 R S (88 R R2 S^3 +
      24 R2 S^4 + R^3 (15 - S S2) + R^2 S (39 + S S2)) - 40 L^6 (-16 R2 S^2 + 3 R R2 S (-9 + 8 R2 S) +
      R^2 (-5 R2 - 4 R4 S^2 + R2 S S2)) + 80 L^5 S (-8 R2 S^2 + 12 R R2 S (-2 + R2 S) + R^2 (-10 R2 -
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2 R4 S^2 + R2 S S2)) + L^4 (320 R2 S^4 - 240 R R2 S^3 (-7 + 2 R2 S) + 40 R^2 S^2 (32 R2 + 2 R4 S^2 -
R2 S S2) + R^3 (15 - 26 S S2 - 6 S^2 S2^2 + S^3 S4)) - L^3 (64 R2 S^5 + 48 R R2 S^4 (15 - 2 R2 S) +
16 R^2 (65 R2 S^3 + R4 S^5) - 30 R^3 S (-3 + S S2) + R^4 (15 - 26 S S2 - 6 S^2 S2^2 + S^3 S4)))) / 120;
```

Switch to vector formation

```
In[ ]:= (*v={L,R,R2,R4,S,S2,S4}*)
Clear[t, ra, ra2, ra4, rb, rb2, rb4];
t[v__] := t[v] = v[[1]];
ra[v__] := ra[v] = v[[2]];
ra2[v__] := ra2[v] = v[[3]];
ra4[v__] := ra4[v] = v[[4]];
rb[v__] := rb[v] = v[[5]];
rb2[v__] := rb2[v] = v[[6]];
rb4[v__] := rb4[v] = v[[7]];
```

```
In[ ]:= Clear[a10, a01, b10, b01, a30, a21, a12, a03, b30, b21,
  b12, b03, a50, a41, a32, a23, a14, a05, b50, b41, b32, b23, b14, b05];
(*Taylor expansion in terms of (x,y)=(s/η, ηu)*)
a10[v__] := a10[v] = va10[t[v], ra[v], rb[v]];
a01[v__] := a01[v] = va01[t[v], ra[v], rb[v]];
b10[v__] := b10[v] = vb10[t[v], ra[v], rb[v]];
b01[v__] := b01[v] = vb01[t[v], ra[v], rb[v]];
(*third order*)
a30[v__] := a30[v] = va30[t[v], ra[v], ra2[v], rb[v], rb2[v]];
a21[v__] := a21[v] = va21[t[v], ra[v], ra2[v], rb[v], rb2[v]];
a12[v__] := a12[v] = va12[t[v], ra[v], ra2[v], rb[v], rb2[v]];
a03[v__] := a03[v] = va03[t[v], ra[v], ra2[v], rb[v], rb2[v]];
b30[v__] := b30[v] = vb30[t[v], ra[v], ra2[v], rb[v], rb2[v]];
b21[v__] := b21[v] = vb21[t[v], ra[v], ra2[v], rb[v], rb2[v]];
b12[v__] := b12[v] = vb12[t[v], ra[v], ra2[v], rb[v], rb2[v]];
b03[v__] := b03[v] = vb03[t[v], ra[v], ra2[v], rb[v], rb2[v]];
(*fifth order*)
a50[v__] := a50[v] = va50[t[v], ra[v], ra2[v], ra4[v], rb[v], rb2[v], rb4[v]];
a41[v__] := a41[v] = va41[t[v], ra[v], ra2[v], ra4[v], rb[v], rb2[v], rb4[v]];
a32[v__] := a32[v] = va32[t[v], ra[v], ra2[v], ra4[v], rb[v], rb2[v], rb4[v]];
a23[v__] := a23[v] = va23[t[v], ra[v], ra2[v], ra4[v], rb[v], rb2[v], rb4[v]];
a14[v__] := a14[v] = va14[t[v], ra[v], ra2[v], ra4[v], rb[v], rb2[v], rb4[v]];
a05[v__] := a05[v] = va05[t[v], ra[v], ra2[v], ra4[v], rb[v], rb2[v], rb4[v]];
b50[v__] := b50[v] = vb50[t[v], ra[v], ra2[v], ra4[v], rb[v], rb2[v], rb4[v]];
b41[v__] := b41[v] = vb41[t[v], ra[v], ra2[v], ra4[v], rb[v], rb2[v], rb4[v]];
b32[v__] := b32[v] = vb32[t[v], ra[v], ra2[v], ra4[v], rb[v], rb2[v], rb4[v]];
b23[v__] := b23[v] = vb23[t[v], ra[v], ra2[v], ra4[v], rb[v], rb2[v], rb4[v]];
b14[v__] := b14[v] = vb14[t[v], ra[v], ra2[v], ra4[v], rb[v], rb2[v], rb4[v]];
b05[v__] := b05[v] = vb05[t[v], ra[v], ra2[v], ra4[v], rb[v], rb2[v], rb4[v]];
```

2. Coefficients of the coordinate transform

$$(x, y) \to (X, Y) = \left(\sum p_{jk}\, x^j\, y^k, \sum q_{jk}\, x^j\, y^k\right)$$

```
In[•]:= Clear[p30, p21, p12, p03, q30, q21, q12, q03];
p30[v__] :=
  p30[v] = -(2 a10[v] × b10[v] (a30[v] + b21[v] - a12[v] - b03[v]) - (a10[v]^2 - b10[v]^2)
        (b30[v] - a21[v] - b12[v] + a03[v])) / (8 b10[v]) +
    (b10[v] (a30[v] - b21[v] - a12[v] + b03[v]) + a10[v] (b30[v] + a21[v] - b12[v] - a03[v])) /
     (32 a10[v] × b10[v]);
p21[v__] := p21[v] = 3 (b10[v] (b30[v] + a21[v] - b12[v] - a03[v]) -
      a10[v] (a30[v] - b21[v] - a12[v] + b03[v])) / (32 a10[v] × b10[v]);
p12[v__] := p12[v] = -3 (2 a10[v] × b10[v] (a30[v] + b21[v] - a12[v] - b03[v]) -
       (a10[v]^2 - b10[v]^2) (b30[v] - a21[v] - b12[v] + a03[v])) / (8 b10[v]) -
    3 (b10[v] (a30[v] - b21[v] - a12[v] + b03[v]) + a10[v] (b30[v] + a21[v] - b12[v] - a03[v])) /
     (32 a10[v] × b10[v]);
p03[v__] := p03[v] = (2 a10[v] × b10[v] (b30[v] - a21[v] - b12[v] + a03[v]) +
       (a10[v]^2 - b10[v]^2) (a30[v] + b21[v] - a12[v] - b03[v])) / (4 b10[v]) -
    (b10[v] (b30[v] + a21[v] - b12[v] - a03[v]) - a10[v] (a30[v] - b21[v] - a12[v] + b03[v])) /
     (32 a10[v] × b10[v]);
q30[v__] := q30[v] = (2 a10[v] × b10[v] (b30[v] - a21[v] - b12[v] + a03[v]) +
       (a10[v]^2 - b10[v]^2) (a30[v] + b21[v] - a12[v] - b03[v])) / (4 b10[v]) +
    (b10[v] (b30[v] + a21[v] - b12[v] - a03[v]) - a10[v] (a30[v] - b21[v] - a12[v] + b03[v])) /
     (32 a10[v] × b10[v]);
q21[v__] := q21[v] = 3 (2 a10[v] × b10[v] (a30[v] + b21[v] - a12[v] - b03[v]) -
       (a10[v]^2 - b10[v]^2) (b30[v] - a21[v] - b12[v] + a03[v])) / (8 b10[v]) -
    3 (b10[v] (a30[v] - b21[v] - a12[v] + b03[v]) + a10[v] (b30[v] + a21[v] - b12[v] - a03[v])) /
     (32 a10[v] × b10[v]);
q12[v__] := q12[v] = -3 (b10[v] (b30[v] + a21[v] - b12[v] - a03[v]) -
      a10[v] (a30[v] - b21[v] - a12[v] + b03[v])) / (32 a10[v] × b10[v]);
q03[v__] := q03[v] = (2 a10[v] × b10[v] (a30[v] + b21[v] - a12[v] - b03[v]) -
       (a10[v]^2 - b10[v]^2) (b30[v] - a21[v] - b12[v] + a03[v])) / (8 b10[v]) +
    (b10[v] (a30[v] - b21[v] - a12[v] + b03[v]) + a10[v] (b30[v] + a21[v] - b12[v] - a03[v])) /
     (32 a10[v] × b10[v]);

In[•]:= Clear[p50, p41, p32, p23, p14, p05, q50, q41, q32, q23, q14, q05];
p50[v__] := p50[v] = p21[v] * q30[v];
p41[v__] := p41[v] = p21[v] * q21[v] + 2 p12[v] * q30[v];
p32[v__] := p32[v] = p21[v] * q12[v] + 2 p12[v] * q21[v] + 3 p03[v] * q30[v];
p23[v__] := p23[v] = p21[v] * q03[v] + 2 p12[v] * q12[v] + 3 p03[v] * q21[v];
p14[v__] := p14[v] = 2 p12[v] * q03[v] + 3 p03[v] * q12[v];
p05[v__] := p05[v] = 3 p03[v] * q03[v]; q50[v__] := q50[v] = q21[v] * q30[v];
q41[v__] := q41[v] = q21[v] * q21[v] + 2 q12[v] * q30[v];
q32[v__] := q32[v] = q21[v] * q12[v] + 2 q12[v] * q21[v] + 3 q03[v] * q30[v];
q23[v__] := q23[v] = q21[v] * q03[v] + 2 q12[v] * q12[v] + 3 q03[v] * q21[v];
q14[v__] := q14[v] = 2 q12[v] * q03[v] + 3 q03[v] * q12[v];
q05[v__] := q05[v] = 3 q03[v] * q03[v];
```

3. Taylor expansion of the billiard map in terms of new coordinate system (X,Y):

$$(X, Y) \to (X_1, Y_1) = \left(\sum A_{jk} X^j Y^k, \sum B_{jk} X^j Y^k\right)$$

```
In[•]:= Clear[aa30, aa21, aa12, aa03, bb30, bb21, bb12, bb03];
aa30[v__] := aa30[v] = a30[v] + b10[v]^3 p03[v] + a10[v] b10[v]^2 p12[v] +
    a10[v]^2 b10[v] * p21[v] + a10[v]^3 p30[v] - (a10[v] * p30[v] + a01[v] * q30[v]);
aa21[v__] := aa21[v] = a21[v] + b01[v]
     (3 b10[v]^2 p03[v] + 2 a10[v] * b10[v] * p12[v] + a10[v]^2 p21[v]) +
   a01[v] (b10[v]^2 p12[v] + 2 a10[v] * b10[v] * p21[v] + 3 a10[v]^2 p30[v]) -
   (a10[v] * p21[v] + a01[v] * q21[v]);
aa12[v__] := aa12[v] = a12[v] + b01[v]^2 (3 b10[v] * p03[v] + a10[v] * p12[v]) +
   2 a01[v] * b01[v] (b10[v] * p12[v] + a10[v] * p21[v]) +
   a01[v]^2 (b10[v] * p21[v] + 3 a10[v] * p30[v]) - (a10[v] * p12[v] + a01[v] * q12[v]);
aa03[v__] := aa03[v] = a03[v] + b01[v]^3 p03[v] + a01[v] b01[v]^2 p12[v] +
   a01[v]^2 b01[v] * p21[v] + a01[v]^3 p30[v] - (a10[v] * p03[v] + a01[v] * q03[v]);
bb30[v__] := bb30[v] = b30[v] + b10[v]^3 q03[v] + a10[v] b10[v]^2 q12[v] +
    a10[v]^2 b10[v] * q21[v] + a10[v]^3 q30[v] - (b10[v] * p30[v] + b01[v] * q30[v]);
bb21[v__] := bb21[v] = b21[v] + b01[v]
     (3 b10[v]^2 q03[v] + 2 a10[v] * b10[v] * q12[v] + a10[v]^2 q21[v]) +
   a01[v] (b10[v]^2 q12[v] + 2 a10[v] * b10[v] * q21[v] + 3 a10[v]^2 q30[v]) -
   (b10[v] * p21[v] + b01[v] * q21[v]);
bb12[v__] := bb12[v] = b12[v] + b01[v]^2 (3 b10[v] * q03[v] + a10[v] * q12[v]) +
   2 a01[v] * b01[v] (b10[v] * q12[v] + a10[v] * q21[v]) +
   a01[v]^2 (b10[v] * q21[v] + 3 a10[v] * q30[v]) - (b10[v] * p12[v] + b01[v] * q12[v]);
bb03[v__] := bb03[v] = b03[v] + b01[v]^3 q03[v] + a01[v] b01[v]^2 q12[v] +
   a01[v]^2 b01[v] * q21[v] + a01[v]^3 q30[v] - (b10[v] * p03[v] + b01[v] * q03[v]);
```

```
In[•]:= Clear[aa50, aa41, aa32, aa23, aa14, aa05];
aa50[v__] := aa50[v] = a50[v] + 3 b10[v]^2 b30[v] × p03[v] + b10[v]^5 p05[v] +
    (a30[v] b10[v]^2 + 2 a10[v] × b10[v] × b30[v]) p12[v] + a10[v] b10[v]^4 p14[v] +
    (2 a10[v] × a30[v] × b10[v] + a10[v]^2 b30[v]) p21[v] + a10[v]^2 b10[v]^3 p23[v] +
    3 a10[v]^2 a30[v] × p30[v] + a10[v]^3 b10[v]^2 p32[v] + a10[v]^4 b10[v] × p41[v] +
    a10[v]^5 p50[v] - (3 aa30[v] × p30[v] + a10[v] × p50[v] + aa21[v] × q30[v] + a01[v] × q50[v]);
aa41[v__] := aa41[v] = a41[v] + b10[v]^4 (5 b01[v] × p05[v] + a01[v] × p14[v]) +
    2 a10[v] b10[v]^3 (2 b01[v] × p14[v] + a01[v] × p23[v]) + b10[v]^2
     (3 b21[v] × p03[v] + a21[v] × p12[v] + 3 a10[v]^2 (b01[v] × p23[v] + a01[v] × p32[v])) +
    2 b10[v] (a10[v] × b21[v] × p12[v] + a01[v] × b30[v] × p12[v] + a10[v] × a21[v] × p21[v] +
       a01[v] × a30[v] × p21[v] + b01[v] (3 b30[v] × p03[v] + a30[v] × p12[v] + a10[v]^3 p32[v]) +
       2 a01[v] a10[v]^3 p41[v]) + a10[v] (a10[v] × b21[v] × p21[v] + 2 a01[v] × b30[v] × p21[v] +
       3 a10[v] × a21[v] × p30[v] + 6 a01[v] × a30[v] × p30[v] + b01[v]
        (2 b30[v] × p12[v] + 2 a30[v] × p21[v] + a10[v]^3 p41[v]) + 5 a01[v] a10[v]^3 p50[v]) -
    (3 aa30[v] × p21[v] + 2 aa21[v] × p30[v] + a10[v] × p41[v] + aa21[v] × q21[v] +
      2 aa12[v] × q30[v] + a01[v] × q41[v]);
```

```
aa32[v__] := aa32[v] = a32[v] + 3 b01[v]^2 b30[v] × p03[v] + a30[v] b01[v]^2 p12[v] +
    2 a10[v] × b01[v] × b21[v] × p12[v] + 2 a01[v] × b01[v] × b30[v] × p12[v] +
    2 a10[v] × a21[v] × b01[v] × p21[v] + 2 a01[v] × a30[v] × b01[v] × p21[v] +
    a10[v]^2 b12[v] × p21[v] + 2 a01[v] × a10[v] × b21[v] × p21[v] + a01[v]^2 b30[v] × p21[v] +
    b10[v]^3 (10 b01[v]^2 p05[v] + 4 a01[v] × b01[v] × p14[v] + a01[v]^2 p23[v]) +
    3 a10[v]^2 a12[v] × p30[v] + 6 a01[v] × a10[v] × a21[v] × p30[v] +
    3 a01[v]^2 a30[v] × p30[v] + a10[v]^3 b01[v]^2 p32[v] +
    b10[v]^2 (3 b12[v] × p03[v] + a12[v] × p12[v] + 6 a10[v] b01[v]^2 p14[v] +
       6 a01[v] × a10[v] × b01[v] × p23[v] + 3 a01[v]^2 a10[v] × p32[v]) +
    4 a01[v] a10[v]^3 b01[v] × p41[v] + b10[v] (3 a10[v]^2 b01[v]^2 p23[v] +
       2 b01[v] (3 b21[v] × p03[v] + a21[v] × p12[v] + 3 a01[v] a10[v]^2 p32[v]) +
       2 (a10[v] × b12[v] × p12[v] + a01[v] × b21[v] × p12[v] + a10[v] × a12[v] × p21[v] +
          a01[v] × a21[v] × p21[v] + 3 a01[v]^2 a10[v]^2 p41[v])) + 10 a01[v]^2 a10[v]^3 p50[v] -
    (3 aa30[v] × p12[v] + 2 aa21[v] × p21[v] + aa12[v] × p30[v] + a10[v] × p32[v] +
      aa21[v] × q12[v] + 2 aa12[v] × q21[v] + 3 aa03[v] × q30[v] + a01[v] × q32[v]);
aa23[v__] := aa23[v] = a23[v] + 6 b01[v] × b10[v] × b12[v] × p03[v] + 3 b01[v]^2 b21[v] × p03[v] +
    10 b01[v]^3 b10[v]^2 p05[v] + a21[v] b01[v]^2 p12[v] + 2 a12[v] × b01[v] × b10[v] × p12[v] +
    a03[v] b10[v]^2 p12[v] + 2 a10[v] × b01[v] × b12[v] × p12[v] +
    2 a01[v] × b10[v] × b12[v] × p12[v] + 2 a01[v] × b01[v] × b21[v] × p12[v] +
    4 a10[v] b01[v]^3 b10[v] × p14[v] + 6 a01[v] b01[v]^2 b10[v]^2 p14[v] +
    2 a10[v] × a12[v] × b01[v] × p21[v] + 2 a01[v] × a21[v] × b01[v] × p21[v] +
    2 a03[v] × a10[v] × b10[v] × p21[v] + 2 a01[v] × a12[v] × b10[v] × p21[v] +
    2 a01[v] × a10[v] × b12[v] × p21[v] + a01[v]^2 b21[v] × p21[v] +
    b03[v] (3 b10[v]^2 p03[v] + 2 a10[v] × b10[v] × p12[v] + a10[v]^2 p21[v]) +
    a10[v]^2 b01[v]^3 p23[v] + 6 a01[v] × a10[v] b01[v]^2 b10[v] × p23[v] +
    3 a01[v]^2 b01[v] b10[v]^2 p23[v] + 3 a03[v] a10[v]^2 p30[v] +
    6 a01[v] × a10[v] × a12[v] × p30[v] + 3 a01[v]^2 a21[v] × p30[v] +
    3 a01[v] a10[v]^2 b01[v]^2 p32[v] + 6 a01[v]^2 a10[v] × b01[v] × b10[v] × p32[v] +
    a01[v]^3 b10[v]^2 p32[v] + 6 a01[v]^2 a10[v]^2 b01[v] × p41[v] +
    4 a01[v]^3 a10[v] × b10[v] × p41[v] + 10 a01[v]^3 a10[v]^2 p50[v] -
    (3 aa30[v] × p03[v] + 2 aa21[v] × p12[v] + aa12[v] × p21[v] + a10[v] × p23[v] +
      aa21[v] × q03[v] + 2 aa12[v] × q12[v] + 3 aa03[v] × q21[v] + a01[v] × q23[v]);
aa14[v__] := aa14[v] = a14[v] + b01[v]^4 (5 b10[v] × p05[v] + a10[v] × p14[v]) +
    2 a01[v] b01[v]^3 (2 b10[v] × p14[v] + a10[v] × p23[v]) + b01[v]^2
     (3 b12[v] × p03[v] + a12[v] × p12[v] + 3 a01[v]^2 (b10[v] × p23[v] + a10[v] × p32[v])) +
    2 b01[v] (3 b03[v] × b10[v] × p03[v] + a10[v] × b03[v] × p12[v] +
       a03[v] × b10[v] × p12[v] + a01[v] × b12[v] × p12[v] + a03[v] × a10[v] × p21[v] +
       a01[v] × a12[v] × p21[v] + a01[v]^3 b10[v] × p32[v] + 2 a01[v]^3 a10[v] × p41[v]) +
    a01[v] (2 a03[v] × b10[v] × p21[v] + a01[v] × b12[v] × p21[v] +
       2 b03[v] (b10[v] × p12[v] + a10[v] × p21[v]) + 6 a03[v] × a10[v] × p30[v] +
       3 a01[v] × a12[v] × p30[v] + a01[v]^3 b10[v] × p41[v] + 5 a01[v]^3 a10[v] × p50[v]) -
    (2 aa21[v] × p03[v] + aa12[v] × p12[v] + a10[v] × p14[v] + 2 aa12[v] × q03[v] +
      3 aa03[v] × q12[v] + a01[v] × q14[v]);
aa05[v__] := aa05[v] = a05[v] + b01[v]^5 p05[v] + a01[v] b01[v]^4 p14[v] +
    a01[v]^2 b01[v]^3 p23[v] + b01[v]^2 (3 b03[v] × p03[v] + a03[v] × p12[v] + a01[v]^3 p32[v]) +
    a01[v] × b01[v] (2 b03[v] × p12[v] + 2 a03[v] × p21[v] + a01[v]^3 p41[v]) +
    a01[v]^2 (b03[v] × p21[v] + 3 a03[v] × p30[v] + a01[v]^3 p50[v]) -
    (aa12[v] × p03[v] + a10[v] × p05[v] + 3 aa03[v] × q03[v] + a01[v] × q05[v]);
```

In[•]:=
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Clear[bb50, bb41, bb32, bb23, bb14, bb05];
bb50[v__] := bb50[v] = b50[v] + 3 b10[v]^2 b30[v] × q03[v] + b10[v]^5 q05[v] +
    (a30[v] b10[v]^2 + 2 a10[v] × b10[v] × b30[v]) q12[v] + a10[v] b10[v]^4 q14[v] +
    (2 a10[v] × a30[v] × b10[v] + a10[v]^2 b30[v]) q21[v] + a10[v]^2 b10[v]^3 q23[v] +
    3 a10[v]^2 a30[v] × q30[v] + a10[v]^3 b10[v]^2 q32[v] + a10[v]^4 b10[v] × q41[v] +
    a10[v]^5 q50[v] - (3 bb30[v] × p30[v] + b10[v] × p50[v] + bb21[v] × q30[v] + b01[v] × q50[v]);
bb41[v__] := bb41[v] = b41[v] + b10[v]^4 (5 b01[v] × q05[v] + a01[v] × q14[v]) +
    2 a10[v] b10[v]^3 (2 b01[v] × q14[v] + a01[v] × q23[v]) + b10[v]^2
     (3 b21[v] × q03[v] + a21[v] × q12[v] + 3 a10[v]^2 (b01[v] × q23[v] + a01[v] × q32[v])) +
    2 b10[v] (a10[v] × b21[v] × q12[v] + a01[v] × b30[v] × q12[v] + a10[v] × a21[v] × q21[v] +
       a01[v] × a30[v] × q21[v] + b01[v] (3 b30[v] × q03[v] + a30[v] × q12[v] + a10[v]^3 q32[v]) +
       2 a01[v] a10[v]^3 q41[v]) + a10[v] (a10[v] × b21[v] × q21[v] + 2 a01[v] × b30[v] × q21[v] +
       3 a10[v] × a21[v] × q30[v] + 6 a01[v] × a30[v] × q30[v] + b01[v]
        (2 b30[v] × q12[v] + 2 a30[v] × q21[v] + a10[v]^3 q41[v]) + 5 a01[v] a10[v]^3 q50[v]) -
    (3 bb30[v] × p21[v] + 2 bb21[v] × p30[v] + b10[v] × p41[v] + bb21[v] × q21[v] +
      2 bb12[v] × q30[v] + b01[v] × q41[v]);
bb32[v__] := bb32[v] = b32[v] + 3 b01[v]^2 b30[v] × q03[v] + a30[v] b01[v]^2 q12[v] +
    2 a10[v] × b01[v] × b21[v] × q12[v] + 2 a01[v] × b01[v] × b30[v] × q12[v] +
    2 a10[v] × a21[v] × b01[v] × q21[v] + 2 a01[v] × a30[v] × b01[v] × q21[v] +
    a10[v]^2 b12[v] × q21[v] + 2 a01[v] × a10[v] × b21[v] × q21[v] + a01[v]^2 b30[v] × q21[v] +
    b10[v]^3 (10 b01[v]^2 q05[v] + 4 a01[v] × b01[v] × q14[v] + a01[v]^2 q23[v]) +
    3 a10[v]^2 a12[v] × q30[v] + 6 a01[v] × a10[v] × a21[v] × q30[v] +
    3 a01[v]^2 a30[v] × q30[v] + a10[v]^3 b01[v]^2 q32[v] +
    b10[v]^2 (3 b12[v] × q03[v] + a12[v] × q12[v] + 6 a10[v] b01[v]^2 q14[v] +
       6 a01[v] × a10[v] × b01[v] × q23[v] + 3 a01[v]^2 a10[v] × q32[v]) +
    4 a01[v] a10[v]^3 b01[v] × q41[v] + b10[v] (3 a10[v]^2 b01[v]^2 q23[v] +
       2 b01[v] (3 b21[v] × q03[v] + a21[v] × q12[v] + 3 a01[v] a10[v]^2 q32[v]) +
       2 (a10[v] × b12[v] × q12[v] + a01[v] × b21[v] × q12[v] + a10[v] × a12[v] × q21[v] +
          a01[v] × a21[v] × q21[v] + 3 a01[v]^2 a10[v]^2 q41[v])) + 10 a01[v]^2 a10[v]^3 q50[v] -
    (3 bb30[v] × p12[v] + 2 bb21[v] × p21[v] + bb12[v] × p30[v] + b10[v] × p32[v] +
      bb21[v] × q12[v] + 2 bb12[v] × q21[v] + 3 bb03[v] × q30[v] + b01[v] × q32[v]);
bb23[v__] := bb23[v] = b23[v] + 6 b01[v] × b10[v] × b12[v] × q03[v] + 3 b01[v]^2 b21[v] × q03[v] +
    10 b01[v]^3 b10[v]^2 q05[v] + a21[v] b01[v]^2 q12[v] + 2 a12[v] × b01[v] × b10[v] × q12[v] +
    a03[v] b10[v]^2 q12[v] + 2 a10[v] × b01[v] × b12[v] × q12[v] +
    2 a01[v] × b10[v] × b12[v] × q12[v] + 2 a01[v] × b01[v] × b21[v] × q12[v] +
    4 a10[v] b01[v]^3 b10[v] × q14[v] + 6 a01[v] b01[v]^2 b10[v]^2 q14[v] +
    2 a10[v] × a12[v] × b01[v] × q21[v] + 2 a01[v] × a21[v] × b01[v] × q21[v] +
    2 a03[v] × a10[v] × b10[v] × q21[v] + 2 a01[v] × a12[v] × b10[v] × q21[v] +
    2 a01[v] × a10[v] × b12[v] × q21[v] + a01[v]^2 b21[v] × q21[v] +
    b03[v] (3 b10[v]^2 q03[v] + 2 a10[v] × b10[v] × q12[v] + a10[v]^2 q21[v]) +
    a10[v]^2 b01[v]^3 q23[v] + 6 a01[v] × a10[v] b01[v]^2 b10[v] × q23[v] +
    3 a01[v]^2 b01[v] b10[v]^2 q23[v] + 3 a03[v] a10[v]^2 q30[v] +
    6 a01[v] × a10[v] × a12[v] × q30[v] + 3 a01[v]^2 a21[v] × q30[v] +
    3 a01[v] a10[v]^2 b01[v]^2 q32[v] + 6 a01[v]^2 a10[v] × b01[v] × b10[v] × q32[v] +
    a01[v]^3 b10[v]^2 q32[v] + 6 a01[v]^2 a10[v]^2 b01[v] × q41[v] +
    4 a01[v]^3 a10[v] × b10[v] × q41[v] + 10 a01[v]^3 a10[v]^2 q50[v] -
    (3 bb30[v] × p03[v] + 2 bb21[v] × p12[v] + bb12[v] × p21[v] + b10[v] × p23[v] +
      bb21[v] × q03[v] + 2 bb12[v] × q12[v] + 3 bb03[v] × q21[v] + b01[v] × q23[v]);
bb14[v__] := bb14[v] = b14[v] + b01[v]^4 (5 b10[v] × q05[v] + a10[v] × q14[v]) +
```

```
    2 a01[v] b01[v]^3 (2 b10[v] × q14[v] + a10[v] × q23[v]) + b01[v]^2
     (3 b12[v] × q03[v] + a12[v] × q12[v] + 3 a01[v]^2 (b10[v] × q23[v] + a10[v] × q32[v])) +
    2 b01[v] (3 b03[v] × b10[v] × q03[v] + a10[v] × b03[v] × q12[v] +
       a03[v] × b10[v] × q12[v] + a01[v] × b12[v] × q12[v] + a03[v] × a10[v] × q21[v] +
       a01[v] × a12[v] × q21[v] + a01[v]^3 b10[v] × q32[v] + 2 a01[v]^3 a10[v] × q41[v]) +
    a01[v] (2 a03[v] × b10[v] × q21[v] + a01[v] × b12[v] × q21[v] +
       2 b03[v] (b10[v] × q12[v] + a10[v] × q21[v]) + 6 a03[v] × a10[v] × q30[v] +
       3 a01[v] × a12[v] × q30[v] + a01[v]^3 b10[v] × q41[v] + 5 a01[v]^3 a10[v] × q50[v]) -
    (2 bb21[v] × p03[v] + bb12[v] × p12[v] + b10[v] × p14[v] + 2 bb12[v] × q03[v] +
      3 bb03[v] × q12[v] + b01[v] × q14[v]);
bb05[v__] := bb05[v] = b05[v] + 3 b01[v]^2 b03[v] × q03[v] + b01[v]^5 q05[v] +
    (a03[v] b01[v]^2 + 2 a01[v] × b01[v] × b03[v]) q12[v] + a01[v] b01[v]^4 q14[v] +
    (2 a01[v] × a03[v] × b01[v] + a01[v]^2 b03[v]) q21[v] + a01[v]^2 b01[v]^3 q23[v] +
    3 a01[v]^2 a03[v] × q30[v] + a01[v]^3 b01[v]^2 q32[v] + a01[v]^4 b01[v] × q41[v] +
    a01[v]^5 q50[v] - (bb12[v] × p03[v] + b10[v] × p05[v] + 3 bb03[v] × q03[v] + b01[v] × q05[v]);
```

4. Twist Coefficients τ_1 and τ_2

```
(*to find τ1 and τ2, use the input of the form v={L,ra,ra2,ra4,rb,rb2,rb4},
where L is the the length, ra and rb are the radii of the two arcs at s=0, ,
ra2 and rb2 are the second-derivatives, and ra4 and rb4 are the four-derivatives*)
```

In[•]:=
```
Clear[twist1, twist2];
twist1[v__] := twist1[v] = (a10[v] (3 b30[v] - a21[v] + b12[v] - 3 a03[v]) -
      b10[v] (3 a30[v] + b21[v] + a12[v] + 3 b03[v])) / 8;
twist2[v__] := twist2[v] = (a10[v] (-2 aa41[v] - 2 aa23[v] -
         10 aa05[v] + 10 bb50[v] + 2 bb32[v] + 2 bb14[v]) -
      b10[v] (10 aa50[v] + 2 aa32[v] + 2 aa14[v] + 2 bb41[v] + 2 bb23[v] + 10 bb05[v])) / 32;
```

Simplified formulas

```
Clear[tau1, denomi0, denomi2, denomi22,
  denomi4, numera0, numeraQ, numeraP, numeraS, numeraT, tau2];
tau1[L_, R_, R2_, S_, S2_] := (R + S)/(8 R S) - L/(8 (R + S - L)) ((S - L)/(R - L) R2 + (R - L)/(S - L) S2);
denomi0[L_, R_, S_] :=
  (512 R^2 S^2 Sqrt[L (R - L) (S - L) (R + S - L)] (2 (R - L) (S - L) - R * S));
denomi22[L_, R_, S_] := 1536 Sqrt[L (R - L) (S - L) (R + S - L)]
   (R - L)^2 (S - L)^2 (2 (R - L) (S - L) - R S) (R + S - L)^2;
denomi2[L_, R_, S_] := 768 Sqrt[L (R - L) (S - L) (R + S - L)] R S
   (R - L) (S - L) (R + S - L) (2 (R - L) (S - L) - R S);
denomi4[L_, R_, S_] := 192 Sqrt[L (R - L) (S - L) (R + S - L)] (R - L) (R + S - L);
numera0[L_, R_, S_] :=
  3 (7 R^2 S^2 (R + S)^2 - 16 R S (R + S)^3 L - 8 R^2 S^2 (R + S) L + 8 (R + S)^4 L^2 +
     16 R S (R + S)^2 L^2 - 16 (R + S)^3 L^3 + 16 R S (R + S) L^3 + 8 (R^2 + S^2) L^4);
numeraP[L_, R_, S_] := L^2 (S - L)^4 (48 R^3 (S - 2 L) + 24 L^2 (S - L)^2 -
     72 L R (2 L^2 - 3 L S + S^2) + R^2 (216 L^2 - 216 S L + 31 S^2));
numeraQ[L_, R_, S_] := -2 R * S L^2 (L - R)^2 (L - S)^2 (32 L^2 - 32 (R + S) L + 17 R * S);
numeraS[L_, R_, S_] := L (S - L)^2 (40 (S - L)^2 L^2 +
     3 R^3 (9 S - 16 L) - 80 R (2 L^2 - 3 L S + S^2) L + 3 R^2 (56 L^2 - 56 L S + 9 S^2));
numeraT[L_, R_, S_] := L^2 R (L - S)^2;
tau2[L_, R_, R2_, R4_, S_, S2_, S4_] := numera0[L, R, S] / denomi0[L, R, S] +
   (numeraP[L, R, S] R2^2 + numeraP[L, S, R] S2^2 + numeraQ[L, R, S] R2 * S2) /
    denomi22[L, R, S] - (numeraS[L, R, S] S R2 + numeraS[L, S, R] R S2) / denomi2[L, R, S] -
   (numeraT[L, R, S] * R4 / denomi4[L, R, S] + numeraT[L, S, R] * S4 / denomi4[L, S, R]);
```

τ_2 when R=S

```
Simplify[numera0[L, R, R] / denomi0[L, R, R]] (*constant term*)
```

Out[•]=
$$\frac{3 \left(2 L^2 - 8 L R + 7 R^2\right)}{128 \sqrt{-L \left(L - 2 R\right) (L - R)^2} R^2}$$

```
Simplify[(numeraS[L, R, R] R) / denomi2[L, R, R]] (* R2+S2*)
```

Out[•]=
$$\frac{L \left(10 L^2 - 40 L R + 27 R^2\right)}{384 \left(L - 2 R\right) \sqrt{-L \left(L - 2 R\right) (L - R)^2} R}$$

In[•]:=
```
Simplify[(numeraP[L, R, R]) / denomi22[L, R, R]] (* R2^2+S2^2*)
```

Out[•]=
$$\frac{L^2 \left(24 L^4 - 192 L^3 R + 456 L^2 R^2 - 384 L R^3 + 79 R^4\right)}{1536 \left(L - 2 R\right)^2 \sqrt{-L \left(L - 2 R\right) (L - R)^2} \left(2 L^2 - 4 L R + R^2\right)}$$

```
Simplify[(numeraQ[L, R, R]) / denomi22[L, R, R]] (* R2*S2 *)
```

Out[•]=
$$-\frac{L^2 R^2 \left(32 L^2-64 L R+17 R^2\right)}{768 (L-2 R)^2 \sqrt{-L (L-2 R) (L-R)^2} \left(2 L^2-4 L R+R^2\right)}$$

In[•]:=
```
Simplify[numeraT[L, R, R] / denomi4[L, R, R]] (* R4 + S4*)
```

Out[•]=
$$\frac{L^3 (L-R)^3 R}{192 \left(-L (L-2 R) (L-R)^2\right)^{3/2}}$$

In[•]:=
```
Clear[tau2RR];
tau2RR[L_, R_, R2_, R4_, S2_, S4_] :=
  1/Sqrt[L (2 R - L) (L - R)^2] ((3 (2 L^2 - 8 L R + 7 R^2))/(128 R^2) - (L (10 L^2 - 40 L R + 27 R^2) (R2 + S2))/(384 (2 R - L) R) +
    (L^2 (24 L^4 - 192 L^3 R + 456 L^2 R^2 - 384 L R^3 + 79 R^4) (R2^2 + S2^2) - 2 L^2 R^2 (32 L^2 - 64 L R + 17 R^2)
      R2 S2) / (1536 (L - 2 R)^2 (2 L^2 - 4 L R + R^2)) - (L^2 (R - L) R (R4 + S4))/(192 (2 R - L)));
```

Integrability test: locally analytically integrable iff convergent Birkhoff Normal form. At $\lambda^4 = 1$: it is necessary to have $c_{03} = 0$

In[•]:=
```
Clear[laiTest];
laiTest[v__] :=
  laiTest[v] = a30[v] + I b30[v] + I a21[v] - b21[v] - a12[v] - I b12[v] - I a03[v] + b03[v];
(*see the definition of c03*)
```

In[•]:=
```
Simplify[laiTest[{L, R, R2, R4, S, S2, S4}], R > L && S > L && L > 0]
```

Out[•]=

$$\frac{1}{3\,R^4\,S^3}\left(-\frac{1}{-L+S}\,i\,(-L+R)\,(-L+R+S)\,\left(4\,L^5-3\,R^2\,S^3-12\,L^4\,(R+S)+\right.\right.$$
$$6\,L\,R\,S^2\,(2\,R+S)-2\,L^2\,S\,(9\,R^2+12\,R\,S+2\,S^2)+L^3\,(30\,R\,S+12\,S^2+R^2\,(9-S\,S2))))+$$
$$3\,i\,(4\,L^6+R^3\,S^2\,(R+S)-4\,L^5\,(5\,R+3\,S)-L\,R^2\,S\,(8\,R^2+16\,R\,S+7\,S^2)+$$
$$L^2\,R\,(44\,R^2\,S+44\,R\,S^2+10\,S^3+R^3\,(9-S\,S2))+L^4\,(50\,R\,S+12\,S^2+R^2\,(37-S\,S2))+$$
$$2\,L^3\,(-37\,R^2\,S-20\,R\,S^2-2\,S^3+R^3\,(-15+S\,S2)))-$$
$$3\,\sqrt{\frac{L\,(-L+R)\,(-L+R+S)}{-L+S}}\,(4\,L^5-4\,R^2\,S^2\,(R+S)-4\,L^4\,(4\,R+3\,S)+L\,R\,S\,(13\,R^2+25\,R\,S+8\,S^2)+$$
$$L^3\,(40\,R\,S+12\,S^2+R^2\,(21-S\,S2))+L^2\,(-42\,R^2\,S-32\,R\,S^2-4\,S^3+R^3\,(-9+S\,S2)))-$$
$$L^{3/2}\,R\,\sqrt{\frac{(-L+R)\,(-L+R+S)}{-L+S}}\,(4\,L^4\,R2-12\,L^3\,R2\,S+3\,R\,S\,(R+S)+$$
$$L^2\,(12\,R2\,S^2+R\,(3+S\,S2))-L\,(6\,R\,S+4\,R2\,S^3+R^2\,(3+S\,S2)))+$$
$$3\,i\,R\,(4\,L^6\,R2+R^2\,S^2\,(R+S)-4\,L^5\,R2\,(R+3\,S)-L\,R\,S\,(2\,R^2+4\,R\,S+S^2)+$$
$$L^2\,R\,(8\,R\,S+2\,S^2\,(2+R2\,S)+R^2\,(3+S\,S2))-$$
$$2\,L^3\,(2\,R2\,S^3+R\,S\,(3+4\,R2\,S)+R^2\,(3+S\,S2))+L^4\,(12\,R2\,S^2+R\,(3+10\,R2\,S+S\,S2)))-$$
$$(i\,L\,R\,(-L+S)\,(4\,L^6\,R2-12\,L^5\,R2\,(R+S)+R^2\,(6\,R^2\,S+S^3+R\,S^2\,(4+R2\,S)+R^3\,(3+S\,S2))-$$
$$L\,R\,(S^3+4\,R\,S^2\,(2+R2\,S)+2\,R^2\,S\,(9+2\,R2\,S)+4\,R^3\,(3+S\,S2))+$$
$$2\,L^2\,R\,(S^2\,(2+3\,R2\,S)+R\,S\,(9+8\,R2\,S)+3\,R^2\,(3+R2\,S+S\,S2))-$$
$$2\,L^3\,(2\,R^3\,R2+2\,R2\,S^3+3\,R\,S\,(1+4\,R2\,S)+2\,R^2\,(3+6\,R2\,S+S\,S2))+$$
$$L^4\,(12\,R^2\,R2+12\,R2\,S^2+R\,(3+30\,R2\,S+S\,S2))))/((-L+R)\,(-L+R+S))+$$
$$\sqrt{\frac{L\,(-L+S)}{(-L+R)\,(-L+R+S)}}\,(4\,L^6+3\,R^3\,S\,(R+S)^2-12\,L^5\,(2\,R+S)+$$
$$L^2\,R\,(68\,R\,S^2+12\,S^3+R^2\,S\,(99-R2\,S)+R^3\,(39-3\,S\,S2))+L^4\,(60\,R\,S+12\,S^2+R^2\,(57-S\,S2))+$$
$$L\,R^2\,(-36\,R^2\,S-11\,S^3+R\,S^2\,(-38+R2\,S)+R^3\,(-9+S\,S2))+$$
$$L^3\,(-114\,R^2\,S-48\,R\,S^2-4\,S^3+R^3\,(-67+3\,S\,S2)))+$$
$$3\,R\,\sqrt{\frac{L\,(-L+S)}{(-L+R)\,(-L+R+S)}}\,(4\,L^6\,R2+R^2\,S\,(R+S)^2-4\,L^5\,R2\,(2\,R+3\,S)+$$
$$L^2\,R\,(4\,S^2\,(1+R2\,S)+R\,S\,(13+5\,R2\,S)+3\,R^2\,(3+S\,S2))-$$
$$L\,R\,(8\,R^2\,S+S^3+R\,S^2\,(6+R2\,S)+R^3\,(3+S\,S2))+L^4\,(4\,R^2\,R2+12\,R2\,S^2+R\,(3+20\,R2\,S+S\,S2))-$$
$$\left.L^3\,(4\,R2\,S^3+2\,R\,S\,(3+8\,R2\,S)+R^2\,(9+8\,R2\,S+3\,S\,S2))\right)$$

```
Simplify[laiTest[{L, R, 0, 0, S, 0, 0}], L > 0 && S > L && R > L] (*asymmetric lemon*)
```

Out[•]=

$$\left(\left(-2\,L^{7/2}+4\,L^{5/2}\,(R+S)+2\,\sqrt{L}\,R\,S\,(R+S)+\right.\right.$$
$$2\,i\,L^2\,\sqrt{(-L+R)\,(-L+S)\,(-L+R+S)}+i\,R\,S\,\sqrt{(-L+R)\,(-L+S)\,(-L+R+S)}-$$
$$\left.2\,i\,L\,(R+S)\,\sqrt{(-L+R)\,(-L+S)\,(-L+R+S)}-2\,L^{3/2}\,(R^2+3\,R\,S+S^2)\right)$$
$$\left.(8\,L^4+9\,R^2\,S\,(R+S)-16\,L^3\,(2\,R+S)+8\,L^2\,(6\,R^2+6\,R\,S+S^2)-8\,L\,R\,(3\,R^2+6\,R\,S+2\,S^2))\right)/$$
$$\left(3\,R^4\,S^3\,\sqrt{(-L+R)\,(-L+S)\,(-L+R+S)}\right)$$

```
Simplify[((((-2 L^(7/2) + 4 L^(5/2) (R + S) + 2 Sqrt[L] R S (R + S) +
          2 I L^2 Sqrt[(-L + R) (-L + S) (-L + R + S)] + I R S Sqrt[(-L + R) (-L + S) (-L + R + S)] -
          2 I L (R + S) Sqrt[(-L + R) (-L + S) (-L + R + S)] - 2 L^(3/2) (R^2 + 3 R S + S^2))
        (8 L^4 + 9 R^2 S (R + S) - 16 L^3 (2 R + S) + 8 L^2 (6 R^2 + 6 R S + S^2) - 8 L R (3 R^2 + 6 R S + 2 S^2)))) /
      (3 R^4 S^3 Sqrt[(-L + R) (-L + S) (-L + R + S)])) /. L -> (R + S - (R^2 + S^2)^(1/2))/2,
 R > 0 && S > 0] (*one case of resonance λ^4=1 for asymmetric
  lemon*)
```

Out[•]= $\frac{1}{R} - \frac{1}{S}$

In[•]:=

```
Simplify[((((-2 L^(7/2) + 4 L^(5/2) (R + S) + 2 Sqrt[L] R S (R + S) +
          2 I L^2 Sqrt[(-L + R) (-L + S) (-L + R + S)] + I R S Sqrt[(-L + R) (-L + S) (-L + R + S)] -
          2 I L (R + S) Sqrt[(-L + R) (-L + S) (-L + R + S)] - 2 L^(3/2) (R^2 + 3 R S + S^2))
        (8 L^4 + 9 R^2 S (R + S) - 16 L^3 (2 R + S) + 8 L^2 (6 R^2 + 6 R S + S^2) - 8 L R (3 R^2 + 6 R S + 2 S^2)))) /
      (3 R^4 S^3 Sqrt[(-L + R) (-L + S) (-L + R + S)])) /. L -> (R + S + (R^2 + S^2)^(1/2))/2,
 R > 0 && S > 0] (*the other case of resonance λ^4=1 for
  asymmetric
  lemon*)
```

Out[•]= $\frac{1}{R} - \frac{1}{S}$