Research Statement

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1. Introduction

My research is in the area of dynamical systems and ergodic theory. I am interested in the study of dynamical systems of geometric origin, including dynamical billiards and geodesic flows. Researchers in this field study how the dynamical properties of such systems, which happen in the phase spaces, are determined by the geometric structure of the underlining configuration spaces, and to what extent their dynamical properties characterize the intrinsic properties of the geometric structure. In this section, we will give an overview of our results. More details will be provided in the following sections.

1.1. **Homoclinic intersections for dynamics of geometric origin.**

Let M be a closed manifold, $f : M \to M$ be a diffeomorphism on M, and p be a hyperbolic periodic point of f. Associated to p are the *stable* and *unstable manifolds* $W^{s,u}(p)$ of p. A point in the intersection $W^s(p) \cap W^u(p) \backslash \{p\}$ is called a *homoclinic point* of p. In [52], Poincaré discovered that every transverse homoclinic intersection is accumulated by other homoclinic intersections, and the self-accumulation of homoclinic intersections leads to complicated dynamical behaviors. Moreover, transverse homoclinic intersections *persist* under small perturbations. This led Poincaré to conjecture that, *generically*, transverse homoclinic intersections exist for *every* hyperbolic periodic point. We have several results in this direction. More precisely, we have proved the existence of transverse homoclinic intersections for *every* hyperbolic periodic point of the following systems:

(1) [68] dynamical billiards on a generic convex domain $Q \subset \mathbb{R}^2$ with smooth boundary;
(2) [69] geodesic flow on a generic convex sphere (S^2, g);
(3) [72] dynamical billiards on a generic geodesically convex domain Q on a convex sphere (S^2, g);
(4) [73] a generic symplectic diffeomorphism close to direct/skew products of symplectic Anosov diffeomorphisms with area-preserving diffeomorphisms.

Moreover, we have proved a Kupka–Smale type property for a generic convex billiards $Q \subset S^2$: every periodic point is non-degenerate, and if any two invariant manifolds intersect, they intersect transversely at some of their intersections.

1.2. **Hyperbolicity of asymmetric lemon billiards.**

Let $Q \subset \mathbb{R}^2$ be a bounded and connected domain with (piecewise) smooth boundary, and consider the dynamical billiards on the domain Q. In [63] Sinaĭ introduced a class of dynamical billiards with dispersing boundary components and proved that they are hyperbolic and ergodic. In [13], Bunimovich observed that there are hyperbolic billiards without any dispersing component. See Section 3 for more background on the development of hyperbolic billiards. In [19] we constructed a new family of dynamical billiards. More precisely, let $Q(b, r, R) = D(O_1, r) \cap D(O_2, R)$ be the intersection of two disks $D(O_1, r)$ and $D(O_2, R)$, where $b = |O_1O_2|$ measures the distance between the two centers. The dynamical billiard on $Q(b, r, R)$ is called an asymmetric lemon billiard since the shape of the table resembles the shape of an asymmetric lemon.

(1) In [19] we showed that the asymmetric lemon billiards have a wide spectrum of dynamical behaviors: they vary from being completely integrable to the coexistence of chaotic seas and elliptic islands.
(2) In [16] we proved that if the circular angle $\angle\Gamma_r > 5\pi/3$, then the asymmetric lemon billiard $Q(b, r, R)$ is hyperbolic as long as the radius R is larger enough.
(3) In [35] we proved that as long as $\angle\Gamma_r > \pi$, the asymmetric lemon billiard $Q(b, r, R)$ is hyperbolic for sufficiently large radius R.

In particular, these hyperbolic asymmetric lemon billiards have positive metric entropy. It is worth pointing out the assumption $\angle\Gamma_r > \pi$ in [35] is necessary. In fact, in the case that $\angle\Gamma_r < \pi$, there are elliptic islands for the dynamical billiards on $Q(b, r, R)$ no matter how large R is.

1.3. **Twist coefficients and nonlinear stability.** Let $U \subset \mathbb{R}^2$ be an open neighborhood of the origin $P = (0,0)$, $f : U \to \mathbb{R}^2$ be a symplectic embedding. Suppose P is an elliptic fixed point of f and is non-resonant up to order $2N+2$. That is, $\lambda^k \neq 1$ for each $1 \leq k \leq 2N+2$, where λ is an eigenvalue of the matrix $D_P f$. Moser's Twist Mapping Theorem [48] states that the fixed point P is nonlinearly stable as long as one of its first N twist coefficients is nonzero.

Let $Q \subset \mathbb{R}^2$ be a convex domain, and F be the billiard map of the dynamical billiards on Q. Suppose there exists an elliptic periodic orbit $O_2^e(Q)$ of period 2. In [38] Kamphorst and Pinto de Carvalho obtained an explicit formula for the first twist coefficient τ_1 of the orbit $O_2^e(Q)$. In [36] we obtained an explicit formula for the second twist coefficient τ_2 for the orbit $O_2^e(Q)$ in terms of geometric characters of the billiard table: the width L of the table Q along the orbit $O_2^e(Q)$ and the radius of curvature $R(s)$ of Q. As an application, we proved the nonlinearity stability and hence the existence of elliptic islands for a variety of dynamical billiards, including lemon billiards, asymmetric lemons and several classes of aspherical lens billiards.

1.4. **Topological entropy of lemon billiards.** Let $Q(b) := Q(b,1,1)$ be the special case of the tables in Section 1.2 where both disks are unit disk. The shape of the domain $Q(b)$ resembles the shape of a (symmetric) lemon. The lemon billiards have been studied by Heller and Tomsovic [33], in which they demonstrated a clear connection between the classical mechanics and the quantum mechanics: the eigenstates of the quantum billiards are large only at where the periodic trajectories of the classical billiards go. Numerical results in [19] suggest that

Conjecture 1.1. For each $b \in (0,2)$, the lemon billiard on $Q(b)$ has positive topological entropy.

In [37] we showed that for a range of parameters b, there exists a pair of hyperbolic periodic orbits of period 6 and a pair of elliptic periodic orbits of period 6 for the billiard maps F_b on the table $Q(b)$. Moreover, the elliptic periodic orbits are nonlinearly stable and the hyperbolic periodic orbits admit crossing heteroclinic and homoclinic intersections. From this we concluded that such lemon billiards have positive topological entropy.

1.5. **Metric properties of partially hyperbolic systems.** Let $\mathrm{PH}^r(M)$ be the set of partially hyperbolic diffeomorphism on M and $\mathrm{PH}^r_m(M)$ be the subset of volume-preserving ones. A map f is said to be topologically transitive if the orbit $\{f^n x : n \in \mathbb{Z}\}$ is dense on M for some point x. Let $\mathrm{Tran}(f) \subset M$ be the set of points $x \in M$ with dense orbits. In [10] Brin proved that if $f \in \mathrm{PH}^r(M)$ is accessible and nonwandering, then f is transitive. In the case $f \in \mathrm{PH}^r_m(M)$, his argument actually implies $\mathrm{Tran}(f)$ is of full measure. Some generalizations are given in [1, 17]. In [70, 71] we obtained some new generalizations of Brin's result.

(1) [70] If $f \in \mathrm{PH}^r(M)$ is accessible and admits an ACIP, then f is transitive, and the set $\mathrm{Tran}(f)$ is essentially dense on M. If f is accessible and center bunched, then either f admits no ACIP, or there is a smooth invariant and ergodic measure for f.
(2) [71] If $f \in \mathrm{PH}^r(M)$ is accessible, then either f is completely dissipative, or f is transitive and the set $\mathrm{Tran}(f)$ is essentially dense on M. If f is accessible and center bunched, then either f is completely dissipative, or the volume is an ergodic measure for f.

Here ACIP is short for Absolutely Continuous Invariant Probability measure. A subset $E \subset M$ is said to be essentially dense on M if $m(E \cap U) > 0$ for any open subset $U \subset M$. A measure μ is said to be ergodic for f if $\mu(E) \in \{0,1\}$ for every f-invariant subset $E \subset M$. Note that in [71] an ergodic measure is not assumed to be f-invariant.

2. Homoclinic intersections in dynamics of geometric origin

The qualitative theory of dynamical systems originated in Poincaré's work on celestial mechanics. Poincaré discovered a robust mechanism for solutions with chaotic behaviors: transverse homoclinic intersections. In [52] Poincaré made the following observation for a generic dynamical system:

Conjecture 2.1. Let M be a closed manifold, μ be a smooth measure on M, $r \geq 1$. Then for a generic diffeomorphism $f \in \mathrm{Diff}^r_\mu(M)$, (P1) periodic points are dense in the space M;

(P2) transverse homoclinic intersections exist for every hyperbolic periodic point p.

The above conjecture is closely related to the following two results/problems about perturbations:

Closing Lemma. Let $x \in M$ be a point with $f^{n_i}x \to x$ for some sequence $n_i \to +\infty$. There is a C^r-small perturbation g of f such that x is a periodic point of g.

Connecting Lemma. Let x and y be two points in M that are on different orbits of f, while the distance $d(f^{n_i}x, f^{-m_i}y)$ goes to zero for some sequences n_i, $m_i \to +\infty$. There is a C^r-small perturbation g of f such that y is on the forward orbit of x under g.

For $r = 1$, the closing lemma was proved by Pugh [53], and the connecting lemma was proved by Hayashi [32]. The property (P1) was proved by Pugh [54], (P2) was proved by Takens [64], and a stronger version was proved by Xia [66]. For $r \geq 2$, most of the results are on surfaces: the C^∞ closing lemma and the property (P1) was proved in [22, 29], see also [34, 4, 58] for related results. In [51, 49, 42] it is proved that (P2) holds for C^r-generic area-preserving surface diffeomorphisms, see also [50, 67] for related results.

Geodesic flows. Let M be a closed manifold and g a Riemannian metric on M. For each $(x, v) \in TM$, let $\gamma(t) := \gamma(t; x, v)$ be the geodesic starting from $x \in M$ with an initial direction $v \in T_xM$. This define a map $\phi_t : TM \to TM$, $(x, v) \mapsto (\gamma(t), \dot\gamma(t))$. Letting t vary, we obtain a flow ϕ_t on TM, which is the so-called *geodesic flow*. Since the geodesic flow preserves the length of the tangent vectors, we can consider the restriction of the geodesic flow ϕ_t on the unit tangent bundle SM. Dynamical billiards can be viewed as geodesic flows on manifolds with boundary: the trajectory follows geodesics in the interior $\mathrm{int}M$ and reflects on the boundary ∂M elastically.

A geodesic $\gamma(t)$ is *closed* if there exists $T > 0$ such that $\gamma(t + T) = \gamma(t)$ for all $t \in \mathbb{R}$. Closed orbits of the geodesic flow are exactly closed geodesics on M. Closing and connecting problems for geodesic flows on Riemannian manifolds were raised by Pugh and Robinson in [55]. Naturally, one may try to create closed geodesics on M using the closing/connecting techniques. However, there is an essential difficulty. In the study of dynamical properties of the geodesic flows, one needs to distinguish two types of returns for a geodesic γ: the *topological returns* when the positions $\gamma(t_i)$ of the geodesic are close to the initial position $\gamma(t_0)$; and the *dynamical returns* when $(\gamma(t_i), \dot\gamma(t_i))$, the positions and the directions, are close to the initial status $(\gamma(t_0), \dot\gamma(t_0))$.

Dynamical obstructions. One *cannot* perturb the geodesic flow directly when closing a dynamically recurrent geodesic: any perturbation of the geodesic flow has to be done through a *deformation of the metric* on the manifold M. However, a local deformation of the Riemannian metric on M changes the dynamics of all geodesics passing through the deformed region. Therefore, such perturbations will involve all the topological returns of the geodesic, making the closing and connecting techniques in [53, 32] not applicable for Riemannian geodesic flows. In fact, the closing and connecting problems for geodesic flows are *still open* even in the C^1 category (that is, among C^2 metrics), more than 50 years after Pugh's proof of C^1 Closing Lemma. This obstruction also exists in the study of dynamical billiards and similar mechanic systems.

2.1. **Planar convex billiards.** Let $Q \subset \mathbb{R}^2$ be a connected compact domain with a (piecewise) smooth boundary $\Gamma = \partial Q$. Consider a point particle at $q \in Q$ moving along a straight line with initial velocity $v \in T_qQ = \mathbb{R}^2$. Upon hitting the boundary Γ, the particle makes an elastic reflection and continues moving inside Q. This defines a flow ϕ_t on the unit tangent bundle T_1Q of Q, the so-call *billiard flow* on Q. The billiard flow ϕ_t on Q admits a natural cross-section $M = \Gamma \times (0, \pi)$, where $s \in \Gamma$ is the arc-length parameter of Γ, and the second coordinate $\theta \in (0, \pi)$ is the angle measured from the tangent vector $\dot\Gamma(s)$ to the direction of the moving particle bouncing off Γ. Then the *billiard map* $F : M \to M$ is defined as the Poincaré map of the flow ϕ_t with respect to the cross-section M.

It is proved in [28, 23, 24] that, if a hyperbolic periodic point of the billiard map admits a tangent homoclinic intersection, then one can deform the table slightly to get a transversal homoclinic intersection. However, it has been an *open* question whether there exists any homoclinic intersection that one can start with. We proved in [68] that

Theorem 2.2. *Let $r \geq 2$. For a C^r generic convex billiard table $Q \subset \mathbb{R}^2$, there exist transverse homoclinic intersections for each hyperbolic periodic point p of the billiard map on Q.*

Our proof is quite different from the classical local perturbation techniques, but utilizes a global theory, the Carathéodory *prime-end compactification* developed by Mather [44], to analyze what can happen to the invariant subsets of a typical surface diffeomorphism. This confirms Part (P2) of Conjecture 2.1 for billiard systems that transversal homoclinic intersections do exist for *every* hyperbolic periodic point of a C^r-generic convex billiard $Q \subset \mathbb{R}^2$.

2.2. **Geodesic flows on convex spheres.** Recall that a Riemannian metric g on S^2 is said to be *convex*, if the Gaussian curvature induced by g is positive. The simple geometry/topology on S^2 may suggest that the dynamics of geodesic flows on S^2 should be "simple". Surprisingly, it is proved in [40] that there exists a convex metric on the sphere S^2 for which the induced geodesic flow has positive topological entropy. A simpler construction is given by Donnay in [27]. In [21] it is proved that there is a C^2 open and dense subset of Riemannian metrics on S^2 whose geodesic flows have positive topological entropy. The C^∞-denseness of positive topological entropy is proved in [41] among the set of convex metrics on S^2. In [69] we proved that

Theorem 2.3. *There is a residual subset $\mathcal{R}$ among the set of convex Riemannian metrics on S^2, such that for each $g \in \mathcal{R}$, for each hyperbolic closed geodesic γ of the geodesic flow ϕ_g^t induced by g, there exist transverse homoclinic intersections for γ.*

The assumption on the convexity of the metric on S^2 is essential in [69]: it enables us to reduce the dimension of the system by 1, that is, from a 3D geodesic flow to a 2D annulus diffeomorphism. Recently, Contreras and Oliveira [20] obtained the generic existence of homoclinic intersections for geodesic flows on general metrics on S^2 and on general surfaces.

2.3. **Convex billiards on convex spheres.** Dynamical billiards on *curved surfaces* are related to the study of quantum magnetic confinement of non-planar 2D electron gases in semiconductors, where the effect of *varying the curvature* of the surface corresponds to a change in the *potential energy* of the system. We prove the existence of transverse homoclinic intersections for billiards on convex surfaces. More precisely, let g be a convex Riemannian metric on S^2. A subset $Q \subset S^2$ is said to be geodesically convex, if the shortest geodesic connecting any two points in Q is contained in Q. On a convex domain $Q \subset S^2$, the billiard map can be defined analogously: the particle moves along a geodesic in Q, and reflects elastically upon hitting the boundary ∂Q. In [72] we proved that

Theorem 2.4. *For a C^r generic convex domain $Q \subset S^2$, $W^s(p) \pitchfork W^u(p) \neq \varnothing$ for each hyperbolic periodic point p of the billiard map on Q.*

Beside the prime-end compactification theory, a new ingredient is Herman's Last Geometric Theorem on Diophantine invariant curves [30], from which we conclude that every elliptic periodic orbit is nonlinearly stable for a dense subset of convex billiards. Then we can show that homoclinic intersections exist for each hyperbolic periodic point for a dense subset of convex billiards. The dynamical nature of the transverse homoclinic intersections guarantees that it actually holds for generic convex billiards.

Problem 2.5. For a generic convex billiard table $Q \subset \mathbb{R}^2$, the set of (hyperbolic) periodic points is dense on the phase space of the billiard system on Q.

A stronger statement might holds that periodic points are dense for *every* planar convex billiard. However, this stronger version relies heavily on the planar geometry, since Gutkin has constructed smooth convex billiards on the round sphere (S^2, g_0) for which periodic points are not dense. Here g_0 is the round metric induced from the embedding $S^2 \subset \mathbb{R}^3$.

2.4. **Transverse homoclinic intersection for symplectic maps.** Let (M, ω) be a closed symplectic manifold, f be a symplectic Anosov diffeomorphism on M, g be a diffeomorphism on a closed surface S preserving an area form μ. Let $M' = M \times S$, and $\omega' = \omega \oplus \mu$. Then the product system $f \times g$ is naturally a symplectic diffeomorphism on M' and is partially hyperbolic if the expansion and contraction of g are weaker than those of f. More generally, let $\phi : M \to \mathrm{Diff}_\mu^r(S)$ be a C^r smooth cocycle over M,

$(f,\phi): M\times S\to M\times S$, $(x,s)\mapsto(fx,\phi(x)(s))$ be the induced skew product of ϕ over f. We proved in [73] the existence of transverse homoclinic intersections for generic partially hyperbolic systems that are close to the above skew products.

Theorem 2.6. *Let (f,ϕ) be the skew product of ϕ over f that is partially hyperbolic and 4-normally hyperbolic. Then there is a C^1-open neighborhood $\mathcal{U}\subset PH^r_{\omega'}(M')$ of (f,ϕ), such that for a C^r-generic map $\Phi\in\mathcal{U}$, there are transverse homoclinic intersections for each hyperbolic periodic point p of Φ.*

3. Hyperbolicity of asymmetric lemon billiards

Boltzmann made the famous *Ergodic Hypothesis* in statistical mechanics about the equilibrium states of ideal gases. However, the validation of the ergodic hypothesis for mechanical systems presented a huge challenge. Sinaĭ reformulated this hypothesis in [61] in terms of billiard systems. Suppose there are two round disks moving on $\mathbb{T}^2$ and bouncing off each other elastically. Then one can eliminate the center of mass as a configuration variable, and reduce the above system to dispersing billiards. In his seminal paper [63] Sinaĭ proved that dispersing billiards are hyperbolic, ergodic, and mixing. Later on, these billiards are called Sinaĭ billiards.

Let $Q\subset\mathbb{R}^2$ be a connected and compact domain with piecewise smooth boundary. Then a boundary arc $\gamma\subset\partial Q$ is said to be *dispersing* if every parallel beam in Q becomes divergent right after one reflection with γ. In comparison, an arc γ is said to be *focusing* if every parallel beam in Q become convergent right after the reflection on γ. Bunimovich constructed in [12, 13] hyperbolic billiards with circular focusing components. In [65] Wojtkowski constructed hyperbolic billiards that allow more general focusing components. Further generalizations have been obtained by Markarian [43], Donnay [26] and Bunimovich [14].

A common feature of the constructions of hyperbolic billiards is that the regular components need to be placed *sufficiently far away* from every focusing component. Generally speaking, the sufficient separation condition is a necessary requirement for constructing hyperbolic billiards with focusing components, see [15]. Our results show that hyperbolic billiards may exist even when there is *no* separation at all. More precisely, pick two points in the plane with distance b, draw a disk around each point with radii $r\le R$, respectively. These two disks intersect each other when $R-r<b<R+r$, and their intersection $Q=Q(b,r,R)$ resembles an *asymmetric lemon*. See Fig. 1. Up to a scaling, one can assume $r=1$ and $R\ge 1$. For clarity, we will fix two corners A and B on the circle $C(O_r,r)$, and let b vary according to R such that the two circles $C(O_r,r)$ and $C(O_R,R)$ intersect exactly at A and B. Then it is clear that b is given by $b_R=(R^2-|AB|^2/4)^{1/2}-(1-|AB|^2/4)^{1/2}$, and we will use the notation $Q(R)=Q(b_R,1,R)$ for short.

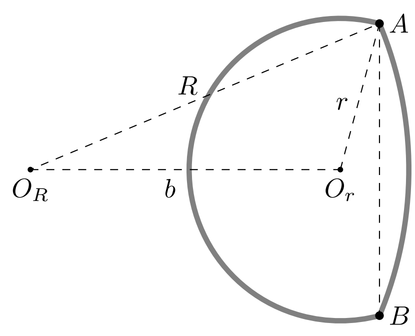

Figure 1. An asymmetric lemon table $Q(b,r,R)$.

Theorem 3.1 ([16])**.** *Assume the length of the chord connecting A and B satisfies $|AB|<1$. Then there exists $R_*>1$ such that the asymmetric lemon $Q(R)$ is hyperbolic for each $R>R_*$.*

The main tool in our proof is the *continued fraction representation* of the curvatures of the wavefronts under the billiard map. We have developed several techniques in the the study of continued fractions to reduce greatly the expressions of the curvatures of the wavefront upon its return moment for a carefully chosen subset $X \subset M$. Then we showed that the return map F_X preserves a strictly invariant cone field on the subset X. Then it follows that the billiard map F preserves an eventually strictly invariant cone field on the whole phase space M. Therefore, the billiard map F is hyperbolic.

It is conjectured in [16] that the condition $|AB| < 1$ can be replaced by the condition

$$b^2 + r^2 < R^2. \tag{3.1}$$

Denote by Γ_r and Γ_R the two boundary components of the table $Q(R)$. Note that $|AB| < 1$ is equivalent to $\angle\Gamma_r > 5\pi/3$, while the assumption (3.1) is equivalent to that $\angle\Gamma_r > \pi$.

Theorem 3.2 ([35])**.** *Suppose $\angle\Gamma_r > \pi$. Then there exists $R_* > 1$ such that the asymmetric lemon $Q(R)$ is hyperbolic for each $R > R_*$.*

Note that the limit $\lim_{R\to\infty} Q(R)$ is the domain Q_* obtained by cutting the disk $D(O_r, r)$ along the chord AB, which is Bunimovich's *one-petal table* [13]. The asymmetric lemon $Q(R)$ can be viewed as a geometric deformation of the table Q_* along the chord AB by circular arcs of small curvature. Then Theorem 3.2 can be interpreted that the one-petal table Q_* is robustly hyperbolic under circular deformations. It is not clear how far we can push this interpretation: are the dynamical billiards hyperbolic if we replace the chord AB by other curves with small curvature?

Recall that an invariant measure μ is said to be ergodic if $\mu(E) = 0$ or 1 for any invariant subset E. One of our projects is to prove the ergodicity of hyperbolic asymmetric lemon billiards:

Problem 3.3. Suppose $\angle\Gamma_r > \pi$. Then there exists $R_* > 0$ such that for all $R \geq R_*$, the asymmetric lemon billiard system on $Q(R)$ is ergodic.

4. Twist coefficients and nonlinear stability for billiards

Let $U \subset \mathbb{R}^2$ be an open neighborhood of the origin $P = (0,0)$, $f : U \to \mathbb{R}^2$ be a symplectic embedding that fixes P. Suppose P is an elliptic fixed point and its eigenvalue λ of the matrix $D_P f$ is non-resonant up to order $2N + 2$. Then there is a symplectic coordinate transformation h on a neighborhood $V \subset U$ of P such that $g = h^{-1} \circ f \circ h$ is of the following form (Birkhoff Normal Form)

$$g\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} \cos\Theta(r^2) & -\sin\Theta(r^2) \\ \sin\Theta(r^2) & \cos\Theta(r^2) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + O(r^{2N+2}), \tag{4.1}$$

where $r^2 = x^2 + y^2$, $\Theta(r^2) = \theta + \tau_1 r^2 + \tau_2 r^4 + \cdots + \tau_N r^{2N}$, and θ is the argument of the eigenvalue λ. The function $\Theta(r^2)$ measures the amount of rotations of points around the fixed point P, whose coefficients τ_k, $1 \leq k \leq N$, are called the twist coefficients of f at P. See [48] for more details.

The existence of Birkhoff Normal Form plays an important role in the study of nonlinear stability and local integrability of elliptic fixed points. Recall that a fixed point P of a map $f : U \to \mathbb{R}^2$ is said to be nonlinearly stable if there is a nesting sequence of f-invariant neighborhoods U_n of P, $n \geq 1$ whose boundaries $S_n = \partial U_n$ are invariant circles, such that $\bigcap_{n\geq 1} U_n = \{P\}$. An elliptic fixed point being nonresonant alone does not guarantee it is nonlinearly stable. In fact, Anosov and Katok [3] constructed an ergodic symplectic diffeomorphism f of the closed unit disk $\bar{D}$ with a nonresonant elliptic fixed point at the origin $P \in \bar{D}$. On the other hand, Moser's Twist Mapping Theorem [48] states that if the elliptic fixed point P is not $(2N+2)$-resonant, then the fixed point P is nonlinearly stable as long as one of its first N twist coefficients is nonzero. See also [45, 46] for applications of the twist coefficients in the study of the stable periodic orbits.

Despite the importance of the twist coefficients, there are only a few cases that explicit formulas for the twist coefficients can be obtained, and all of them are limited to the coefficient τ_1, see [47, 38]. In [36] we obtained an explicit formula of the coefficient τ_2 of some periodic orbits of dynamical billiards in terms of the geometric characteristics of the billiard table. More precisely, let $L > 0$, $a(t)$ and $b(t)$ be two smooth functions with $a(0) = b(0) = 0$ and $a'(0) = b'(0) = 0$, and consider the smooth arcs given by $\gamma_0(t) = (a(t), t)$ and $\gamma_1(t) = (L - b(t), t)$. Let $R_j(s)$ be the radius of curvature of the arc γ_j with respect to the arc-length parameter, $j = 1, 2$, and $\mathcal{O}_2(a, b, L) = \{P, F(P)\}$ be the periodic orbit of period 2 that bounces back and forth between the two vertices $(0,0)$ and $(L, 0)$. In [36] we started with the symmetric case that $a(t) = b(t)$

is an even function, and denote $R(s)$ the radius of curvature of the two arcs. We will use $R = R(0)$ for short. It follows that the $\mathcal{O}_2(a, a, L)$ is elliptic if $0 < \frac{L}{R} < 2$. We will need the following nonresonance assumptions:

(A1) $\lambda^4 \neq 1$, or equally, $\frac{L}{R} \in (0,2)\backslash\{1\}$;
(A2) $\lambda^6 \neq 1$, or equally, $\frac{L}{R} \in (0,2)\backslash\{\frac{1}{2}, \frac{3}{2}\}$.

Theorem 4.1. *Assuming (A1), the first twist coefficient $\tau_1(F,P)$ of the one-step billiard map F at P is given by*

$$\tau_1(F,P) = \frac{1}{8R} - \frac{L}{8(2R-L)}R''. \tag{4.2}$$

Assuming (A1) and (A2), the second twist coefficient $\tau_2(F,P)$ of the one-step billiard map F at P is given by

$$\begin{aligned}\tau_2(F,P) =& \frac{1}{64}\cdot\Big(\frac{3(7R^2-8RL+2L^2)}{4R^2(R-L)\sqrt{(2R-L)L}} - \frac{\sqrt{L}(27R^2-40RL+10L^2)}{6R(R-L)(2R-L)^{3/2}}R'' \\ &+ \frac{L^{3/2}(31R^2-36RL+6L^2)}{12(R-L)(2R-L)^{5/2}}(R'')^2 - \frac{L^{3/2}R}{3(2R-L)^{3/2}}R^{(4)}\Big).\end{aligned} \tag{4.3}$$

In the asymmetric case we will need the following nonresonance assumptions for the iterate F^2 along the periodic orbit $\mathcal{O}_2$:

(B1) $\lambda^4 \neq 1$, or equally, $(\frac{L}{R_0}-1)(\frac{L}{R_1}-1) \in (0,1)\backslash\{\frac{1}{2}\}$;
(B2) $\lambda^6 \neq 1$, or equally, $(\frac{L}{R_0}-1)(\frac{L}{R_1}-1) \in (0,1)\backslash\{\frac{1}{4}, \frac{3}{4}\}$.

Theorem 4.2. *Assuming (B1), the first twist coefficient of the billiard map F along the orbit $\mathcal{O}_2 = \{P, F(P)\}$ is given by*

$$\tau_1(F^2,P) = \frac{1}{8}\Big(\frac{R_0+R_1}{R_0R_1} - \frac{L}{R_0+R_1-L}\Big(\frac{R_1-L}{R_0-L}R_0'' + \frac{R_0-L}{R_1-L}R_1''\Big)\Big). \tag{4.4}$$

Let $k=0$ if $L < \min\{R_0,R_1\}$ and $k=1$ if $\max\{R_0,R_1\} < L < R_0+R_1$. Assuming (B1) and (B2), the second twist coefficient of the billiard map F along the orbit $\mathcal{O}_2 = \{P, F(P)\}$ is given by

$$\begin{aligned}\tau_2 =& \frac{(-1)^k}{\sqrt{\Delta}}\Big(\frac{3N(L,R_0,R_1)}{512R_0^2R_1^2\Gamma} + \frac{P(L,R_0,R_1)(R_0'')^2 + P(L,R_1,R_0)(R_1'')^2}{1536(R_0-L)^2(R_1-L)^2(R_0+R_1-L)^2\Gamma} \\ &+ \frac{Q(L,R_0,R_1)R_0''R_1''}{768(R_0+R_1-L)^2\Gamma} - \frac{S(L,R_0,R_1)R_1R_0'' + S(L,R_1,R_0)R_0R_1''}{768R_0R_1(R_0-L)(R_1-L)(R_0+R_1-L)\Gamma} \\ &- \frac{T(L,R_0,R_1)(R_1-L)R_0^{(4)} + T(L,R_1,R_0)(R_0-L)R_1^{(4)}}{192(R_0-L)(R_1-L)(R_0+R_1-L)}\Big),\end{aligned} \tag{4.5}$$

where $\Delta = L(R_0-L)(R_1-L)(R_0+R_1-L)$, $\Gamma = 2(R_0-L)(R_1-L) - R_0R_1$, and the functions $N(L,R_0,R_1)$, $P(L,R_0,R_1)$, $Q(L,R_0,R_1)$, $S(L,R_0,R_1)$ and $T(L,R_0,R_1)$ are given by

$$\begin{aligned}N(L,R_0,R_1) =& 8L^4(R_0^2+R_1^2) - 16L^3(R_0^3+2R_0^2R_1+2R_0R_1^2+R_1^3) \\ &+ 8L^2(R_0+R_1)^2(R_0^2+4R_0R_1+R_1^2) \\ &- 8LR_0R_1(2R_0^3+7R_0^2R_1+7R_0R_1^2+2R_1^3) + 7R_0^2R_1^2(R_0+R_1)^2;\end{aligned} \tag{4.6}$$

$$\begin{aligned}P(L,R_0,R_1) =& L^2(R_1-L)^4(48R_0^3(R_1-2L) + 24L^2(R_1-L)^2 \\ &- 72LR_0(R_1-L)(R_1-2L) + R_0^2(216L^2-216R_1L+31R_1^2));\end{aligned} \tag{4.7}$$

$$Q(L,R_0,R_1) = -L^2R_0R_1(32L^2+17R_0R_1-32L(R_0+R_1)); \tag{4.8}$$

$$\begin{aligned}S(L,R_0,R_1) =& L(R_1-L)^2(40(R_1-L)^2L^2 + 3R_0^3(9R_1-16L) \\ &- 80R_0(2L^2-3R_1L+R_1^2)L + 3R_0^2(56L^2-56R_1L+9R_1^2));\end{aligned} \tag{4.9}$$

$$T(L,R_0,R_1) = L^2R_0(R_1-L)^2. \tag{4.10}$$

Note that for the symmetric lemon billiards $Q(b)$, the periodic orbit $\mathcal{O}_2(b)$ of period 2 is elliptic (for the one-step map). Then it follows from Theorem 4.1 that the periodic orbit $\mathcal{O}_2(b)$ is nonlinearly stable except one resonance case $b = 1$. Note that our result does cover the resonance case $b = 1.5$ due to the extra symmetry of the billiard table. Similarly, for the asymmetric lemon billiards $Q(b, r, R)$, the periodic orbit of period 2 is elliptic when $b < r$ or $b > R$. Then it follows from Theorem 4.2 that the periodic orbit $\mathcal{O}_2(b)$ is nonlinearly stable except one resonance case $2(b - r)(b - R) = rR$. Our result again covers the resonance case $4(b - r)(b - R) = rR$. For more applications, see [36].

5. Topological entropy of lemon billiards

Let M_b be the phase space of the lemon billiard on $Q(b) = Q(b, 1, 1)$, and F_b be the billiard map on M_b. Note that the map F_b is not defined on the whole space M_b due to the existence of two corners on the lemon table. Let $S_b \subset M_b$ be the set of singularities of F_b, which consists of four segments on M_b. A nature way to define the topological entropy $h_{top}(F_b)$ is to use the Variational Principle:

$$h_{top}(F_b) = \sup\{h_{\mu(F_b)} : \mu \in \mathcal{M}(F_b)\}, \tag{5.1}$$

where $\mathcal{M}(F_b)$ consists of probability measures on M_b with $\mu(S_b) = 0$ that is invariant under F_b.

As we have seen in Section 4, the periodic orbit $\mathcal{O}_2(b)$ is nonlinearly stable for each $b \in (0, 2)\backslash\{1\}$. As b increases and passes $b = 1.5$, the rotation number $\rho(b)$ decreases and passes $\rho = 1/3$. The bifurcation theory in [45] states that generically, there will be a periodic orbit of period 6 bifurcating from the orbit $\mathcal{O}_2(b)$ as b passes $b = 1.5$. However, a simple calculation in [37] shows that the family $\mathcal{O}_2(b)$ does not satisfy the genericity assumption due to the presence of symmetry. In fact, we obtained that

Lemma 5.1. *For each $b \in (1.5, \frac{1+\sqrt{5}}{2})$, there exist two hyperbolic periodic orbits of period 6 and two elliptic periodic orbits of period 6 bifurcating from $\mathcal{O}_2(b)$.*

Denote by $\mathcal{O}^h_j(b)$ and by $\mathcal{O}^e_j(b)$, $j = 1, 2$, the periodic orbits given in Lemma 5.1. Note that the topological entropy $h_{top}(F_b)$ would be positive if F_b were a diffeomorphism on M_b and some of its hyperbolic periodic points admitted transverse homoclinic intersections, which could be achieved by an arbitrarily small perturbation, see [39]. The main difficulty in the study of the lemon billiards is that a generic perturbation of the billiard map F_b is not a billiard map anymore, not to mention being a lemon billiard. This forces us to have a detailed characterization of the phase portrait of the billiard map F_b:

Proposition 5.2. *There exist a Diophantine number $\rho_0 > 1/3$ and a constant $\delta_0 > 0$ such that for any $b \in (1.5, 1.5 + \delta_0)$,*

1. *there exists an F_b^2-invariant curve $C_b(\rho_0)$ of rotation number ρ_0 surrounding the periodic point P and it depends continuously on the parameter b;*
2. *$\mathcal{O}^e_j(b)$ and $\mathcal{O}^h_j(b)$, $j = 1, 2$, are the only periodic orbits of period 6 contained in the union of the domains bounded by $C_b(\rho_0)$ and $F_bC_b(\rho_0)$, respectively;*
3. *the periodic orbits $\mathcal{O}^e_j(b)$, $j = 1, 2$, are nonlinearly stable.*

Using Mather's results on prime-end compactifications and the symmetry of the lemon billiards, we are able to show that there are heteroclinic and homoclinic intersections for the hyperbolic periodic orbit $\mathcal{O}^h_6(b)$. We proved in [37] that

Theorem 5.3. *Let $E \subset (0, 2)$ be the set of parameters b with $h_{top}(F_b) = 0$. Then $E \cap (1.5, 1.5 + \delta_0)$ has no accumulation point in $(1.5, 1.5 + \delta_0)$.*

In order to show that F_b have positive topological entropy, it suffices to show that some of the heteroclinic and/or homoclinic homoclinic intersections are topologically crossing. It is not easy to prove this directly since there is no explicit formula for these invariant manifolds. We have to prove it by contradiction: suppose on the contrary that $E \cap (1.5, 1.5 + \delta_0)$ has an accumulation point. Then all the heteroclinic intersections are saddle connections between the hyperbolic periodic points. Using the analytic dependence of the invariant manifolds on the parameter b, we were able to extend the existence of saddle connections far beyond the interval $(1.5, 1.5 + \delta_0)$ and to obtain a contradiction from this extension.

Theorem 5.3 provides an important progress towards Conjecture 1.1. One of our projects is to prove the general case that $h_{top}(F_b) > 0$ for every $0 < b < 2$.

6. Metric properties of partially hyperbolic systems

The *partially hyperbolic systems* are introduced by Pugh–Shub [56] and Brin–Pesin [11] and are generalizations of the uniformly hyperbolic system, also known as Anosov system (named by Smale). Anosov proved in [2] that every volume-preserving Anosov system is ergodic. In [62] Sinaĭ proved that each transitive Anosov diffeomorphism f admits a unique Gibbs measure μ_f such that its support $\mathrm{supp}(\mu_f) = M$, and its basin $B(\mu_f)$ is of full volume. Therefore, for each transitive Anosov diffeomorphism, the orbit for *almost every* point is dense on M. The existence of above special measure has later been proved for Axiom A attractors by Bowen and Ruelle [9] and been named as the SRB measure.

Note that Anosov systems form an open set and their ergodicity is automatically *stable* under perturbations. Clearly there are many partially hyperbolic systems that are not ergodic. However, the nonergodic ones are *not* typical among the set of partially hyperbolic systems, as conjectured by Pugh and Shub [57]:

Stable Ergodicity Conjecture. Stably ergodic systems are C^r-dense among $\mathrm{PH}^r_m(M)$ (for each $r \geq 1$).

They also designed a program for proving above conjecture:

(PS1) Stably accessible diffeomorphisms are C^r-dense among $\mathrm{PH}^r(M)$ for each $r \geq 1$.
(PS2) If $f \in \mathrm{PH}^2_m(M)$ is essentially accessible, then it is ergodic.

Definition. Let $f : M \to M$ be a partially hyperbolic diffeomorphism on M, $\mathcal{W}^u$ and $\mathcal{W}^s$ be the unstable and stable foliations of f. The *accessibility class* of x, $A_f(x)$, is the set of points that can be connected to x via an *su*-path $\gamma = \gamma_1 \# \cdots \# \gamma_k$, where each γ_i lies in a single leaf of $\mathcal{W}^u$ or $\mathcal{W}^s$. Then f is said to be *accessible* if $A_f(x) = M$ for some x.

There has been some significant progress on the C^1 Stable Ergodicity Conjecture, see [17, 25, 18, 60, 5, 6] and finally proved completely in [7]. To prove ergodicity of volume-preserving systems, a key tool is *Hopf argument*. It has been noticed for quite a while (see [59]) that Hopf's argument only works for smooth measures, since it is based on *Birkhoff ergodic theorem* and relies heavily on the invariance of a smooth measure m. There has been a lot of interest in whether analogous results hold for general dissipative systems.

To state our theorem for dissipative partially hyperbolic systems, we need to introduce some relevant notations. A subset $W \subset M$ is said to be *wandering* with respect to f if all of its iterates $f^n W$, $n \in \mathbb{Z}$, are mutually disjoint. The *dissipative part* D_f of f is the measurable union of the collection of wandering sets of positive measure (let $D_f = \varnothing$ if no such set exists). The complement $C_f = M \backslash D_f$ is called the *conservative part* of f. Then the partition $\{C_f, D_f\}$ is called the *Hopf decomposition* of the diffeomorphism f on M.

We proved in [71] that

Theorem 6.1. *Let $f \in \mathrm{PH}^2(M)$ be essentially accessible, $\{C_f, D_f\}$ be the Hopf decomposition of M with respect to f. If $m(C_f) > 0$, then f is transitive, and the orbit of x is dense for m-a.e. $x \in C_f$.*

Now assume that f admits some ACIP μ. Let $E_\mu = \{x \in M : \frac{d\mu}{dm}(x) > 0\}$ be the essential support of μ. Then it is easy to see $m(E_\mu) > 0$ and $E_\mu \subset C_f$. Therefore, such a map f is transitive, and μ-a.e. x has a dense orbit on M. Moreover, we have

Theorem 6.2. *Suppose $f \in \mathrm{PH}^2(M)$ is essentially accessible and center-bunched. Then we have*

(1) [70] *If there is an ACIP $\mu \ll m$, then μ is equivalent to m and is ergodic.*
(2) [71] *Either f is completely dissipative: $m(C_f) = 0$, or (f, m) is ergodic.*

This generalizes Gurevich and Oseledets's results for Anosov systems in [31]. In [71] we also obtained some intersecting results for Anosov diffeomorphisms. Recall that Bowen [8] gave a new definition of the topological entropy $h_B(f, E)$ of subsets $E \subset X$ that resembles the definition of Hausdorff dimension $\dim_H(E)$. We proved in [71] that for any transitive Anosov diffeomorphism $f : M \to M$, for any two invariant measures

μ, ν of f, there exists an invariant subset $E = E(\mu, \nu) \subset M$ such that the *forward entropy* and *backward entropy* of f on E satisfy

$$h_B(f, E) = h_\mu(f), \quad h_B(f^{-1}, E) = h_\nu(f). \tag{6.1}$$

By the affine property of metric entropy, for any $0 \le a \le h_{top}(f)$, there is an invariant (not necessarily ergodic) measure μ with metric entropy $h_\mu(f) = a$. Combining (6.1), we conclude that for any pair of numbers $(a, b) \in [0, h_{top}(f)] \times [0, h_{top}(f)]$, there exists an invariant subset $E = E(a, b) \subset M$, such that the forward entropy and backward entropy of f on E satisfy

$$h_B(f, E) = a, \quad h_B(f^{-1}, E) = b. \tag{6.2}$$

In particular, the entropy $h_B(f, E)$ of a subset E is asymmetric with respect to time-reversal $f \leftrightarrow f^{-1}$.

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