PDE (Math 4163) Spring 16

4. Separation of Variables and Sturm-Liouville Problems

So far we have not dealt with two boundary conditions, even though it seems that this is the most common situation.
The reason for that is that explicit solutions are hard to get, if that's at all possible. So we try to do the next best thing to solving the problem, that is to approximate the solution and make sure that, in the limit, what we get are solutions.
The main idea is to look at solutions of the type

$$u(x,t) = X(x)T(t), \text{ respectively } u(x,y) = X(x)Y(y).$$

As an introductory example lets look at the IBVP for the heat equation.

$$\begin{cases} PDE & u_t - k\, u_{xx} = 0 \quad \text{for} \quad (x,t)\epsilon(0,\pi)\times(0,\infty) \\ IC & u(x,0) = \phi(x) \\ BC's & u(0,t) = u(\pi,t) = 0\,. \end{cases}$$

For $u(x,t) = X(x)T(t)$ we have

$$u_t = XT', \quad u_{xx} = X''T\,.$$

inserting into the PDE we get

$$XT' - kX''T = 0\,,$$

"Dividing" by kXT, we get

$$\frac{X''}{X}(x) = \frac{T'}{kT}(t)$$

Now we observe that this can be only true if both fractions are the same constant, because each depend on only one, but a different variable.
So let's say this constant is $-\lambda$; (the $''-''$ sign is anticipating the later use).
We get

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda$$

or

$$X'' + \lambda\, X = 0, \qquad \text{and} \qquad T' + k\, T = 0$$

which have solutions for each λ. That's to many, so we try to find out which are a more reasonable set of solutions, in particular it would be convenient

if they would fit the B.C. of the problem:.
If

$$0 = u(0,t) = X(0)T(t)\,,$$

then we get a solution different from the trivial solution only if $T(t) \not\equiv 0$, but then $X(0)$ has to be zero; the same argument yields $X(\pi) = 0$.
So we have to consider the boundary value problem

$$\begin{cases} X'' + \lambda\, X = 0 \\ X(0) = 0, X(\pi) = 0\,. \end{cases}.$$

Which has the trivial solution (in which we are not interested.)
So dead end? (We have two auxiliary conditions so we expect the solution to be unique!)
Yes, most of the time! But contrary to the initial conditions (where two conditions determine the solution uniquely) for BV problems there are exceptions. Lets find out what those are.
1) For $\lambda\ <\ 0$,

a general solution is given by

$$X(x) = C_1\ \cosh(\nu x) + C_2\ \sinh(\nu x) \qquad\qquad \text{with}\quad \nu = \sqrt{-\lambda}$$

and the B.C require
i) $0 = X(0) = C_1 \cdot 1 + C_2$ and hence $C_1 = 0\,.$

ii) $0 = X(\pi) = C_2 \sinh(v\pi) \qquad \Rightarrow \qquad C_2 = 0$

Since $\sinh(v\pi) \neq 0$ for all positive ν. Consequently there is no solution except the trivial one.
For $\lambda = 0$,

we have the general solution

$$X(x) = C_1 + C_2 x,$$

and the BC require
i)) $0 = X(0) = C_1 + C_2 0$ and hence $C_1 = 0\,.$

ii)) $0 = X(\pi) = C_2 \cdot \pi\,, \qquad \Rightarrow C_2 = 0.$

No luck here either! Finally for
$\lambda > 0$ with $\nu = \sqrt{\lambda}$ we have that

$$X(x) = C_1\, \cos(\nu x) + C_2\, \sin(\nu x)\ \text{ with }\ \nu = \sqrt{\lambda}\,,$$

is a general solution. Again the BC require

i)) $0 = X(0) = C_1 \cdot 1 \qquad + C_2 \cdot 0 \qquad \Rightarrow C_1 = 0\,.$

ii)) $0 = X(\pi) = C_2 \sin \nu\pi\,;$

this is true if either $C_2 = 0$ or $\sin(\nu\pi) = 0$, i.e. if

$$\nu = 1, 2, 3, 4\ ...$$

and so we get non trivial solutions

$$X(x) = C \sin \nu x \quad \text{for } \nu = 1, 2, 3, 4\ ...$$

We found that for $\lambda_n = n^2$, the B.V. problem has the solution

$$X_n = C_n \sin nx$$

The associated ODE for the function T now reads:

$$T' + n^2 k\, T = 0\,,$$

which has the solution

$$T_u = C_n\, e^{-n^2 kt}\,.$$

Altogether we get countably many solutions

$$u_n(x,t) = X_n T_n$$

of

$$\begin{cases} u_t - k\, u_{xx}\,. \\ u(x,0) = 0\,, \text{ and } \ u(x,\pi) = 0\,. \end{cases}$$

Now the question arises, if it is possible to superpose those solutions such that the result satisfies the initial condition, i.e.: can we find constants C_n such that

$$u(x,t) = \sum_{n=1}^{\infty} C_n\, u_n(x,t)$$

satisfies

$$\phi(0) = \sum_{n=1}^{\infty} C_n\, u_n(x,0)\,?$$

In our case we have to find C_n such that

$$\phi(x) = \sum_{n=1}^{\infty}\ C_n \sin nx$$

We will learn more explicit methods to determine these constants later, but the following one will always work, if it is adjusted to the specific problem at hand.

Multiply the last equation by a particular function of this series, say $\sin kx$, and integrate over the interval:

$$\int_0^\pi \phi(x) \sin kx = \int_0^\pi \sum_{n=1}^\infty C_n \sin(nx) \cdot \sin kx \, dx$$

$$= \sum_{n=1}^\infty C_n \int_0^\pi \sin nx \cdot \sin kx \, dr$$

$$= C_k \int_0^\pi \sin^2 kx \, dx\,.$$

Since

$$\int_0^\pi \sin(kx) \sin nx \, dx = 0\,;$$

if $k \neq n$. Indeed

$$\int_0^\pi \sin kx \cdot \sin nx \, dx = -\frac{1}{2} \int_0^\pi (\cos)\,(k-n)\,x - \cos\,(k+n)\,x \, dx =$$

$$\tfrac{1}{2}\left(\tfrac{1}{k-n} \sin\left((k-n)\,x\right) + \tfrac{1}{k+n} \sin\left((k+n)x\right)\right)\Big|_0^\pi = 0\,;$$

if $k \neq 0$. In case $k = n$ we get

$$\int_0^\pi \sin^2 kx \, dx = \frac{1}{2} \int_0^\pi 1 - \cos 2kx \, dx = \frac{\pi}{2} - \frac{1}{2k} \sin 2kx \Big|_0^\pi = \frac{\pi}{2}\,;$$

and so

$$C_k = \tfrac{2}{\pi} \int\limits_0^\pi \phi \sin kx \; dx.$$

This indicates that we should be able to write the function ϕ in the form

$$\phi(x) = \sum_{x=1}^{\infty} C_k \sin k \; dx$$

and therefore finding the solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} C_n u_n(x,t).$$

Remark:
For a given function f on the interval $(0, \pi)$ the series

$$\sum_{n=1}^{\infty} a_n \sin nx,$$

with

$$a_n = \frac{2}{\pi} \int\limits_0^\pi f(x) \sin nx \; dx$$

is called the Fourier Sine series of f.

Example:
For $f(x) = x$ we get

$$\frac{\pi}{2} a_n = \int_0^\pi x \sin nx dx = -\frac{x}{n} \cos nx \Big|_0^\pi + \frac{1}{n} \int_0^\pi \cos x dx$$

$$= -\frac{\pi}{n}(-1)^n$$

So $a_n = \dfrac{2(-1)^{n+1}}{n}$ and the Fourier Sine series of f is given by

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx \; .$$

Sturm-Liouville Problems

Recall that the major problem to justify the approach outlined above is the justification of the representation of the functions as infinite series; for example can we write.

$$f(x) = \sum_{k=1}^{\infty} a_n \sin nx,$$

for $0 \leq \ x \ \leq \pi,$, with

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

where the functions $\sin nx$ have been nontrivial solution of the B.V. problem

$$B.V. \begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 \\ X(\pi) = 0 \end{cases} .$$

That is a special case of a so called Sturm-Liouville problem, and in this particular case we call the series Fourier Sine Series. We will discuss some of the issues raised in the context of a more general type of problems:

Definition

A boundary value problem for a second order ODE of the type

$$\begin{cases} (pX')' - qX + \lambda m X = 0 & \text{on } (a, b)\,, \\ \alpha X(a) + \beta X'(a) = 0 & \text{with } \alpha^2 + \beta^2 > 0\,, \\ \gamma X(b) + \delta X'(b) = 0 & \text{with } \gamma^2 + \delta^2 > 0\,, \end{cases}$$

is called Sturm-Liouville problem with homogeneous boundary conditions. In general we assume that $p,\ p',\ q,\ m$ are continuous functions and that p, m are positive in (a, b).

Remarks:

1) If the homogeneous boundary conditions are replaced by the periodic boundary conditions

   $$X(a) = X(b) \text{ and } X'(a) = X'(b)$$

   and if in addition the coefficient of the highest term is periodic i.e.:

$$p(a) = p(b)$$

then we speak of a periodic Sturm-Liouville problem.

2) The problems are called regular if p, q, m are bounded functions and p is positive and continuous on $[a, b]$, (up to the boundary). Otherwise the problem is called singular.
3) For singular Sturm-Lionville problems, quite often other auxiliary conditions are introduced.

Definition:
Nontrivial solutions of a Sturm-Lionville problem are called eigenfunctions. The numbers λ for which eigenfunctions do exists are called eigenvalues.

In oder to discuss the the properties of the eigenvalues and eigen functions, it is sometimes helpful to rewrite the ODE of the Sturm-Lionville problem as

$$L(X) = \lambda mX$$

with

$$L(X) = -(pX')' + qX$$

The eigenfunction and eigenvalues of a regular Sturm-Lionville problem have the following properties.

1) As both the ODE and the boundary conditions are linear, nontrivial linear combinations of eigenfunctions with respect to the same eigenvalue are eigenfunctions with respect to that eigenvalue.

Remark:

i) That statement is true for all Sturm-Lionville problems even if an eigenvalue has linearly independent eigenfunctions.
ii) Note again, that $\varphi \equiv 0$ is always a solution of the Sturm-Louisville problem since both the PDE and the auxiliary conditions are linear and homogeneous, regardless of the value of λ. But it is not considered to be an eigenfunction. (Also it would not help us to solve the problems, because superposing zeros we don't get anything else as zero again).

2) Two eigenfunctions φ_1, and φ_2, say, with respect to two different eigenvalues λ_1 and λ_2, are orthogonal with respect to the weight function $m(x)$, i.e.:

$$0 \;=< \varphi_1, \varphi_2 >_m : \;= \int_a^b \varphi_1 \, \varphi_2 \, m \, dx.$$

Proof:

We have

i) $(p \cdot \varphi_1')' \;-\; q \cdot \varphi_1 \;+\; \lambda_1 \, m \, \varphi_1 \;=\; 0,$

ii) $(p \cdot \varphi_2')' \;-\; q\varphi_2 \;+\; \lambda_2 \, m \, \varphi_2 \;=\; 0\,.$

Multiplying the first equation by φ_2 and the second by φ_1, and subsequently subtracting (ii) from (i) yields

$$(\lambda_2 - \lambda_1) m \, \varphi_1 \varphi_2 = \varphi_1 L(\varphi_2) - \varphi_2 L(\varphi_1) =$$

$$-(p \, \varphi_2')' \varphi_1 + q \varphi_2 \varphi_1 \; + (p \, \varphi_1')' \varphi_2 \; - q \varphi_1 \varphi_2$$

$$= -(p \, \varphi_2')' \varphi_1 - p \, \varphi_2' \varphi_1' + (p \, \varphi_1')' \varphi_2 \; + p \, \varphi_1' \varphi_2'$$

$$= \frac{d}{dx}(-(p(\varphi_2' \varphi_1 - p(\varphi_1' \varphi_2))$$

$$= -\frac{d}{dx}(p \, W(\varphi_1 \varphi_2))\,.$$

This calculation proves a so-called Lagrange identity

$$\varphi_2 L(\varphi_1) - \varphi_1 L(\varphi_2) = \frac{d}{dx}(p \, W(\varphi_1 \varphi_2))$$

Integrating both over (a, b) we get Green's identity

$$= \int_a^b \varphi_2 L(\varphi_1) - \varphi_1 L(\varphi_2) dx = (p \, W(\varphi_1 \varphi_2) \Big|_a^b$$

and so

$$(\lambda_2 \;-\; \lambda_1) \cdot \int_a^b \; \varphi_1 \cdot \varphi_2 \, m \, dx \;= -(p \, W(\varphi_1 \varphi_2) \Big|_a^b$$

$$- \, p(\varphi_1' \cdot \varphi_2 \;-\; \varphi_2' \cdot \varphi_1) \Big|_b^a$$

$$= p\,(\varphi_1'\,(b)\cdot\varphi_2\,(b)\ -\ \varphi_2'\,(b)\cdot\varphi_1\,(b))-$$

$$p\,\big(\varphi_1'\,(a)\,\varphi_2\,(a)\ -\ \varphi_2'\,(a)\,\varphi\,(a)\big)$$

$$= I_1\ +\ I_2$$

If $\alpha\cdot\beta = 0$ then either $\alpha = 0$ or $\beta = 0$, in the first case

$$\varphi_1'\,(a)\ =\ \varphi_2'\,(a)\ =\ 0\,,$$

in the second case

$$\varphi_1\,(a)\ =\ \varphi_2\,(a)\ =\ 0$$

If $\alpha\cdot\beta \neq\ 0$, then

$$\varphi_i\,(a)\ =\ \tfrac{\beta}{\alpha}\cdot\varphi_i'\,(a) \qquad i = 1,2$$

and we get

$$\varphi_1'\,(a)\,\varphi_2\,(a)\ -\ \varphi_2'\,(a)\ \cdot\ \varphi_1\,(a)$$

$$= \varphi_1'\,(a)\cdot\ \tfrac{\beta}{\alpha}\,\varphi_2'\,(a)\ -\ \varphi_2'\,(a)\,\tfrac{\beta}{\alpha}\,\varphi_1'\,(a)\ =\ 0\,.$$

Hence in all possible cases, $I_2\ =\ 0$.
Likewise we find $I_1\ =\ 0$, hence

$$(\lambda_2\ -\ \lambda_1)\cdot\int_a^b\ \varphi_1\cdot\varphi_2\,m\,dx\ =\ 0\,.$$

Since $(\lambda_2\ -\ \lambda_1)\ \neq\ 0$ the orthogonality follows.

3) The eigenvalues of a regular Sturm-Liouville problem are nonnegative if

i) $q\ \geq\ 0$ and

ii) $B\colon = p\,(a)\,\varphi'\,(a)\,\varphi\,(a)\ -\ p\,(b)\,\varphi'\,(b)\,\varphi\,(b) \geq\ 0\,.$

Indeed we have

$$\int_a^b (p\,\varphi')'\,\varphi\,dx\ -\int_a^b\ q\,\varphi\cdot\ \varphi\,dx + \ \lambda\int_a^b\ \varphi\ \cdot\ \varphi\,dx\ =\ 0$$

or

$$\lambda\int_a^b\ \varphi^2 m\,dx\ =\int_a^b\ q\,\varphi^2\,dx\ +\int_a^b\ p\,\varphi'\ \cdot\ \varphi'$$

$$+\, p\,(a)\,\varphi'\,(a)\,\varphi(a)\;-\;p\,(b)\,\varphi'\,(b)\,\varphi(b)\,.$$

Under the assumption of the statement the terms on the right-hand side are nonnegative, hence λ has to be nonnegative.

4) From i) and ii) above it also follows that eigenfunctions of one eigenvalue are unique up to a constant:

With $\lambda_2 = \lambda_1$ the above calculation gives

$$0 = \frac{d}{dx}(p(\varphi_1'\varphi_2 - \varphi_2'\varphi_1))'\,.$$

Hence $(p(\varphi_1'\varphi_2 - \varphi_2'\varphi_1))$ is constant. So we may evaluate that function at one boundary, say at a and get

$$(p(\varphi_1'\varphi_2 - \varphi_2'\varphi_1) = I_2\,.$$

But we argued above that $I_2 = 0$, as a consequence of the boundary conditions for a regular Sturm- Liouville problem. That yields

$$\varphi_1'\varphi_2 - \varphi_2'\varphi_1 = 0\,.$$

But this is the numerator of $(\frac{\varphi_1}{\varphi_2})'$. Hence the quotient is a constant and that's what we claimed. (Here we have tacitly assumed that the solutions of the eigenvalue problem are twice differentiable. A fact which to prove is beyond our means.)

5) Uniqueness of the generalized Fourier coefficients.

If (φ_n) is a sequence of orthogonal (eigen)functions with respect to the weight function m and if some functions f is given by

$$(*)\qquad f \;=\; \sum_{n=1}^{\infty} a_n\varphi_n$$

then we have

$$a_n \;=\; \left(\int_a^b \varphi_n^2\; m\; dx\right)^{-1} \int_a^b f\;\varphi_n\; m\; dx$$

provided the integration of

$$\int_a^b \left(\sum_{n=1}^{\infty} a_n \, \varphi_n \right) \varphi_k \, m \, dx$$

commutes with the series. Consequently, the coefficients a_n in $(*)$ are uniquely determined and given by the above formula.

Indeed multiplying the equation (*) with $\varphi_k m$ and integrating over (a, b) yields

$$\int_a^b f \, \varphi_k m \, dx \; = \; \sum_{n=1}^{\infty} a_n \int_a^b \varphi_n \, \varphi_k \, m \, dx \; =$$

$$= a_k \int_a^b \varphi_k^2 \, m \, dx.$$

6) " Best Approximation."

If a_n for $n = 1, \ldots, k$ is chosen as above, then $\sum\limits_{n=1}^{k} a_n \, \varphi_n$ is the best possible approximation of f in the following sense:
If

$$g \; = \; \sum_{n=1}^{k} d_n \, \varphi_n$$

is any linear combination of the (eigen-) functions

$$\varphi_1, \ldots, \varphi_k$$

then

$$\int_a^b \left(f - \sum_{n=1}^{k} a_n \varphi_n \right)^2 m \, dx \; \leq \; \int_a^b (f - g)^2 \, m \, dx.$$

Note: in analogy to the norm of vectors in $\mathbb{R}^n$, we introduce the "norm" for functions.

$$|| f ||_m \ = \ (< f, f >_m)^{\frac{1}{2}} \ = \ \left(\int_a^b f^2 \, m \, dx \right)^{\frac{1}{2}} .$$

With this definition and taking roots of both sides, the above inequality reads

$$|| f - \sum_{n=1}^{k} a_n \, \varphi_n ||_m \ \leq \ || f - \sum_{n=1}^{k} \, d_n \, \varphi_n ||_m .$$

Lets denote with

$$E_n \ = \ || \, f \ - \ \sum_{n=1}^{k} d_n \, \varphi_n \, ||_m^2$$

the "mean square distance of f with the approximation". We get

$$E_n \ = \ \int_a^b \, [f^2 \, -$$

$$- \, 2 \, f \, \sum_{n=1}^{k} d_n \, \varphi_n \ + \ \left(\sum_{n=1}^{n} d_n \varphi_n \right)^2 \,] \, m \, dx$$

$$= \ \int_a^b f^2 m \, dx \ - \ \sum_{n=1}^{k} 2 d_n \ \int_a^b (f \varphi_n) \, m \, dx$$

$$+ \sum_{n=1}^{k} d_n \int_a^b \left(\varphi_n \sum_{l=1}^{k} \varphi_l \right) \ m \, dx$$

$$= \ \int_a^b f^2 \, m \, dx \ - 2 \, \sum_{n=1}^{k} \ d_n \int_a^b (f \varphi_n) \, m \, dx$$

$$+ \sum_{n=1}^{k} d_n^2 \int_a^b (\varphi_n \ \cdot \varphi_n) \, m \, dx \, ,$$

using the orthogonality of the sequence. We continue now writing the integrals in terms of the inner product and norms introduced above, and note that in this notation

$$a_n = \frac{<f,\varphi_n>}{||\varphi_n||^2}\ .$$

We get

$$E_n \ = \ ||\ f\ ||_m^2 - \sum_{n=1}^{k} 2d_n < f,\varphi_n >_m + \sum_{n=1}^{k} d_n^2 < \varphi_n,\varphi_n >_m$$

$$= ||\ f\ ||_m^2$$

$$+ \sum_{n=1}^{k} ||\ \varphi_n||_m^2\ [d_n^2 \ - \ 2d_n \frac{< f,\varphi_n >_m}{||\ \varphi_n\ ||_m^2} \ + \ \left(\frac{< f,\varphi_n >_m}{||\ \varphi_n\ ||_m^2}\right)^2]$$

$$- \sum_{n=1}^{k} \frac{< f,\varphi_n >_m^2}{||\ \varphi_n\ ||_m^2}$$

$$= ||\ f\ ||_m^2 \ - \ \sum_{n=1}^{k} a_n^2\ ||\varphi_n\ ||_m^2 \ + \sum_{n=1}^{k} \ (d_n \ - \ a_n)^2\ ||\ \varphi_n\ ||_m^2$$

Now E_n becomes the least possible number if

$$d_n \ = a_n = \left(\int_a^b (\varphi_n)^2 m\ dx\right)^{-1} \int_a^b f\varphi_n\ m\ dx$$

7) Bessel's inequality:

For all function, for which $||\ f\ ||_m^2$ exist, we have the inequality

$$\sum_{n=1}^{k} a_n^2\ ||\ \varphi_n\ ||^2\ _m \leq \ ||\ f\ ||_m^2$$

or

$$\sum_{n=1}^{k} \frac{(< f,\varphi_n >_m)^2}{||\ \varphi_n\ ||^2} \leq \ ||\ f\ ||_m^2$$

for all k.

The inequality is called Bessel's inequality.

It is a consequence of the above calculation with $d_n = a_n$. We have

$$0 < E_n = \| f \|_m^2 - \sum_{n=1}^{k} a_n^2 \| \varphi_n \|^2 ,$$

Bringing $\sum_{n=1}^{k} a_n^2 \| \varphi_n \|_m^2$ to the left hand side yields Bessel's inequality.

Gram-Schmidt orthogonalization procedure:

If $\mathbf{v}_1 , \mathbf{v}_2 , \ldots , \mathbf{v}_n ,$ are n linearly independent vectors in a vector space with inner product $<,>$ then we obtain n orthogonal vectors

$\mathbf{w}_1 , \mathbf{w}_2 , \ldots , \mathbf{w}_n ,$ setting

$$\mathbf{w}_1 = v_1 , \ \mathbf{w}_2 = \mathbf{v}_2 - \frac{< \mathbf{v}_2, \mathbf{w}_1 >}{< \mathbf{w}_1, \mathbf{w}_1 >} \mathbf{w}_1 , \ldots ,$$

$$\mathbf{w}_n = \mathbf{v}_n - \sum_{i=1}^{n-1} \frac{< \mathbf{v}_n, \mathbf{w}_i >}{< \mathbf{w}_i, \mathbf{w}_i >} \mathbf{w}_i .$$

8) If there is a infinte sequence of orthogonal (eigen-) functions then we have

$$E_n \to 0$$

if and only if

$$\sum_{n=1}^{\infty} a_n^2 \| \varphi_n \|_m^2 = \| f \|_m^2 ,$$

or

$$\sum_{n=1}^{\infty} \frac{< f, \varphi_n >_m^2}{\| \varphi_n \|_m^2} = \| f \|_m^2 .$$

This equation is known as Parseval's identity.

Definition:

A sequence of pairwise orthogonal functions $f_1, f_2, \ldots,$ is called complete (with respect to the inner product $<,>_m$ if for all functions f with $\| f \|_m <$

∞ we have Parseval's identity.

Remark:

A sequence of pairwise orthogonal functions $\varphi_1, \ \varphi_2, \ldots$ is complete if the generalized Fourier series $\sum_{n=1}^{k} a_n \, \varphi_n$ approximates every f, with $\| \, f \, \|_m < \infty$, in the norm $\| \quad \|_m$.

Theorem:

A regular Sturm-Liouville problem has a countable number of eigen values.
The eigenfunctions to different eigenvalues a pairwise orthogonal.
The eigen functions are complete with respect to the inner product $<,>_m$.

Distribution of the eigenvalues of the generic ST-L

$$\begin{cases} X'' \, + \, \lambda \, X \; = \; 0 \, , & \text{on } (0, l) \, ; \\ \alpha \, X \, (0) \; + \; \beta X' \, (0) \; = \; 0 \, , & \\ \gamma \, X \, (l) \; + \; \delta X' \, (l) \; = \; 0 & \end{cases}$$

Firstly we note that have no negative eigenvalues. for Dirichlet and Neumann B.C. because then we have have $q \; = \; 0$, and $B = \; 0$ hence (i) and (ii) are satisfied.

Robin boundary condition

$$\begin{cases} X'' \, + \, \lambda \, X \; = \; 0 \, , & \text{on } (0, l) \, ; \\ \alpha \, X \, (0) \; + \; X' \, (0) \; = \; 0 \, , & \\ \gamma \, X \, (l) \; + \; X' \, (l) \; = \; 0 & \end{cases}$$

Here B is not necessarily zero: We have

$$X' \, (0) \; = \; - \, \alpha \, X(0) \quad \text{and} \quad x' \, (b) \; = \; -\lambda \, x(l) \, .$$

So

$$B = - \, \alpha \, \big(x(0)\big)^2 + \gamma \, \big(x(l)\big)^2 \, ,$$

may be negative. If, however, $\alpha \; < \; 0, \; \gamma \; > \; 0,$ then $B \geq 0$ and there are no negative eigen values!

We want now to investigate the situation in more detail.

For $\lambda \; < \; 0$,

the general solution is

$$\varphi\,(x) \;=\; C_1\;\cosh\;\nu\,x\;+\;C_2\;\sinh\;\nu\,x\,.$$

with $\nu \;=\; \sqrt{-\lambda}\,.$

We have

$$X'\,(x) \;= \lambda\,(C_1 \sinh\;\nu\,x\;+\;C_2\;\cosh\;\nu\,x)$$

hence the $B.C.s$ give

$$0 \;=\; \alpha\,C_1 \;+\;\nu\,C_2 \qquad \Rightarrow \qquad \nu\,C_2 \;=\; -\,\alpha\,C_1\,,$$

and

$$0 \;=\; \gamma\,(C_1\;\cosh\;\nu\,l\;+\;C_2\;\sinh\;\nu\,l)\,+$$

$$\nu\,(C_1\;\sinh\;\nu\,l\;+\;C_2\;\cosh\;\nu\,l)$$

$$=\;(\gamma\,C_1\;+\;\nu\,C_2)\;\cosh\;\nu\,l\;+\;(\gamma\,C_2\;+\;\nu\;C_1)\;\sinh\;\nu\,l$$

$$=\;C_1(\gamma\;-\;\alpha)\cosh\;\nu\,l + C_1(\nu - \frac{\gamma\alpha}{\nu})\;\sinh\;\nu\,l\,:$$

that is

$$(\alpha\;-\;\gamma)\;\cosh\;\nu\,l\;=\;\frac{\nu^2 - \gamma\alpha}{\nu}\;\sinh\;\nu\,l\,; \tag{1'}$$

or

$$\tanh\;\nu\,l\;=\;\frac{\nu(\alpha\;-\;\gamma)}{\nu^2\;-\gamma\alpha}\,, \tag{1}$$

if $\nu^2\;-\gamma\alpha\;\neq\;0\,.$ If a solution ν_k of (1) respectively (1') exist then the eigenfunctions for $\lambda_k = -\nu_k^2$ are of the form

$$\varphi_k = \cosh\;\nu_k\;x\;-\;\frac{\alpha}{\nu_k}\sinh\;\nu_k\;x\,.$$

For $\lambda = \mathbf{0}$

The general solutions are

$$X(x) = C_1 x + C_2,$$

with $X'(x) = C_1$ the B.C.'s give

$$0 = \alpha(C_1\;\cdot\;0\;+\;C_2) + C_1,$$

and

$$0 = \gamma(C_1\;\cdot\;l +\;C_2) + C_1$$

so $C_2 = -\frac{C_i}{\alpha}$ and

$$C_1(\gamma \cdot (l - \tfrac{\gamma}{\alpha}) = -C_1$$

or

$$l = \frac{\gamma - \alpha}{\gamma\alpha}\,.$$

Hence zero is an eigenvalue if and only if

$$(\gamma\ \alpha)l = \gamma - \alpha. \tag{2}$$

Then the eigenfunction is of the form $\varphi(x) = x - \dfrac{1}{\alpha}\,.$

$\lambda > \mathbf{0}$

The general solution are

$$X(x) = C_1 \cos\ \gamma x + C_2 \sin \nu x\ , \quad \text{with } \gamma = \sqrt{\lambda}\,.$$

With

$$X'(x) = \gamma(C_1(-\sin \nu x) + C_2(\cos \nu x))\,,$$

the B.C.s give

$$0 = \alpha C_1 + \nu C_2 \quad \Rightarrow C_2 = -\tfrac{\alpha}{\nu} C_1$$

and

$$0 = \gamma\big(C_1 \cos \nu \cdot l\ +\ \nu C_1(\sin \nu l)\big) +\ \gamma C_2 \cos \nu l - \nu C_1 \sin \nu l$$

$$= C_1[(\gamma - \alpha) \cos \nu l\ -\ (\tfrac{\gamma\alpha}{\nu} + \nu) \sin \nu l\,]$$

which allows for eigenvalues ν_k as the solutions of

$$(\gamma - \alpha) \cos \nu l\ -\ \tfrac{\gamma\alpha + \nu^2}{\nu} \sin \nu l = 0 \tag{3'}$$

or

$$\tan \nu l = \frac{\nu(\gamma - \alpha)}{\nu^2 + \gamma\alpha}\,, \tag{3}$$

if $\nu^2 + \gamma\alpha) \neq 0$. The eigenfunctions to $\lambda_k = \nu_k^2$ are of the form

$$\varphi_k(x) = \cos \nu_k x\ -\ \frac{\alpha}{\nu_k} \sin \nu_k x\,.$$

Summarizing we have:
For $\lambda < \mathbf{0}$, $\nu = \sqrt{-\lambda}\,$.

We have eigenvalues for solutions of

$$(\alpha - \gamma) \cosh \nu l = \frac{\nu^2 - \gamma\alpha}{\nu} \sinh \nu l \, ; \tag{1'}$$

or

$$\tanh \nu l = \frac{\nu(\alpha - \gamma)}{\nu^2 - \gamma\alpha} \, , \tag{1}$$

if $\nu^2 - \gamma\alpha \neq 0$.

If a solution ν_k of (1) respectively (1') exist then the eigenfunctions for $\lambda_k = -\nu_k^2$ are of the form

$$\varphi_k = \cosh \nu_k x - \frac{\alpha}{\nu_k} \sinh \nu_k x \, .$$

For $\lambda = \mathbf{0}$. $\lambda = 0$, is an eigenvalue iff

$$(\gamma \, \alpha) l = \gamma - \alpha. \tag{2}$$

The eigen function is of the form $\varphi(x) = x - \frac{1}{\alpha}$.

$\lambda > \mathbf{0}$, $\nu = \sqrt{\lambda}$.

We have eigenvalues for solutions of

$$(\gamma - \alpha) \cos \nu l - \tfrac{\gamma\alpha + \nu^2}{\nu} \sin \nu l = 0 \tag{3'}$$

or

$$\tan \nu l = \frac{\nu(\gamma - \alpha)}{\nu^2 + \gamma\alpha} \, , \tag{3}$$

if $\nu^2 + \gamma\alpha) \neq 0$.

The eigenfunctions to $\lambda_k = \nu_k^2$ are of the form

$$\varphi_k(x) = \cos \nu_k x - \frac{\alpha}{\nu_k} \sin \nu_k x \, .$$

Let us now consider some cases in detail:

1. Case $\alpha < 0, \gamma > 0$

i) $\lambda < 0$.

For $\alpha < 0$, $\gamma > 0$ the right hand side of (1) is positive and the lefthand side is negative, hence there are no negative eigenvalues.

ii) $\lambda = 0$.

Now the right hand side of (2) is negative and the left hand side is positive; hence 0 is not an eigenvalue.

iii) $\lambda > 0$.

We write (3) in the form

$$g(\nu) = f(\nu)\,,$$

with $g(\nu) = \tan \nu l$ and $f(\nu) = \dfrac{\nu(\gamma - \alpha)}{\nu^2 + \gamma\alpha}$,

There is a pole at $\nu = \sqrt{-\gamma\alpha}$ of f on the positive ν axis. In case $\sqrt{-\gamma\alpha}$ is not one of the pole of $\tan(l\nu)$, we have

iii, α) $\sqrt{-\gamma\alpha} \neq \dfrac{\pi}{l}(k + \dfrac{1}{2})$ $k = 0, 1, \ldots,$

and there is a nonnegative integer k_0 such that

$$(k_0 - \frac{1}{2})\pi < l\sqrt{-\gamma\alpha} < (k_0 + \frac{1}{2})\pi\,.$$

Observing that $f(0) = 0$ and that $f(\nu)$ is negative for $\nu < \sqrt{-\alpha\gamma}$ we get the following qualitative figure of the graphs of g and f .

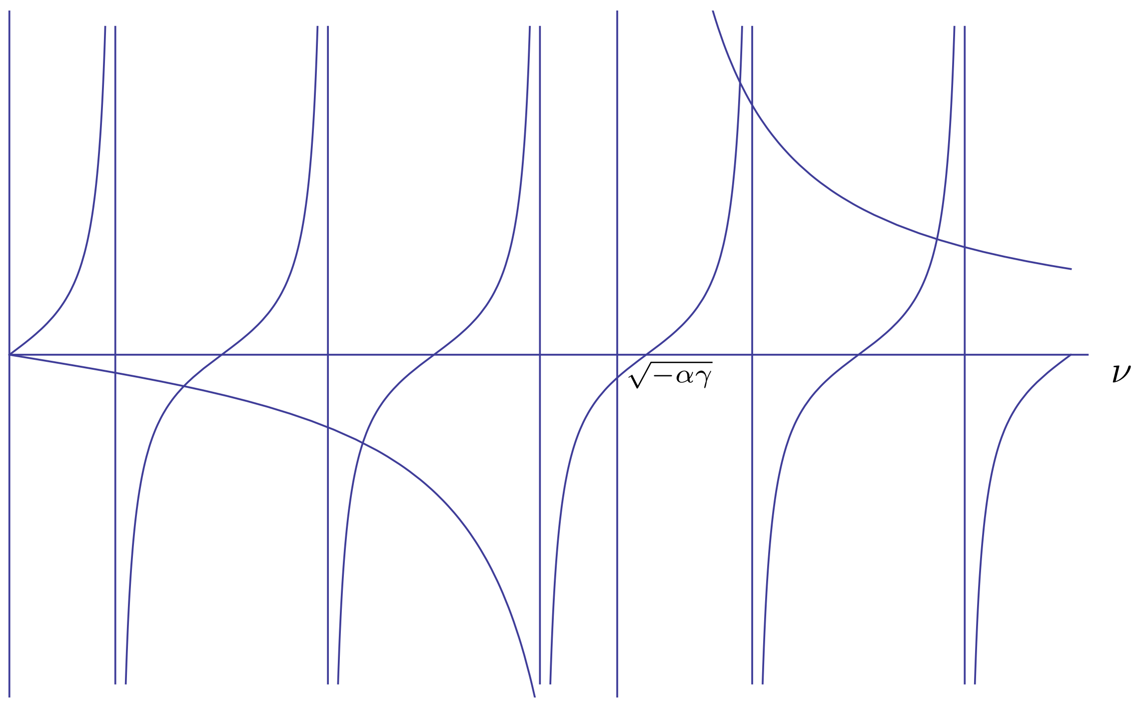

Fig. 9

Then we have k_0 solutions ν_k one each in the intervals

$$(\frac{\pi}{l}((k-\frac{1}{2}),\frac{\pi}{l}k)\ \ k=1,\ldots,k_0\,.$$

and countably many solutions ν_k each of the intervals

$$((\frac{\pi}{l}k-1,\frac{\pi}{l}(k-\frac{1}{2}))$$

for $k=k_0+1,k_0+2\ldots$.

So we have only positive eigenvalues $\lambda_k=\nu_k^2$, $k=1,2,\ldots$, with the eigenfunctions

$$\phi_k(x)=\cos\nu_k x\ -\ -\frac{\alpha}{\nu_k}\sin\nu_k x\,.$$

Now if $\sqrt{-\gamma\alpha}$ is at a pole of $\tan(x)$ we have

iii, β) $l\sqrt{-\gamma\alpha}=(k_0+\frac{1}{2})\pi$:

for some nonnegative integer k_0 .

We get k_0 eigenvalues one each in the intervals

$$(\frac{\pi}{l}(k-\frac{1}{2}),\frac{\pi}{l}k)\,,\ k\leq k_0\,.$$

To see what happens at $\nu=\sqrt{-\gamma\alpha}$, where both function g and f have now a pole, we consider the equation (3') instead of (3) and find that $\nu_{k_0+1}=\sqrt{-\gamma\alpha}$ is a solution of (3') hence we get an eigenvalue $\lambda_{k_0+1}=-\gamma\alpha$ for this value.

Finally we get countably many solutions ν_k of (3) one each in the intervals

$$((\frac{\pi}{l}k-1,\frac{\pi}{l}(k-\frac{1}{2}))\ \text{for}\ k=k_0+2,k_0+3,\ldots\ .$$

Altogether again, we have only positive eigenvalues $\lambda_k=\nu_k^2$, $k=1,2,\ldots$, with the eigenvectors

$$\phi_k(x)=\cos\nu_k x\ -\ \frac{\alpha)}{\nu_k a}\sin\nu_k x\,.$$

2. Case $\alpha<\gamma<0$

We get started with

i) $\lambda<0$.

Note that for $\nu = \sqrt{-\lambda}$ the function $g(\nu) = \tanh \nu l$ is positive and concave down (in ν).

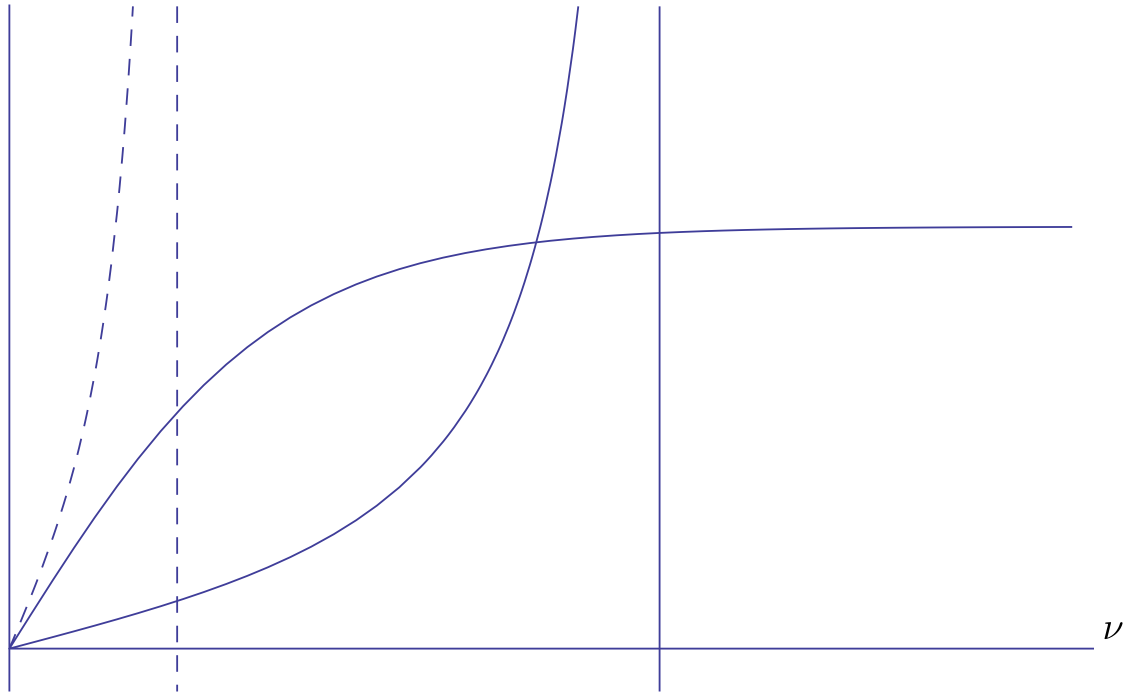

Fig. 10

Since $f(\nu) = \dfrac{\nu(\alpha - \gamma)}{\nu^2 - \gamma\alpha} < 0$ if $\nu^2 > \gamma\,\alpha$.

Hence no eigen value for $\nu > \sqrt{\gamma\alpha}$.
For $\nu^2 = \gamma\,\alpha$, equation (1') has no positve solution.
For $\nu^2 < \gamma\,\alpha$, we note that

$$f = \frac{\nu(\alpha - \gamma)}{\nu^2 - \gamma\alpha},$$

$$f' = \frac{(\gamma - \alpha)(\nu^2 + \gamma\,\alpha)}{(\nu^2 - \gamma\,\alpha)^2},$$

$$f'' = (\alpha - \gamma)\frac{4\,\nu\,\gamma^{\alpha}}{(\nu^2 - \gamma\alpha)^3},$$

and with $\nu^2 < \gamma\,\alpha$, we get $\frac{(\alpha - \gamma)}{(\nu^2 - \gamma\,\alpha)^3} > 0$.

Consequently, the function $f(\nu)$ is concave up for $0 < \nu < \sqrt{\gamma\alpha}$,
On the other hand $g(\nu) = \tanh \nu l$ is concave down and hence f and g can have at most two common points. One common point is the origin because

$$0 = f(0) = g(0).$$

So there might be at most one eigenvalue for

$$0 < \nu < \sqrt{\alpha\gamma}.$$

Observing that $f(\nu) \to \infty$ as $\nu \to \sqrt{\alpha\gamma}$, we conclude that there is exactly one solution if and only if

$$f'(\nu)|_{\nu=0} < g'(\nu)|_{\nu=0}.$$

Calculating

$$f'(0) = \tfrac{\gamma-\alpha}{\gamma\alpha} \quad \text{and} \quad g'(0) = l,$$

we obtain that there is exactly one negative eigenvalue in the interval $(0, \sqrt{\alpha\gamma})$, if and only if

$$(\gamma - \alpha) < l\,\gamma\,\alpha. \qquad (*)$$

ii) $\lambda = \mathbf{0}$

From (2) we get that

0 is an eigenvalue if and only if

$$(\gamma\,\alpha)l = \gamma - \alpha.$$

ii) $\lambda > \mathbf{0}$.

Let $g(v) = \tan(lv)$ and $f(v) = (\gamma - \alpha)\frac{\nu}{(\nu^2+\gamma\alpha)}$.

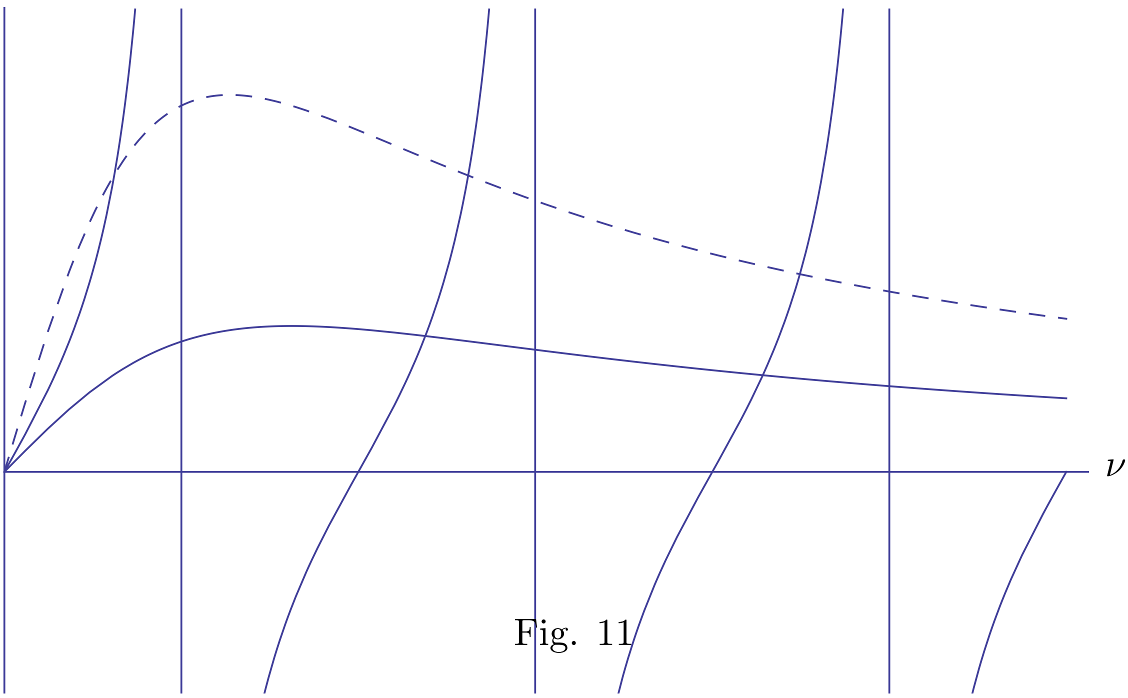

Fig. 11

If $(\gamma - \alpha) > l\,\gamma\,\alpha$ (then we have no non positive eigen values) and we note that

$$\gamma'(\nu)|_{\nu=0} = (\tan\nu\, l)'|_{\nu=0} = \frac{1}{\cos^2 \nu l} l|_{\nu=0} = l.$$

and

$$f'(\nu) = [(\gamma-\alpha)\frac{\nu}{(\nu^2+\gamma\alpha)}]'|_{\nu=u}$$

$$= (\gamma-\alpha)\frac{(\nu^2+\gamma\alpha)-\gamma(2\nu)}{(\nu^2+\gamma\alpha)^2}|_{\nu=0}$$

$$= (\gamma-a)\frac{\gamma\alpha-\gamma^2}{(\nu^2+\gamma\alpha)^2}|_{\nu=0} = \frac{\gamma-\alpha}{\gamma\alpha}\,.$$

That is $g'(0) < f'(0)$.

Hence we have that the smallest solution ν_0 of (3) is in $(0, \frac{\pi}{2})$, if $(\gamma-\alpha) > l\,\gamma\,\alpha$,
and there is no solution of (3) in $(0, \frac{\pi}{2})$, if $(\gamma-\alpha) \le l\,\gamma\,\alpha$.

So now, we have located the smallest eigenvalue for each possible relation between $(\gamma-\alpha)$ and $l\gamma\alpha$.

The location of the solutions ν_k, $k = 1, 2, 3, , \ldots$ of (3) are in the intervals

$$(k\frac{\pi}{l}, (k + \frac{1}{2})\frac{\pi}{l}),$$

as can been seen from the above sketch, note that $f(\nu) \geq 0$ for $\nu > 0$.

The following results on the most basic Sturm-Liouville problems will be supposed to be known threafter and can be used without further justifications.

Fourier Series, Fourier Sine series, Fourier Cosine Series:

Fourier Sine Series:
The regular Sturm-Liouville problem

$$\begin{cases} X'' + \lambda X & 0 < x < l \\ X(0) = 0, \\ X(l) = 0, \end{cases}$$

has the the eigenvalues

$$\lambda_n = (\frac{n\pi}{l})^2, \; n = 1, 2, 3, \ldots,$$

with eigenfunctions

$$\phi_n = \sin((\frac{n\pi}{l}x).$$

We have

$$\sum_{1=n}^{\infty} b_n \sin(\frac{n\pi}{l}x)$$

is the Fourier Sine representation of a function f if

$$b_n = \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi}{l}x)dx.$$

Fourier Cosine Series:
The regular Sturm-Liouville problem

$$\begin{cases} X'' + \lambda X \\ X'(0) = 0, \\ X'(l) = 0, \end{cases}$$

has the the eigenvalues

$$\lambda_0 = 0, \quad \lambda_n = (\frac{n\pi}{l})^2, \; n = 1, 2, 3, \ldots,$$

with the eigenfunctions

$$\phi_0 = 1, \quad \phi_n = \cos(\frac{n\pi}{l} x).$$

We have

$$a_0 + \sum_{1=n}^{\infty} a_n \cos(\frac{n\pi}{l} x)$$

is the Fourier Cosine representation of a function f if

$$a_0 = \frac{1}{l} \int_0^l f(x) dx, \quad a_n = \frac{2}{l} \int_0^l f(x) \cos((\frac{n\pi}{l} x) dx.$$

Fourier Series:
The periodic Sturm-Liouville problem

$$\begin{cases} X'' + \lambda X \\ X(-l) = X(l); \\ X'(-l) = X'(l); \end{cases}$$

has the the eigenvalues

$$\lambda_0 = 0, \quad \lambda_n = (\frac{n\pi}{l})^2, \; n = 1, 2, 3, \ldots,$$

with eigenfunctions

$$\phi_0 = 1, \text{ for } \lambda_0;$$

For λ_n we have two orthogonal eigenfunctions

$$\phi_n(x) = \cos((\frac{n\pi}{l} x) \quad \text{and} \quad \psi_n(x) = \sin((\frac{n\pi}{l} x); \; n = 1, 2, 3, \ldots.$$

We have that

$$a_0 + \sum_{1=n}^{\infty} [a_n \cos(\frac{n\pi}{l} x) + b_n \sin(\frac{n\pi}{l} x)]$$

is the Fourier representation of a function f if

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx,$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos((\frac{n\pi}{l}x)dx\,,$$

and

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin((\frac{n\pi}{l}x)dx\,.$$

The Fourier Sine Sequence and Fourier Cosine Sequence are complete those are regular Sturm-Liouville problems. As a corollary we get that the Fourier Sequence is complete:

(Note that for a function $f(x)$ on $(-l, l)$ we have that

$$g(x) = f(x) + f(-x), \quad \text{and } h(x) = f(x) - f(-x)$$

are even and odd functions, respectively, and we have

$$f(x) = \frac{1}{2}(g(x) + h(x))\,.$$

On the point wise convergence of the eigen function expansion.

For many Sturm-Liouville problems in addition to the mean square convergence of the eigen function expansion further convergence properties are known.

For instance, for the Fourier series we have if $f(x)$ and $f'(x)$ are continuous periodic function then the Fourier series converges point wise at all x.

If $f(x)$ and $f'(x)$ are periodic piecewise continuous function, then the Fourier series converges point wise to $\frac{1}{2}\big(f(x-0) + f(x+0)\big)$.

Recall a function is called piecewise continuous on an open interval if $f(x+0)$ and $f(x-0)$ exist for all $x\ \xi(a, g)$ and except for possibly a infinite number of exceptional point ((points of a "jump") we have $f(x+\nu) = f(x-0)$. (i.e. $f(x)$ is continuous at those points).

At a jump Gibbs phenomenon occurs, i.e. the approximation will always over shoot the one sided limit by about $\frac{1}{9}$ the height of the jump.

Analogous statements hold for the Fourier Sine Series and Fourier Cosine series.

Example

Earlier we found

$$x = \sum_{n=1}^{\infty_1} \frac{2}{n}(-1)^{n+1} \sin n\, x \text{ in } (0,\pi)$$

For $x = \frac{\pi}{2}$

$$\frac{\pi}{2} = \sum_{n=1}^{\infty_1} \frac{2}{n}(-1)^{n+1} \sin \frac{n\pi}{2}$$

$$\sin\ \frac{n\pi}{2} = \begin{cases} 0 & \text{if } n \text{ even} \\ (-1)^{\frac{n-1}{2}} & \text{if } n \text{ odd} \end{cases} .$$

So

$$\frac{\pi}{2} = \sum_{\substack{n=1 \\ n\, odd}}^{\infty} \frac{2}{n}(-1)^{\frac{n-1}{2}} = \sum_{m=1}^{\infty} \frac{2}{2m-1}(-1)^{m-1}$$

for $n = 2m-1$, or

$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \dots\ .$$