5. Boundary Value Problems (using separation of variables).

Seven steps of the approach of separation of Variables:

1) Separate the variables:

   (by writing e.g. $u(x,t) = X(x)T(t)$ etc..
2) Find the ODE for each "variable".
3) Determine homogenous boundary values to stet up a Sturm- Liouville problem.
4) Find the eigenvalues and eigenfunctions.
5) Solve the ODE for the other variables for all different eigenvalues.
6) Superpose the obtained solutions
7) Determine the constants to satisfy the boundary condition.

Second example: Initial boundary value problem for the wave equation with periodic boundary conditions on $D = (-\pi, \pi) \times (0, \infty)$

$$\text{(IBVP)} \begin{cases} u_{tt} - c^2\, u_{xx} = 0 & \text{(Hyperb. PDE)} \\ u(x,0) = f(x) & \text{IC for } u\,, \\ u_t(x,0) = g(x) & \text{IC for } u'\,, \\ u(-\pi,t) = u(\pi,t) & \text{BC for } u \\ u_x(-\pi,t) = u_x(\pi,t) & \text{BC for } u_x \end{cases}$$

Step 1.

$$u(x,t) = X(x)T(t)$$

$$u_{xx}(x,t) = X''(x)T(t)$$

and

$$u_{tt}(x,t) = X(x)T''(t)$$

Step 2.

Substituting that into the equation and and dividing the result by

$$c^2 X(x) T(t)$$

yields the ODEs

$$X'' + \lambda X = 0 \text{ and } T'' + \lambda c^2 T = 0\,.$$

Step 3.

We obtain the B.C:

$$X(-\pi) = X(\pi)\,, \text{ and } \; X'(-\pi) = X'(\pi)$$

Step 4.

The eigenvalues and eigenfunctions of

$$\begin{cases} X'' + \lambda X = 0\,, \\ X(-\pi) = X(\pi)\,, \\ X'(-\pi) = X'(\pi) \end{cases}$$

are

$$\lambda_0 = 0 \text{ with eigenfunction } \phi(x) \equiv 1$$

and

$$\lambda_n = n^2 \text{ with eigenfunctions } \phi_n(x) = \cos nx \text{ and } \psi_n(x) = \sin nx\,.$$

Step 5.

1) For $\lambda_0 = 0\,,$ the ODE for T_0 reads:

$$T_0'' = 0;$$

with general solution $T_0 = A_0 + B_0 t\,.$

2) For $\lambda_n = n^2\,,$ the ODE for T_n reads:

$$T_n'' + n^2c^2T_n = 0;$$

with general solution $\tau_n(t) = A_n \cos nct + B_n \sin nct$.

Step 6.

We get the following solution of the PDE satisfying the boundary conditions:

$$u_0(x,t) = (\tilde{A}_0 + \tilde{B}_0 t)C \quad \text{and}$$

$$u_n(x,t) = (\tilde{A}_n \cos nct + \tilde{B}_n \sin nct)(\tilde{C}_n \cos nx + \tilde{D}_n \sin nx)$$

"Streamlining" the constants, we write these as

$$u_0(x,t) = (A_0 + B_0 t) \quad \text{and}$$

$$u_n(x,t) =$$

$$(A_n \cos nct + B_n \sin nct) \cos nx + (C_n \cos nct + D_n \sin nct) \sin nx \, .$$

Superposing these functions we get

$$u(x,t) = (A_0 + B_0 t) +$$

$$\sum_{n=1}^{\infty} [(A_n \cos nct + C_n \sin nct) \cos nx + (B_n \cos nct + D_n \sin nct) \sin nx]$$

which is a solution of the PDE satisfying the homogeneous boundary condition, provided the coefficients allow for an appropriate convergence of the sequence.

Step 7.

Determine the coefficients so that $u(x,t)$ satisfies the initial conditions.

1) First initial condition gives:

$$f(x) = A_0 + \sum_{n=1}^{\infty} [A_n \cos nx + B_n \sin nx]$$

If we choose these constants to be the coefficients of the Fourier Series of f , then the first IC is satisfied:

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f dx\,, \quad A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f \cos nx dx \quad \text{and}$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f \sin nx dx$$

2) Second initial condition gives:

$$g(x) = u_t(x,0) = B_0 +$$

$$\sum_{n=1}^{\infty}[(A_n - nc\sin nct + C_n nc\cos nct)\cos nx +$$

$$(B_n - nc\sin nct + D_n nc\cos nct)\sin nx]\big|_{t=0}$$

$$= B_0 + \sum_{n=1}^{\infty}[ncC_n \cos nx + ncD_n \sin nx]$$

If we choose the constants B_0 , ncC_n and ncD_n , to be the coefficients of the Fourier Series of g then the second IC is satisfied: so

$$B_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g dx\,, \quad C_n = \frac{1}{nc\pi}\int_{-\pi}^{\pi} g \cos nx dx \quad \text{and}$$

$$D_n = \frac{1}{nc\pi}\int_{-\pi}^{\pi} g \sin nx dx$$

With the above choice of constants $u(x,t)$ is the solution of the problem.

Example 3.

Now we deal with a mixed boundary value problem for the Laplace equation.:

$$
\text{(IBVP)} \begin{cases} u_{xx} + u_{yy} = 0 & \text{Laplace EQ. in } (0,l) \times (0,m) \\ u_y(x,0) = 0 & \text{NC at } y = 0\,, \\ u(x,m) = 0 & \text{DC at } y = m\,, \\ u(0,y) = g(y) & \text{nonhomogeneous DC at } x = 0 \\ u(l,y) = 0 & \text{DC at } x = l \end{cases}
$$

Step 1.

$$u(x,y) = X(x)Y(y)$$

$$u_{xx} = X''Y$$

and

$$u_{yy} = XY''$$

Step 2.

Substituting that into the equation and and dividing the result by XY yields the ODEs

$$X'' \pm \lambda X = 0 \text{ and } Y'' \mp \lambda Y = 0\,.$$

Step 3.

We obtain the B.C:

$$Y'(0) = Y(m) = 0\,, \text{ and } \; X(l) = 0\,, \;\; X(0)Y(y) = g(y)\,.$$

So we have Y as the unkown in the Sturm- Liouville problem.

Step 4.

We have to find the eigenvalues and eigenfunctions of

$$
\begin{cases} Y'' + \lambda Y = 0\,, \\ Y'(0) = Y(m) = 0 \end{cases}
$$

i) $\lambda < 0$

No negative eigen values since $\mathcal{B} = 0$.

ii) $\lambda = 0$

The general solution is $\phi(y) = A + By$, and we get from the first BC

$$0 = \phi'(0) = B \quad \text{and then from the second}$$

$$0 = \phi(m) = A$$

hence zero is not an eigenvalue.

iii) $\lambda > 0$ The general solution is $\phi(y) = A\cos\nu y + B\sin\nu y$. with $\nu = \sqrt{\lambda}$ and we get from the first B.C

$$0 = \nu(A(-\sin\nu y) + B\cos\nu y)\big|_{y=0} = \nu B, \quad \Rightarrow \quad B = 0,$$

and then from the second BC

$$0 = A\cos\nu m,$$

which allows for eigenvalues if $\cos\nu m = 0$ or $\nu m = (n - \frac{1}{2})\pi$, that is

$$\lambda_n = (\frac{(2n-1)\pi}{2m})^2, \quad n = 1, 2, \ldots \;;$$

with eigenfunction

$$\phi_n(y) = \cos(\frac{(2n-1)\pi y}{2m}).$$

Step 5.

With $\lambda_n = (\frac{(2n-1)\pi}{2m})^2 =: \nu_n^2$, the ODE

$$X_n'' - \lambda X_n = 0;$$

has the general solution $\psi_n(x) = A_n\cosh\nu_n x + B_n\sinh\nu_n x$.

Step 6.

So the solutions

$$u_n(x,t) = (A_n\cosh\nu_n x + B_n\sinh\nu_n x)(\cos\nu_n y)$$

of the PDE also satisfy the boundery conditions $u_y(x,0) = u(x,m) = 0$.

Superposing these functions we get

$$u(x,y) = \sum_{n=1}^{\infty} (A_n \cosh \nu_n x + B_n \sinh \nu_n x)(\cos \nu_n y)$$

which is a solution of the PDE satisfying the homogeneous boundary condition for y, provided the coeffcients allow for an appropriate convergence of the sequence.

Step 7.

Determine the coefficients so that the PDE satisfies the other boundary conditions:

In order to deal firstly with the homogeneous boundary condition we write

$$u(x,y) = \sum_{n=1}^{\infty} (A_n \cosh \nu_n (x-l) + B_n \sinh \nu_n (x-l)(\cos \nu y) .$$

(Recall: Solutions of ODEs with constant coefficients are translation invariant.) We get from the second B.C:

$$0 = u(l,y) = \sum_{n=1}^{\infty} A_n$$

Hence $A_n = 0, \quad n = 1, 2, \dots$, and the first BC requires

$$g(y) = \sum_{n=1}^{\infty} B_n \sinh \nu_n (-l)](\cos(\nu_n y)$$

So if we choose the constants $B_n \sinh \nu_n(-l)$ to be the coefficients of the generalized Fourier Series of g, then the BC for $x = 0$ is satisfied.

Consequently:

$$B_n = \frac{2}{m \sinh \nu_n(-l)} \int_0^m g(y) \cos \frac{(2n-1)\pi y}{2m} dy .$$

With the above choice of constants B_n the solution of the above boundary value problem for the Laplace equation is:

$$u(x,y)=\sum_{n=1}^{\infty}\sinh(\frac{(2n-1)\pi}{2m}(x-l))\cos(\frac{(2n-1)\pi}{2m}y)\,.$$

Remark:

If the boundary conditions are inhomogeneous at more than one side of the rectangle $(0,l)\times(0,m)$ then we separate the problem into problems with inhomogeneous BC given at one side only, and we obtain the solution by superposing the solution of the separated problems.

Inomogeneous problems and higher dimensional problems

As a first example let us consider

$$\begin{cases} u_t - k\,u_{xx} = f(x,t) & \text{(inhom. PDE)}\\ u(x,0)=0 & \text{Initial condition for } u\,,\\ u(0,t)=0 & \text{BC at } x=0\\ u(\pi,t)=0 & \text{BC at } x=\pi \end{cases}$$

Our first approach is to find solution of the kind

$$u(x,t)=\sum_{n=1}^{\infty}\quad A_n(t)\varphi_n(x)\,,$$

Where (φ_n) is a sequence of eigenfunctions of Sturm- Liouville problem of the associated homogeneous problem.

In our case that is the sequence of eigenfunctions of the Fourier Sine Series. So here we have:

$$u(x,t)=\sum_{n=1}^{\infty}\quad A_n(t)\sin nx\,.$$

Assuming that we can interchange the differentiation with the infinite sum, we calculate

$$u_t(x,t) = \sum_{n=1}^{\infty} \quad A'_n(t)\sin nx\,.$$

$$u_{xx}(x,t) = \sum_{n=1}^{\infty} \quad A_n(t)(-n^2\sin nx)\,.$$

With those derivatives the PDE reads

$$\sum_{n=1}^{\infty} \quad A'_n(t)\sin nx + k\sum_{n=1}^{\infty} \quad A_n(t)n^2\sin nx = f(x,t)$$

Now we multiply the inhomogeneous PDE with an eigenfunction $\varphi_k = \sin(kx)$, say, and integrate over the domain $(0,\pi)$. Because of the orthogonality of the eigenfunction we get

$$A'_n(t)\int_0^{\pi} \sin^2 nxdx + kn^2 \; A_n(t)\int_0^{\pi} \sin^2 nxdx$$

$$= \int_a^b f(x,t)\sin nx\,.$$

Dividing by $\int_0^{\pi} \sin^2 nxdx$ we get on the right hand side a function in t say,

$$\psi_n(t) = (\int_0^{\pi} \sin^2 nxdx)^{-1}\int_0^{\pi} f(x,t)\sin nxdx\,.$$

and we get the first order ODE for $A_n(t)$ as

$$A'_n(t) + kn^2 A_n(t) = \psi_n(t)\,. \qquad (*)$$

The initial condition

$$u(x,0) = 0\,,$$

is satisfies if we add the initial condition

$$A_n(0) = 0$$

to the ODE. This is a first order ODE for which we can write down the solution in the form

$$A_n(t) = e^{-n^2kt}\int_0^t \psi_n(\tau)e^{n^2k\tau}d\tau\,,$$

which in turn we can write as

$$A_n(t) = \int_0^t \psi_n(\tau)e^{-n^2k(t-\tau)}d\tau\,.$$

Example: $f(x,t) = t\,,$ so

$$\psi_n(t) = \frac{2}{\pi}\int_0^\pi t\sin nx dx = \begin{cases} 0 & \text{if } n\,, \text{ is even} \\ \dfrac{4t}{n\pi} & \text{if } n\,, \text{ is odd} \end{cases}\,.$$

In that case a more direct approach to solve the ODE's is possible. We have that $A_n(t) = C_n e^{-kn^2t}$ and is a solution of the homogeneous equation for $(*)$. That is the case if n is even.

If n is odd, we use as trial solutions for a particular solution of (*)

$$(A_n)_p(t) = \alpha t + \beta\,, \text{ with } (A_n)_p'(t) = \alpha$$

From the Ode we get

$$\alpha + kn^2(\alpha t + \beta) = \frac{4t}{n\pi}$$

Comparing coefficients we get

$$kn^2\alpha t = \frac{4t}{n\pi}\,, \quad \text{and } \alpha + kn^2\beta = 0$$

or

$$\alpha = \frac{4}{\pi k n^3}\,, \quad \text{and } \beta = -\frac{4}{\pi k^2 n^5}\,.$$

Finally the initial condition $A_n(0) = 0\,,$ gives

$$C_n = 0 \text{ if } n \text{ even and } C_n = -\beta_n = \frac{4}{\pi k^2 n^5}\,, \text{ if } n \text{ is odd.}$$

Altogether, with $2m-1 = n$ we have

$$u(x,t) = \sum_{m=1}^{\infty} \quad A_n(t)\sin(2m-1)x\,.$$

with

$$A(t) = \left(\frac{4}{\pi k^2(2m-1)^5} e^{-k(2m-1)^2 t} + \frac{4}{\pi k(2m-1)^3} t - \frac{4}{\pi k^2(2m-1)^5}\right).$$

Inhomogeneous boundary conditions

As an example let us deal here with the one-dimensionnal wave equation (vibrating string equation) on $(0,l)\times(0,\infty)$:

$$\begin{cases} u_{tt} - c^2\, u_{xx} = 0 & \text{(inhom. PDE)} \\ u(x,0) = 0 & \text{Initial cond. for } u\,, \\ u_t(x,0) = 0 & \text{Initial cond. for } u_t\,, \\ (u(0,t) = g(t) & \text{BC at } x = 0 \\ u(l,t) = h(t) & \text{BC at } x = l \end{cases}$$

The most straight forward approach here is to write $u(x,t) = w(x,t) + v(x,t)$ where $v(x,t)$ is a “simple function ” satisfying the boundary conditions.

Then $w(x,t) = u(x,t) - v(x,t)$ satisfies the inhomogeneous PDE with homogeneous B.C.:

Usually we set

$$v(x,t) = g(t)(1 - x\frac{1}{l}) + h(t)(x\frac{1}{l})\,.$$

We calculate

$$v_{tt} = g''(t)(1 - x\frac{1}{l}) + h''(t)(x\frac{1}{l})\,,$$

and

$$v_{xx} = 0\,.$$

So we need w to be a solution of

$$\begin{cases} w_{tt} - c^2\, w_{xx} = -(g''(t)(1-\frac{1}{l}x) + h''(t)(\frac{1}{l}x); \\ w(x,0) = -(g(0)(1-x\frac{1}{l}) + h(0)(x\frac{1}{l})), \\ w_t(x,0) = -(g'(0)(1-x\frac{1}{l}) + h'(0)(x\frac{1}{l})), \\ (w(0,t) = w(l,t) = 0\,. \end{cases}$$

which we can solve using the discussed methods.

Elliptic equations on spherical domains.

$$(1)\begin{cases} u_{xx} + u_{yy} = 0 & \text{on} \quad \sqrt{x^2+y^2} < R \\ (x,y) = f(x,y) & \text{for} \quad \sqrt{x^2+y^2} = R \end{cases}$$

We introduce a change of variables setting

$$U(r,\varphi) = u(x,y)\Big|_{\substack{x=r\cos\varphi \\ y=r\sin\varphi}}$$

then $u_{xx} + u_{yy} = 0$ for $\sqrt{x^2+y^2} < R$, imply that

$$(2)\ \begin{cases} U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\varphi\varphi} = 0 & \text{for } 0 < r < R,\ -\pi < \varphi < \pi\,, \\ U(R,\varphi) = F(\varphi), & \text{with } F(\varphi) = f(R\cos\varphi, R\sin\varphi) \end{cases}$$

Somehow the set of auxiliary conditions does not seem to be complete but if we separate the variables first, this will give us a better picture of what will be needed.

Step 1:We set

$$U(r,\varphi) = \mathcal{R}(r)\,\Phi(\varphi)\,, \quad \text{and get}$$

$$U_r = \mathcal{R}'\Phi\,, \quad U_{rr} = \mathcal{R}''\Phi \quad \text{and} \quad U_{\varphi\varphi} = \mathcal{R}\phi''\,.$$

The PDE of (2) implies

$$\mathcal{R}''\Phi + \frac{1}{r}\mathcal{R}\Phi + \frac{1}{r^2}\mathcal{R}\Phi'' = 0\,,;$$

Step 2: Multiplying with $\dfrac{r^2}{\mathcal{R}\Phi}$. yields

$$\frac{r^2\mathcal{R}'' + r\mathcal{R}'}{\mathcal{R}} + \frac{\Phi''}{\phi} = 0\,,$$

which gives us the ODE's

$$(3) \quad r^2\mathcal{R}'' + r\mathcal{R}' \mp \lambda\mathcal{R} = 0,,, \qquad (2) \quad \Phi'' \pm \lambda\Phi = 0\,.$$

Step 3. We would like to work with Φ as the ODE for a Sturm-Lionville problem. In order to do this we observe that the "reasonable" requirement that the solution is continuously differentiable implies

$$U(r,-\pi) = U(r,\pi) \quad \text{and} \quad U_\varphi(r,-\pi) = U_\varphi(r,\pi)\,.$$

Step 4 For Φ, we get the Sturm-Liouville problem

$$ST.L. \begin{cases} \Phi'' + \lambda\Phi = 0 \\ \Phi(-\pi) = \Phi(\pi) \\ \Phi'(-\pi) = \Phi'(\pi) \end{cases}$$

with the eigen values $\lambda_0 = 0, \quad \lambda_n = n^2, \quad n = 1, 2, \ldots$ and the eigen function $1, \cos\ x, \sin\ x,\ \cos\ 2\ x\ \sin\ 2\ x\ \ldots.$

Step 5 We have to consider the ODE's

$$r^2\mathcal{R}'' + r\mathcal{R}' = 0 \qquad (\text{for } \lambda = 0)$$

and

$$r^2\mathcal{R} + r\mathcal{R}' - n^2\mathcal{R} = 0 \qquad (\text{for } \lambda_n = n^2)\,.$$

For $\lambda = 0$ we get with $w = \mathcal{R}'$

$$w'(\pi) + \tfrac{1}{r}w(r) = 0\,.$$

The solution can be found by a "separation of variables approach". We get

$$\frac{w'}{w} = -\frac{1}{r}$$

$$\ln\ w = \ln(-r) + C$$

$$w = C_1\frac{1}{r}\cdot$$

or

$$\varphi_0(r) = A_0 + B_0 \ \ln(r)$$

The second ODE an Euler equation has the solutions

$$\varphi_n(r) = A_n r^n + B_n r^{-n}$$

Step 6

Now superposing the solution of (1) and (2) we get

$$U(r,\phi) = A_0 + \sum_{n=1}^{\infty} r^n \ (A_n \ \cos \ n\varphi + B_n \ \sin \ n\varphi)$$

$$+C_0 \left(\ln(r)\right) + \sum_{n=1}^{\infty} (\frac{1}{r})^n \ (C_n \ \cos \ n\varphi + D_n \ \sin \ n\varphi)$$

Step 7

Since $\ln(r)$ and $\frac{1}{r^n}$ are unbounded the functions $u_n(r,\theta)$ which need to superpose are not solutions of the PDE if those terms a part of $u_n(r,\theta)$. So we set the coefficients C_n , $n = 0, 1, 2, \ldots$ and D_n, $n = 1, 2, \ldots$ zero. Then

$$u(r,\varphi) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \ \cos \ n\varphi + B_n \ \sin \ n\varphi)$$

Is a solution of the PDE in the disk and we get we get from the initial condition

$$U(R,\varphi) = F(\varphi) = A_0 + \sum_{n=1}^{\infty} R^n A_n \ \cos \ n\varphi + \sum_{n=1}^{\infty} R^n B_n \ \sin \ n\varphi$$

which are satisfied if A_0 , $R^n A_n$, $R^n B_n$ are the Fourier coefficients of F, i.e.:

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\varphi) d\varphi$$

$$A_n = \frac{1}{\pi R^n} \int\limits_{-\pi}^{+\pi} F(\varphi)\ \cos\ n\varphi d\varphi$$

$$B_n = \frac{1}{\pi R^n} \int\limits_{-\pi}^{+\pi} F(\varphi)\ \sin\ n\varphi\, d\varphi$$

Note, that we can not set the coefficients $C_n\ ,\ n = 0, 1, 2, \ldots$ and $D_n,\ n = 1, 2, \ldots$ the be zero in case the domain is a "washer", say

$$D = \{(x, y) \,\big|\, 0 < a^2 < x^2 + y^2 < R^2\}\,.$$

In that case there will be an additional boundary condition for $x^2+y^2 = a^2$. and we get from the BC's two equation for each pair of coeffiecents A_n, C_n and two equation for the pair of coeffiecents B_n, D_n, respectively, c.f. the related Homework problem.

We want to consider the solution for the Disk somewhat closer:

We have

$$u(r, \phi) = \frac{1}{2\pi} \int\limits_{-pi}^{\pi} F(\theta) d\theta$$

$$+ \sum_{n=1}^{\infty} \frac{r^n}{\pi R^n} \int\limits_{-\pi}^{+\pi} F(\theta)(\cos\ n\phi \cos\ n\theta + \sin\phi \sin\ n\theta) d\theta$$

$$= \frac{1}{2\pi} \int\limits_{-pi}^{\pi} F(\theta)(1 + 2\sum_{n=1}^{\infty} (\frac{r}{R})^n \cos\ n(\phi - \theta)) d\theta$$

$$= \frac{1}{2\pi} \int\limits_{-pi}^{\pi} F(\theta)(1 + \sum_{n=1}^{\infty} (\frac{r}{R})^n (e^{in(\phi-\theta)} + e^{-in(\phi-\theta)})) d\theta$$

$$= \frac{1}{2\pi}\int\limits_{-pi}^{\pi} F(\theta)(1 + (\frac{re^{i(\phi-\theta)}}{R - re^{i(\phi-\theta)}} + \frac{re^{-i(\phi-\theta)}}{R - re^{-i(\phi-\theta)}}))d\theta$$

and so we get Poissons integral formula

$$u(r,\phi) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} F(\theta)\frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(\phi-\theta)}d\theta$$

For $r = 0$ we get

$$u(0,\phi) = \frac{1}{2\pi R}\int\limits_{-\pi}^{\pi} U(R,\theta)R\,d\theta\,,$$

That is the vaule of u at the center is the average of the values of u at the boundary of the disk.

Remark:

Consequences: Mean Value Theorem and Maximum Principle for solutions of the Laplace equations.

Higher dimensional problems

Cooling of the sphere:

Under the assumption of a rotational symmetric initial temperature distribution and an constant outer temperature distribution (normalized to zero the inital boundary value problem for heat distribution in a sphere

$$\begin{cases} u_t - k\Delta u = 0\,, & t > 0,\ \| \underline{x} \| < \pi\,, \\ u(\underline{x},0) = f(\| \underline{x} \|)\,, & \\ u(\underline{x},t) = 0\,, & \text{if } \| \underline{x} \| = \pi\,, \end{cases}$$

becomes

$$\begin{cases} u_t - k(u_{\rho\rho} + \dfrac{2}{\rho} u_\rho) = 0\,, & t > 0\,,\ \rho < \pi\,, \\ u(\rho, 0) = f(\rho)\,, \\ u(\pi, t) = 0\,. \end{cases}$$

(Assuming that the heat distribution is only a matter of the distance to the center the derivatives with respect to ϕ are vanishing.)

Separating the variables $u(\rho, t) = \mathcal{R}T$ we get

$$T'\mathcal{R} + k(\mathcal{R}''T + \frac{2}{\rho}(\mathcal{R}'T)) = 0\,, \text{ and so}$$

$$\frac{T'}{kT} = -\frac{\mathcal{R}'' + \frac{2}{\rho}(\mathcal{R}')}{\mathcal{R}} = \lambda$$

Since we have now choice for which variable we might use for a Sturm - Liouville problem, we need to consider

$$\mathcal{R}'' + \frac{2}{\rho}\mathcal{R}' + \lambda\mathcal{R} = 0, \quad 0 < \rho < \pi$$

But this is not in the form

$$(pX')' - qX + \lambda mX = 0\,.$$

Further more the boundary conditions $0 = u(\pi, t) = \mathcal{R}(\pi)T(t)$, provides one homogeneous BC

$$\mathcal{R}(\pi) = 0$$

but not a second one.

so we are not dealing with a regular Sturm - Liouville problem, here.

Introducing the functions $X(\rho) = \rho\mathcal{R}(\rho)$ we get for X the ODE

$$X'' + \lambda X = 0$$

and so for $\nu^2 = \lambda > 0$ we have the solutions

$$\mathcal{R}(\rho) = \frac{1}{\rho} C_1 \cos(\nu\rho) + \frac{1}{\rho} C_1 \sin(\nu\rho)\,.$$

Since we can use only bounded solutions we would like to disregard the unbounded functions in this sum. So replacing the second boundary condition by the condition that the solution need to be bounded might replace the second boundary condition. Then multipying the ODE with ρ^2 gives the singular Sturm-Liouville problem

$$\begin{cases} \rho^2\mathcal{R}'' + 2\rho\mathcal{R}' + \rho^2\lambda\mathcal{R} = 0, & 0 < x < \pi\,, \\ \mathcal{R}(\pi) = 0\,, & \\ \mathcal{R} \text{ a bounded function in } (0,\pi) & \end{cases},$$

whose eigenfunctions have the properties discussed in the last chapter with respect to the weigt function $m(\rho) = \rho^2$.

Is it an exercise using the above transformation to show that all eigen values are $\lambda_n = n^2$, $n = 1, 2, 3, \ldots$ with eigenfunctions

$$\phi_n(\rho) = \frac{\sin(n\rho)}{\rho}\,.$$

The solutions of the ODE for T are e^{-n^2kt} so that we have

$$u_n(\rho, t) = A_n e^{-n^2kt} \frac{\sin(n\rho)}{\rho}\,.$$

are bounded solutions of the PDE with $u_n(\pi, t) = 0$.

For $u(\rho, t) = \sum_{n=1}^{\infty} u_n(\rho, t)$, we determine the coefficents from the remaing auxilliary condition:

$$f(\rho) = u(\rho, 0) = \sum_{n=1}^{\infty} \frac{A_n}{\rho} \sin(n\rho)$$

With the generalized Fourier cofficient of the eigenfunctions $\phi_n(\rho)$ that is

$$A_n = \left(\int_0^\pi \left(\frac{\sin(\rho)}{\rho}\right)^2 \rho^2\, d\rho\right)^{-1} \int_0^\pi f(\rho) \frac{\sin\rho}{\rho} \rho^2\, d\rho\,.$$

$$= (\int_0^\pi \sin^2(\rho)\, d\rho \int_0^\pi f(\rho)\rho \sin\rho\, d\rho\,.$$

Note that that A_n are the Fourier Sine coefficients of ρf,

which we would get if we work the entire problem with the transformed functions.

Two dimensional Wave equation

Example

$$\begin{cases} u_{tt} - c^2\Delta\, w = 0\,, & (x,y,t) \in (0,L)\times(0,M)\times\mathbb{R}\,, \\ u(x,y,0) = f(x,y)\,, \\ u_t(x,0) = g(x,y)\,, \\ u(0,y,t) = u(L,y,t) = 0\,, \\ u_y(x,0,t) = u_y(0,M,t) = 0\,, \end{cases}$$

Separation of variables:

Substiuting $u(x,y,t) = X(x)Y(y)T(t)$ into the PDE we get

$$X(x)Y(y)T''(t) - c^2(X''(x)Y(y)T(t) + X(x)Y''(y)T(t)) = 0\,,$$

or

$$\frac{T''}{c^2T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$

this gives us first the ODE's for T, X and Y :

$$T'' + c^2\lambda T = 0$$

$$X'' + \mu X = 0$$

$$Y'' + \kappa Y = 0 \text{ with } \lambda = \mu + \kappa\,.$$

From the boundary condition we get the Sturm-Liuoville problems

$$\begin{cases} X'' + \mu X = 0 \\ X(0) = X(L) = 0\,. \end{cases}$$

and

$$\begin{cases} Y'' + \kappa Y = 0 \\ Y'(0) = Y'(M) = 0\,. \end{cases}$$

with are the Sturm-Liuoville problems for the Fourier Sine and Fourier Cosine Series, respectively. So we have the eigen values $\mu_m = (\dfrac{m\pi}{L})^2$ with eigen functions

$$\phi_m = \sin(\sqrt{\mu_m}\, x\,, \quad m = 1, 2, 3, 4, \dots\ ,$$

for the first Sturm-Liuoville problem.

For the second Sturm-Liuoville problem we have the eigenvalues $\kappa_k = (\dfrac{k\pi}{M})^2$ with eigen functions

$$\psi_k = \cos(\sqrt{\kappa_k}\, y),, \quad k = 1, 2, 3, 4, \dots\ ,$$

and

$$\psi_0 \equiv 1\,.$$

For each combination $\lambda_{mk} = \mu_m + \kappa_k$ we get the solution

$$\tau_{mk} = A_{mk}\cos(c\sqrt{\mu_m + \kappa_k}\, t) + B_{m,k}\sin(c\sqrt{\mu_m + \kappa_k}\, t)$$

of the ODE's for T

$$T'' + c^2\lambda_{mk} T = 0\,.$$

(Step 6) So the functions

$$u_{m,k}(x, y, t) = A_{m,k}\cos(c\sqrt{\mu_m + \kappa_k}\, t)\sin(\sqrt{\mu_m}x)\cos(\sqrt{\kappa_k}y)$$

$$+B_{m,k}\sin(c\sqrt{\mu_m + \kappa_k}\, t)\sin(\sqrt{\mu_m}x)\cos(\sqrt{\kappa_k}y)$$

for $m, k = 1, 2, 3, \dots$ and

$$u_{m,0} = A_{m,0}\cos(c\mu_m t)\sin(\sqrt{\mu_m}x) + B_{m,0}\sin(c\sqrt{\mu_m}t)\sin(\sqrt{\mu_m}x)$$

for $m = 1, 2, 3, \ldots$ are solutions of the PDE together with the homogeneous boundary condition. We superpose those function to get

$$u(x, y, t)$$

$$= \sum_{m=1}^{\infty} \quad [A_{m,0}\cos(c\sqrt{\mu_m}t)\sin(\sqrt{\mu_m}x) +$$

$$B_{m,0}\sin(c\sqrt{\mu_m}t)\sin(\sqrt{\mu_m}x)]$$

$$+\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \quad [A_{m,k}\cos(c\sqrt{\mu_m + \kappa_k}\,t)\sin(\sqrt{\mu_m}\mu_m x)\cos(\sqrt{\kappa_k}y)$$

$$+B_{m,k}\sin(c\sqrt{\mu_m + \kappa_k}\,t)\sin(\sqrt{\mu_m}\mu_m x)\cos(\sqrt{\kappa_k}y)] \ .$$

We determine the constants using the orthogonality relations of those eigenfunctions. We get from the first initial conditions:

$$f(x) = u(x, y, 0) =$$

$$= \sum_{m=1}^{\infty} \quad A_{m,0}\sin(\sqrt{\mu_m}x) + \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \quad A_{m,k}\sin(\sqrt{\mu_m}x)\cos(\sqrt{\kappa_k}y)$$

so

$$A_{m,k} = \frac{4}{LM}\int_0^L \int_0^M \ f(x, y))\sin(\sqrt{\mu_m}x)\cos(\sqrt{\kappa_k}y)dydx\,,$$

and

$$A_{m,0} = \frac{2}{LM}\int_0^L \int_0^M \ f(x, y)\sin(\sqrt{\mu_m})dydx\,.$$

The second initial condition gives

$$g(x) = u_t(x, y, 0) =$$

$$= \sum_{m=1}^{\infty} \quad c\sqrt{\mu_m} B_{m,0} \sin(\sqrt{\mu_m} x)$$

$$+ \sum_{m=1}^{\infty} \quad \sum_{k=1}^{\infty} \quad c\sqrt{\mu_m + \kappa_k} B_{m,k} \sin(\sqrt{\mu_m} x) \cos(\sqrt{\kappa_k} y) \quad ,$$

hence

$$B_{m,k} =$$

$$\frac{4}{c\sqrt{\mu_m + \kappa_k} LM} \int_0^L \int_0^M g(x,y) \sin(\sqrt{\mu_m} x) \cos(\sqrt{\kappa_k} y) dy dx \, ,$$

for $m, k = 1, 2, 3, \ldots$ and

$$B_{m,0} = \frac{2}{LM\sqrt{\mu_m}} \int_0^L \int_0^M g(x,y) \sin(\sqrt{\mu_m} x) dy dx \, ,$$

for $m = 1, 2, 3, \ldots$.