$L$-functions of $S_3(\Gamma_2(2, 4, 8))$

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van Geemen and van Straten [B. van Geemen, D. van Straten, The cuspform of weight 3 on $\Gamma_2(2, 4, 8)$, Math. Comp. 61 (1993) 849–872] showed that the space of Siegel modular cusp forms of degree 2 of weight 3 with respect to the so-called Igusa group $\Gamma_2(2, 4, 8)$ is generated by 6-tuple products of Igusa theta constants, and each of them are Hecke eigenforms. They conjectured that some of these products generate Saito–Kurokawa representations, weak endoscopic lifts, or $D$-critical representations. In this paper, we prove these conjectures. Additionally, we obtain holomorphic Hermitian modular eigenforms of $\text{GU}(2, 2)$ of weight 4 from these representations.

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1. Introduction

Let $S_2 = \{Z = ^tZ \in M_2(\mathbb{C}) \mid \Im(Z) > 0\}$ be the Siegel upper half space of degree 2. Let

$$\theta_m(Z) = \sum_{x \in \mathbb{Z}^2} \exp \left( 2\pi i \left( \frac{1}{2} \left( x + \frac{m'}{2} \right) Z \left( x + \frac{m'}{2} \right) + \left( x + \frac{m'}{2} \right) \left( \frac{m''}{2} \right) \right) \right)$$

be the Igusa theta constant with $m = (m', m'') \in \mathbb{Q}^2 \times \mathbb{Q}^2$. For a congruence subgroup $\Gamma$ of $\text{Sp}_4(\mathbb{Z})$ ($\subset \text{SL}_4(\mathbb{Z})$), let $S_\Gamma$ denote the Siegel modular 3-fold and $S_3(\Gamma)$ denote the space of Siegel modular cusp forms of weight 3 with respect to $\Gamma$. van Geemen and van Straten showed that $S_3(\Gamma_2(2, 4, 8))$
is spanned by certain 6-tuple products $\prod_{j=1}^6 g_{m_j}(n_j Z)$ with $m_j \in \{0, 1\}^4$, $n_j \in \{1, 2\}$ using the theta embedding of $S_\Gamma(2, 4, 8)$ into $\mathbb{P}^{13}$ (cf. [6]), where $\Gamma(2, 4, 8) = \Gamma(2, 4, 8)$ is defined by

$$I_4 + 4 \left[ \begin{array}{cc} A & B \\ C & D \end{array} \right] \in \text{Sp}_4(\mathbb{Z}) \quad | \quad A, B, C, D \in M_2(\mathbb{Z}), \quad \text{diag}(B) \equiv \text{diag}(C) \equiv 0, \quad \text{tr}(A) \equiv 0 \pmod{2} \right\}. \quad (1.1)$$

Through Igusa’s transformation formula, $\text{Sp}_4(\mathbb{Z})$ acts on these 6-tuple products. They showed that $S_3(\Gamma(2, 4, 8))$ is decomposed into eleven irreducible $\text{Sp}_4(\mathbb{Z})$-modules, and each module is generated by acting $\text{Sp}_4(\mathbb{Z})$ a 6-tuple product of Igusa theta constants. Further, they showed that these 6-tuple products is associated to irreducible cuspidal automorphic representations of $\text{PGSp}_4(\mathbb{A})$ (cf. Proposition 2.2). Computing some eigenvalues of Evdokimov’s Hecke operators on

$$g_1(Z) := \theta(0, 0, 0, 0)(2Z)\theta(1, 0, 0, 0)(Z)\theta(0, 1, 0, 0)(Z)\theta(0, 0, 1, 0)(Z)\theta(0, 0, 0, 1)(Z),$$

$$g_4(Z) := \theta(0, 0, 0, 0)(2Z)\theta(1, 0, 0, 0)(2Z)\theta(0, 1, 0, 0)(2Z)\theta(0, 0, 1, 0)(2Z)\theta(0, 0, 0, 1)(Z)\theta(0, 0, 1, 1)(Z),$$

they gave:

**Conjecture.** (See van Geemen and van Straten [7].) Let $\Pi_{g_i}$ be the irreducible cuspidal automorphic representation of $\text{PGSp}_4(\mathbb{A})$ associated to $g_i$. Then the spinor $L$-functions (of degree 4) are

$$L(s, \Pi_{g_1}; \text{spin}) = L(s, \lambda), \quad L(s, \Pi_{g_4}; \text{spin}) = L\left(s - \frac{1}{2}, \left( \begin{array}{c} 2 \\ \ast \end{array} \right) \right),$$

up to the Euler factors at 2. Here $\lambda$ is a größencharacter of the bi-quadratic CM-field $\mathbb{Q}(i, \sqrt{2})$ of conductor 2, $\rho_1$ is an irreducible cuspidal automorphic representation of $\text{PGL}_2(\mathbb{A})$ of lowest weight 4 of level 8, and $(\frac{2}{\ast})$ is the Legendre symbol.

In this paper, we prove

**Theorem A.** The conjecture is true.

More precisely, their conjecture referred to Andrianov–Evdokimov’s $L$-functions $L(s, g_i; \text{AE})$. However, $L(s, g_i; \text{AE})$ is essentially equal to the (partial) spinor $L$-functions of $\Pi_{g_i}$ (cf. Proposition 2.1). Anyway, Theorem A means that $\Pi_{g_i}$ is a D-critical representation in the sense of Weissauer [31], and $\Pi_{g_4}$ is the $(\frac{-2}{\ast})$-twist of a Saito–Kurokawa representation associated to $\rho_1 \otimes (\frac{-2}{\ast})$. Let $\text{Gr}^W_3 H^3(S_{\Gamma_{g_i}}, \mathbb{C})$ be the graded quotient of degree 3 of a mixed Hodge structure on $H^3(S_{\Gamma_{g_i}}, \mathbb{C})$. Theorem A also means that $g_i$ corresponds to a generator of the 1-dimensional space $\text{Gr}^W_3 H^3(S_{\Gamma_{g_i}}, \mathbb{C})$ associated to a quotient $S_{\Gamma_{g_i}}$ of $S_{\Gamma(2, 4, 8)}$ (cf. Proposition 2.3). We are interested in the quotients $S_{\Gamma_{g_1}}$, $S_{\Gamma_{g_2}}$ of $S_{\Gamma(2, 4, 8)}$, for various reasons. Let $S'_{\Gamma_{g_5}}$ be a resolution of the Satake compactification of $S_{\Gamma_{g_5}}$. Van Geemen and Nygaard [6] calculated the Hodge numbers $h^{3, 0}$ and $h^{2, 1}$ of $S'_{\Gamma_{g_5}}$ are both equal to one and showed that the $L$-function of the third etale cohomology of $S'_{\Gamma_{g_5}}$ is equal to $L(s - \frac{3}{2}, \mu)L(s - \frac{3}{2}, \mu^3)$, up to the Euler factors at 2, where $\mu$ is the unitary größencharacter related to the CM-elliptic curve $E/\mathbb{Q} : y^2 = x^3 - x$. Because $f_5$ corresponds to the generator of $H^{3, 0}(S'_{\Gamma_{g_5}})$, it was conjectured in [7,6] and verified in [17] that $L(s, \Pi_{f_5}; \text{spin}) = L(s, \mu)L(s, \mu^3)$, up to the Euler factors at 2. Thus, $\Pi_{f_5}$ is a weak endoscopic lift of $(\pi(\mu), \pi(\mu^3))$ in the sense of [31] and we have

$$L(s, H^3_{\text{et}}(S'_{\Gamma_{g_5}}; \mathbb{Q}_2)) = L\left(s - \frac{3}{2}, \Pi_{f_5}; \text{spin} \right).$$
up to the Euler factors at 2, where \( \pi(\mu) \) indicates the irreducible cuspidal automorphic representation of \( \text{PGL}_2(\mathbb{A}) \) associated to \( \mu \). From the above Hodge numbers and these \( L \)-functions, it is natural to guess that a weak endoscopic lift of \( (\pi(\mu), \pi(\mu^3)) \) contributes to \( H^{2,1}(G^W_2 H^3(S_{\Gamma^*}, \mathbb{C})) \). In Section 3.3, we will give the desired weak endoscopic lift.

We have verified in [17] their conjectures on \( L(s, \Pi_{f_i}; \text{spin}) \) for \( 1 \leq i \leq 6 \), and we will verify in another work in preparation their conjectures for \( \Pi_{f_7} \) and \( \Pi_{\delta_3} \). Here \( f_i, g_j \) with \( 1 \leq i \leq 7, 1 \leq j \leq 4 \) are certain 6-tuple products of Igusa theta constants. Combining all these works, we will complete the proof for the conjectures given in [7].

By the way, our result means that there are irreducible automorphic representations of \( \text{GSO}(6) \) related to these representations of \( \text{GSp}(4) \) with the \( \theta \)-correspondence. We find holomorphic Hermitian modular forms of \( \text{GU}(2,2) \) of weight 4 from the Siegel modular forms of weight 3 by the following theorem.

**Theorem B.** Let \( K = \mathbb{Q}(\sqrt{-d}) \) be an imaginary quadratic field. Let \( B/\mathbb{Q} \) be a definite quaternion algebra such that \( B \otimes K \simeq M_2(K) \). Put \( V = K + B/\mathbb{Q} \). Suppose that a Siegel modular eigen-cusp form \( F \) of degree 2 of weight 3 is given by a \( \theta \)-lift from \( \text{PGSO}_v \). Then, there is a holomorphic Hermitian modular form \( \tilde{F} \) of \( \text{PGL}_{2,2}(K) \) of weight 4 with

\[
L(s, \tilde{F}; \wedge^2) = \zeta(s)L\left(s, F, \left( \begin{smallmatrix} -d \\ 0 \\ 0 \\ 1 \end{smallmatrix} \right); r_5 \right),
\]

outside of finitely many bad places. If \( F \) satisfies the generalized Ramanujan conjecture at almost all good places, then \( \tilde{F} \) is a cusp form. Here \( L(s, F, (\frac{-d}{*}); r_5) \) is the \( (\frac{-d}{*}) \)-twist of the \( L \)-function of degree five, and \( L(s, \tilde{F}; \wedge^2) \) is the \( L \)-function of \( \tilde{F} \) with respect to the twisted exterior square map from the \( L \)-group \( \text{GU}_{2,2}(\mathbb{C}) \) to \( \text{GL}_6(\mathbb{C}) \) introduced by Kim and Krishnamurthy [11].

Notice that a holomorphic Hermitian cusp form of \( \text{GU}(2,2) \) of weight 4 is canonically identified with a holomorphic differential 4-form on a modular 4-fold. A globally generic weak endoscopic lift of \( \eta(\nu) \) where \( \nu = [\nu_i] \) and \( \nu(g) \in R^\times \) is the similitude norm of \( g \). We will denote by \( Z(R)(\simeq R^\times) \) the center of \( \text{GSp}_{2n}(R) \). For a quasi-character \( \chi \) and a representation \( \tau \) of \( \text{GSp}_{2n}(R) \), let \( \chi \tau \) denote the representation sending \( g \) to \( \chi(\nu(g))\tau(g) \).
2. Preliminaries

2.1. Review of van Geemen and van Straten's result

van Geemen and van Straten computed some local factors of Evdokimov’s $L$-functions of the 6-tuple products $f_i, g_j$ of Igusa theta constants. To begin with, we will compare Evdokimov’s $L$-function of a Siegel modular cusp form of degree 2 with the spinor $L$-function of a unitary irreducible cuspidal automorphic representation of $GSp_4(A)$. We will relate Siegel modular forms to automorphic forms, in order to regard Evdokimov’s Hecke operator for Siegel modular forms as an operator for automorphic forms. For $Z \in S_2$ and $g = [A B] \in Sp_4(\mathbb{R})$, let $j(g, Z) = det(CZ + D)$ and $g \cdot Z = (AZ + B)(CZ + D)^{-1}$. For a function $f$ on $S_2$, an element $g \in Sp_4(\mathbb{R})$, and a positive integer $\kappa$, we define

$$f|_\kappa g(Z) = j(g, Z)^{-\kappa} f(g \cdot Z).$$

Let $K = \{g \in Sp_4(\mathbb{R}) \mid g \cdot i_2 = i_2 \}$, where $i_2$ is $iI_2$. For a congruence subgroup $\Gamma \subset Sp_4(\mathbb{Z})$, let

$$\Gamma_\kappa = K \otimes_{p < \infty} \Gamma_p,$$

where $\Gamma_p$ is the $p$-adic completion of $\Gamma$. For a Siegel modular form $f$ of degree 2 of weight $\kappa$ with respect to a congruence subgroup $\Gamma \subset Sp_4(\mathbb{Z})$, we put $f^2(g) = f(g \cdot i_2) f(g, i_2)^{-\kappa}$ with $g \in Sp_4(\mathbb{R})$. Through the isomorphism: $\Gamma \otimes_{p < \infty} \Gamma_p \simeq Sp_4(\mathbb{R})/K \simeq Sp_4(\mathbb{Q}) \setminus Sp_4(\mathbb{A})/\Gamma_h$, we extend $f^2$ to an automorphic form on $Sp_4(\mathbb{A})$, which is also denoted by $f^2$. Let $\tilde{\Gamma}_p$ be the compact subgroup of $GSp_4(\mathbb{Z})$ generated by elements of $\Gamma_p$ and $[i_2 zI_2]$ with $z \in \mathbb{Z}_p$. Let $\tilde{\Gamma}_\kappa = (Z(\mathbb{R})K_{\kappa}) \otimes_{p < \infty} \tilde{\Gamma}_p$. Because $Sp_4(\mathbb{Q}) \setminus Sp_4(\mathbb{A})/\Gamma_h \simeq GSp_4(\mathbb{Q}) \setminus GSp_4(\mathbb{A})/\tilde{\Gamma}_\kappa$, we can write an element $g \in GSp_4(\mathbb{A})$ as $\gamma_1 g_1[1 zI_2]$ with $g_1 \in Sp_4(\mathbb{A}), \gamma \in GSp_4(\mathbb{Q}), t \in Z(\mathbb{R}), z \in \otimes_p \mathbb{Z}_p$. We put

$$\tilde{f}(g) = f^2(g_1).$$

Then, $\tilde{f}$ is an automorphic form on $GSp_4(\mathbb{A})$. Let $\chi^\Gamma$ be a congruence character of $\Gamma/\Gamma(N)$. Let

$$S_\kappa(\chi^\Gamma) = \{ f \in S_\kappa(\Gamma(N)) \mid f|_\kappa \gamma = \chi^\Gamma(\gamma) f(\gamma \in \Gamma) \}.$$

We identify $\chi^\Gamma$ with a character $\chi^\Gamma = 1_\infty \otimes_p \chi^\Gamma_p$ on $\Gamma_h$. For an integer $N$, let $\Gamma^\times(N)_p$ be the subgroup generated by elements of $\Gamma^\times(N)_p$ and $[i_2 zI_2]$ with $z \in \mathbb{Z}_p$. Define $\Gamma^\times(N) = Sp_4(\mathbb{Q}) \cap \bigotimes_p \Gamma^\times(N)_p$. For a character $\chi^\Gamma(N)$ on $\Gamma^\times(N)/\Gamma(N)$, we define $\tilde{\chi}^\Gamma(N)(u) = \chi^\Gamma(N)(u[1 zI_2])$ and $\chi^\Gamma(N) = 1_\infty \otimes_p \tilde{\chi}^\Gamma(N)_p$. Let

$$A_\kappa(\chi^\Gamma(N)) = \{ f \in A(GSp_4(\mathbb{A})) \mid \varphi(u) f = j(u_{\infty, i_2})^{-\kappa} \otimes_p \tilde{\chi}^\Gamma_p(u_p) f \text{ for } u \in \tilde{\Gamma}(N) \}. \quad (2.2)$$

Note that the central character of each $f \in A_\kappa(\chi^\Gamma(N))$ is unitary. If $f \in S_\kappa(\chi^\Gamma(N))$, then $\tilde{f} \in A_\kappa(\chi^\Gamma(N))$. Now, we can regard Evdokimov’s Hecke operators (cf. (2.13) of [5]) for Siegel modular forms as the following operator $T^\prime_p = A_\kappa(\chi^\Gamma(N))$ with $p \mid N$:

$$T^\prime_p \tilde{f}(g) = p^{\kappa-3} \sum_j \tilde{f}(i_{\infty}(h_j) g) = p^{\kappa-3} \sum_j \tilde{f}(g i_{\infty}(h_j) h_j^{-1})$$

where $g \in Sp_4(\mathbb{R}), i_v$ denotes the embedding $GSp_4(\mathbb{Q})$ to $GSp_4(\mathbb{Q}_v)$, and $h_j \in GSp_4(\mathbb{Q}) \cap M_4(\mathbb{Z})$ is taken so that
Suppose that \( f \in S_\kappa(\chi \Gamma(N)) \) is a common eigenform and that \( \tilde{f} \) lies in a (unitary) irreducible cuspidal automorphic representation \( \pi \). Let \( \lambda_{p^n} \) denote the eigenvalue of \( T_{p^n} \) on \( f \). The \( p \)-factor of Evdokimov’s \( L \)-function of \( f \) is

\[
(1 - \lambda_{p}^{-s} + (\lambda_{p}^{-2} - \lambda_{p^2}^{-2} - \omega_{\pi_p}(p)^{-1} p^{2\kappa-4}) p^{-2s} - \omega_{\pi_p}(p)^{-1} \lambda_{p}^{-3} p^{2\kappa-3-3s} + \omega_{\pi_p}(p)^{-2} p^{4\kappa-6-4s})^{-1}.
\]

Let \( \lambda_{p^n} \) be the eigenvalue of the Hecke operator

\[
T_{p^n} \tilde{f}(g) = \sum_j \tilde{f}(g_i p(h_j)) = \sum_j \omega_{\pi_p}(p^n) \tilde{f}(g_i p(h_j)^{-1}).
\]

The spinor \( L \)-function of unramified \( \pi_p \) is

\[
(1 - p^{-3/2} \lambda_{p}^{-s} + p^{-3} (\lambda_{p}^{-2} - \lambda_{p^2}^{-2} - p^{-2} \omega_{\pi_p}(p)) p^{-2s} - p^{-3/2} \omega_{\pi_p}(p) \lambda_{p}^{-3} p^{-3s} + \omega_{\pi_p}(p)^2 p^{-4s})^{-1}.
\]

In order to compare these \( L \)-functions, we recall generalized Whittaker function. Let \( F \) be a Siegel modular cusp form, and \( \tilde{F} \) be the automorphic form on \( GSp_4(\mathbb{A}) \) related to \( F \) as above. Let \( \mathfrak{S}_2(\mathbb{Q}) = \{ T = \overline{T} \in M_2(\mathbb{Q}) \} \). For a \( T \in \mathfrak{S}_2(\mathbb{Q}) \), the Fourier coefficient \( \tilde{F}_T \) with respect to \( \psi \) of \( \tilde{F} \) is

\[
\tilde{F}_T(g) = \int_{\mathfrak{S}_2(\mathbb{Q}) \setminus \mathfrak{S}_2(\mathbb{A})} \psi(\text{Trace}(Ts))^{-1} \tilde{F}\left(\begin{bmatrix} I_2 & s \\ t & I_2 \end{bmatrix} g\right) \, ds,
\]

and that of \( F \) is \( \tilde{F}_T(1) \). Because \( F \) is a cusp form, some \( \tilde{F}_T(1) \) is not zero for some \( T \) with \( \det T \neq 0 \). For a character \( \mu \) of \( SO_{T}(\mathbb{Q}) \setminus SO_{T}(\mathbb{A}) \), the generalized Whittaker function \( \tilde{F}_T^\mu \) is defined by

\[
\tilde{F}_T^\mu(g) = \int_{\mathfrak{S}_T(\mathbb{Q}) \setminus \mathfrak{S}_T(\mathbb{A})} \mu(z)^{-1} \tilde{F}_T\left(\begin{bmatrix} z \\ t z^{-1} \end{bmatrix} g\right) \, dz
\]

and factors as \( \otimes_v \tilde{F}_T^\mu_v \) (cf. [19]). Because \( \tilde{F}_T = \sum_{\mu} \tilde{F}_T^\mu \), some \( \tilde{F}_T^\mu(1) \) is not zero.

**Proposition 2.1.** Suppose that a Siegel modular form \( f \in S_\kappa(\Gamma(N)) \) of degree 2 is a common eigenfunction with respect to Evdokimov’s Hecke operators. Suppose that \( f \) lies in a (unitary) irreducible cuspidal automorphic representation \( \pi \). Then, for \( p \nmid N \),

\[
L(s, f; AE)_p = L\left(s - \kappa + \frac{3}{2}, \omega_{\pi_p, p}^{-1} \omega_{\pi, p}; \text{spin}\right).
\]

**Proof.** It suffices to show that

\[
\lambda_{p^n} = p^{n(\kappa - 3)} \omega_{\pi, p}(p)^{-n} \lambda_{p^n}
\]
for \(n = 1, 2\). To do it, we will observe the actions of the operators on \(\tilde{\mathbf{f}}_T^\mu \overline{\mathbf{f}}_T^\mu \) with \(T \in \mathfrak{S}_2(\mathbb{Z})\) such that \(\tilde{\mathbf{f}}_T^\mu(1) \neq 0\). Then \(\tilde{\mathbf{f}}_T^\mu p(1) \neq 0\). Abbreviate \(\tilde{\mathbf{f}}_T^\mu\) as \(B_p\). In the case \(n = 1\), as a complete system \(\{h_j\}\) in (2.3), we can take the following types:

\[
\begin{bmatrix}
1 & * & * & * \\
1 & * & * & * \\
p & p & 1 & 1 \\
p & p & 1 & 1
\end{bmatrix},
\]

where * indicate elements of \(\mathbb{Z}\). But one can show \(B_p(h_j^{-1}) = 0\), if \(h_j\) is not of the first type. Indeed, for example, using the property

\[
B_p\left(\begin{bmatrix}
pI_2 & \\
I_2 & \end{bmatrix}^{-1}
\right) = B_p\left(\begin{bmatrix}
pI_2 & \\
I_2 & \end{bmatrix}^{-n(s)}\right)
\]

\[
= B_p\left(n\left(p^{-1}s\right)\begin{bmatrix}
pI_2 & \\
I_2 & \end{bmatrix}^{-1}\right)
\]

\[
= \psi_p\left(\frac{\text{Trace}(Ts)}{p}\right)B_p\left(\begin{bmatrix}
pI_2 & \\
I_2 & \end{bmatrix}^{-1}\right)
\]

for \(s \in S_2(\mathbb{Z}_p)\), one can show that \(B_p\left(\begin{bmatrix}
pI_2 & \\
I_2 & \end{bmatrix}^{-1}\right) = 0\). Here \(n(s) = \left[1, 2 s \right]\), and note that \(B_p\) is right \(GSp_4(\mathbb{Z}_p)\)-invariant. Then, (2.6) is derived from (2.1). The argument for the case \(n = 2\) is similar to that for the case \(n = 1\) and omitted. □

Next, we recall the result of Sections 6, 7 of van Geemen and van Straten [7]. Let

\[
\Gamma'(2) = \left\{ \begin{bmatrix} A & B \\ 2C' & D \end{bmatrix} \in \Gamma(2) \left( \subset \text{Sp}_4(\mathbb{Z}) \right) \mid \text{diag}(C') \equiv 0 \pmod{2} \right\},
\]

\[
\Gamma'(4, 8) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma(4) \left( \subset \text{Sp}_4(\mathbb{Z}) \right) \mid \text{diag}(B) \equiv \text{diag}(C) \equiv 0 \pmod{8} \right\}.
\]

Let \(f_i, g_j\) with \(1 \leq i \leq 7, 1 \leq j \leq 4\) be the 6-tuple products of Igusa theta constants in the table on p. 864 of [7]. We will abbreviate \(f_i[\gamma, g_j]_{\gamma'}\) for some \(\gamma, \gamma' \in \text{Sp}_4(\mathbb{Z})\) as \(f_i', g_j'\). Through Igusa’s transformation formula, from \(F = f_i'\) (resp. \(g_j'\)), we obtain a congruence character \(\chi_F\) of \(\Gamma'(2)\) (resp. \(\Gamma'(4, 8)\)). In Theorem 6.4 of [7], they showed that \(S_3(\chi_F)\) is 1-dimensional and

\[
S_3(\Gamma(4)) = \sum_{f_i'} S_3(\chi_{f_i'}),
\]

\[
S_3(\Gamma(4, 8)) = S_3(\Gamma(4)) + \sum_{i=2}^7 \sum_{f_i'} S_3(\chi_{f_i'}),
\]

\[
S_3(\Gamma(2, 4, 8)) = S_3(\Gamma(4, 8)) + \sum_{j=1}^4 \sum_{g_j'} S_3(\chi_{g_j'}).
\]

**Proposition 2.2.** (See van Geemen and van Straten [7].) Let \(\tilde{f}_i, \tilde{g}_j\) be the automorphic forms related to \(f_i, g_j\) as above. Then each \(\tilde{f}_i\) (resp. \(\tilde{g}_j\)) lies in an irreducible cuspidal automorphic representation of \(\text{PGSp}_4(\mathbb{A})\).
Proof. Let \( f = f_i \). Write \( \tilde{f} = \sum h_i \in \sum \pi_i \) where \( \pi_i \)'s are irreducible cuspidal automorphic representations. From (2.1), it follows that \( \varrho \left( \begin{bmatrix} I_2 & z \iota z_2 \end{bmatrix} \right) \tilde{f} = \tilde{f} \) for any \( z \in \mathbb{Z}_\ast^\times \). Thus,

\[
\text{vol}(\mathbb{Z}_\ast^\times)^{-1} \int_{\mathbb{Z}_\ast^\times} \varrho \left( \begin{bmatrix} I_2 & z \iota z_2 \end{bmatrix} \right) h_i \, dz = \sum_{i} h_i.
\]

Hence, we can assume that

\[
\varrho \left( \begin{bmatrix} I_2 & z \iota z_2 \end{bmatrix} \right) h_i = h_i, \quad z \in \mathbb{Z}_\ast^\times.
\] (2.7)

With the similar argument, we can assume that

\[
\varrho(u_0) h_i = \chi_f(u_0) h_i, \quad u_0 \in \Gamma(2)_{\ast,0}
\] (2.8)

\[
\varrho(u_{\infty}) h_i = \det(-Bi_2 + A)^{-3} h_i, \quad u_{\infty} = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in \mathbb{K}_{\infty}.
\] (2.9)

Using Proposition 6.2 of [7], we find that \( \chi_{f,p}(\begin{bmatrix} z \iota z_2 & -1 \iota \iota z_2 \end{bmatrix}) = 1 \) for any \( z \in \mathbb{Z}_p^\times \). It follows that the central character of \( \tilde{f} \) is trivial. Hence \( \omega_{\pi_i} \) is also trivial. Consulting Eq. (2) of p. 505 of Oda and Schwermer [16], we find that \( \pi_{l,\infty}|_{\text{Sp}_4} \) is the holomorphic discrete series representation with Blattner parameter \((3,3)\). Define the function \( h_i^0 \) on \( Z \subset S_2 \) by \( h_i^0(Z) = h_i(g_{\infty}, i_2)^3 \), where \( g_{\infty} \in \text{Sp}_4(\mathbb{R}) \) is taken so that \( g_{\infty}^* \cdot i_2 = Z \). Then, \( h_i^0 \in S_3(\chi_f) \). Because \( S_3(\chi_f) \) is 1-dimensional, \( h_i^0 \in \mathbb{C}f \). One can show that \( h_i^0 \in \mathbb{C}\tilde{f} \), noting (2.7), (2.8), (2.9) and \( \omega_{\pi_i} = 1 \). This completes the proof for \( f_i \). The proof for \( \tilde{g}_j \) is similar. \( \Box \)

We will denote by \( \Pi_{f_i} \) (resp. \( \Pi_{\tilde{g}_j} \)) the irreducible cuspidal automorphic representation of \( \text{PGSp}_4(\mathbb{A}) \) containing \( f_i \) (resp. \( \tilde{g}_j \)).

Noting that \( \Gamma'(2) \), \( \Gamma(2, 4, 8) \) are normal subgroups of \( \Gamma(2) \) and \( \Gamma(2)/\Gamma'(2) \simeq (\mathbb{Z}/2\mathbb{Z})^2 \), one can extend \( \chi_{g_i} \) in 4 ways, \( \chi_{g_i,l} \) with \( 1 \leq l \leq 4 \). Then \( S_3(\chi_{g_i}) = \sum S_3(\chi_{g_i,l}) \). However, because \( \dim_{\mathbb{C}} S_3(\chi_{g_i}) = 1 \), \( \dim_{\mathbb{C}} S_3(\chi_{g_i,l}) = 1 \) for an \( l \) and \( \dim_{\mathbb{C}} S_3(\chi_{g_i}) = 0 \) for other \( l \). We define the character \( \chi_{g_i} \) on \( \Gamma(2) \) by \( \dim_{\mathbb{C}} S_3(\chi_{g_i}) = 1 \).

**Proposition 2.3.** For \( \Gamma = \ker(\chi_f), \ker(\tilde{g}_j), \text{Sp}_4^h \) \( (S_{\Gamma, \mathbb{C}}) \) is 1-dimensional.

**Proof.** We give the proof for \( \Gamma = \ker(\chi_f) \). The proof for \( \Gamma = \ker(\tilde{g}_j) \) is similar and omitted. To prove \( H^3(\text{Sp}_4^h \mathbb{H}^3(S_{\Gamma, \mathbb{C}})) \simeq S_3(\ker(\chi_f)) \) is 1-dimensional, it suffices to show that \( \ker(\chi_f) \subseteq \ker(\chi_{f,l}) \) for any \( f_l \) and \( \ker(\chi_{g,f}) \subseteq \ker(\chi_{g,l}) \) for any \( g_l \neq g \). Using the tables in Proposition 6.2 of [7], we find that \( \chi_{f,l} \) is \{±1\}-valued, and that \( \chi_{f,l} \) for \( i \neq 1 \) and \( \chi_{g,l} \) are \{±1, ±i\}-valued. Thus

\[
\Gamma(2)/\ker(\chi_{f,l}) \simeq \mathbb{Z}/2\mathbb{Z}, \quad \Gamma'(2)/\ker(\chi_{g,l}) \simeq \Gamma(2)/\ker(\chi_{f,l}|_{\Gamma'(2)}) \simeq \mathbb{Z}/4\mathbb{Z} \quad (i \neq 1).
\]

Because the commutator subgroup of \( \Gamma(2) \) is \( \Gamma(4, 8) \), and \( g \neq g_l \), \( g \neq g_l \), it is impossible to extend \( \chi_{g} \) to a character on \( \Gamma(2) \). Hence, \( \chi_{f,l}|_{\Gamma'(2)} \neq \chi_{g}, \chi_{g} \) and \( \ker(\chi_{g}) \subseteq \ker(\chi_{f,l}|_{\Gamma'(2)}) \) for \( i \neq 1 \). As described in the proof of Proposition 7.5 in [7], and \( \chi_{g,l} \neq \chi_{g}, \tilde{g} \tilde{g} \). Hence \( \ker(\chi_{g}) \subseteq \ker(\chi_{g,l}) \) for \( g_l \neq g \). Finally, assume that \( \ker(\chi_{g}) \subseteq \ker(\chi_{f,l}) \) for some \( f_l \). Then, \( \chi_{g} = \tilde{g}_{f,l}, \) and hence \( \chi_{g} \subseteq \Gamma(4) \). But, this conflicts to the table of Proposition 6.2(b) in [7]. Hence \( \ker(\chi_{g}) \not\subseteq \ker(\chi_{f,l}). \) This completes the proof. \( \Box \)
2.2. $\theta$-lifts

In this section, we summarize the $\theta$-correspondence for GSO(4) and GSp(4). Let $X/\mathbb{Q}$ be a $2m$-dimensional space defined over $\mathbb{Q}$ with a nondegenerate quadratic form $(\cdot, \cdot)$. For $x = (x_i)$, $y = (y_i) \in X^n$, we denote $((x_i, y_j))$ also by $(x, y)$. Let $d_X$ be the discriminant of $X$. Let $\chi_X(\ast) = \{\ast, (-1)^m d_X\}_\nu$ where $\{\ast, \ast\}_\nu$ denotes the Hilbert symbol. We fix the standard additive character $\psi$ on $\mathbb{Q}\setminus \mathbb{A}$. Let $S(X(Q_v)^n)$ be the space of Schwartz-Bruhat functions of $X(Q_v)^n$. The Weil representation $r^n_v$ of $\text{Sp}_{2n}(Q_v) \times \text{O}_X(Q_v)$ with respect to $\psi_v$ is the unitary representation on $S(X(Q_v)^n)$ given by

$$r^n_v(1, h)\varphi_v(x) = \varphi_v(h^{-1}x), \quad (2.10)$$
$$r^n_v\left( \begin{bmatrix} a & 0 \\ 0 & i a^{-1} \end{bmatrix}, 1 \right)\varphi_v(x) = \chi_X(\det a)|\det a|^m\varphi_v(xa), \quad (2.11)$$
$$r^n_v\left( \begin{bmatrix} I_n & b \\ 0 & I_n \end{bmatrix}, 1 \right)\varphi_v(x) = \varphi_v\left( \frac{\text{Trace}(b(x, x))}{2} \right)\varphi_v(x), \quad (2.12)$$
$$r^n_v\left( \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, 1 \right)\varphi_v(x) = \gamma\varphi^\gamma_v(x). \quad (2.13)$$

The Weil constant $\gamma$ is a fourth root of unity depending on the anisotropic kernel of $X$, $n$ and $\psi$. The Fourier transformation $\varphi^\gamma$ of $\varphi$ is defined by

$$\varphi^\gamma(x) = \int_{X(Q_v)^n} \psi_v(\text{Trace}(x, y))\varphi(y) \, dy$$

where $dy$ is the self-dual Haar measure. As in [21], we extend $r^n_v$ to the group $(g, h) \in \text{GSp}_n(Q_v) \times \text{GO}_X(Q_v) | \nu(g) = \nu(h))$, where $\nu(h)$ denotes the similitude norm of $h$. Let $r^n = \otimes_v r^n_v$. For $\varphi = \otimes_v \varphi_v \in S(X(\mathbb{A})^n)$, we put

$$\theta_n(\varphi)(g, h) = \sum_{x \in X(Q_v)^n} r(g, h)\varphi(x).$$

This series converges absolutely. Let $dh$ be a right Haar measure on $\text{SO}_X(Q) \setminus \text{SO}_X(\mathbb{A})$. Let $\sigma$ be an irreducible cuspidal automorphic representation of $\text{GSO}_X(\mathbb{A})$. Take an $f \in \sigma$. We define the $\theta$-lift of $f$ to $\text{GSp}_n(\mathbb{A})$ with respect to $\varphi$ by

$$\theta_n(\varphi, f)(g) = \int_{\text{SO}_X(Q)\setminus \text{SO}_X(\mathbb{A})} \theta_n(\varphi)(g, h)f(h h_0) \, dh, \quad (2.14)$$

where $h_0$ is chosen so that $\nu(g) = \nu(h_0)$, and the value of $\theta_n(\varphi, f)(g)$ is independent of the choice of $h_0$. This integral converges absolutely and is an automorphic forms on $\text{GSp}_{2n}(\mathbb{A})$. We will denote by $\theta_n(\sigma)$ the subspace of $\mathcal{A}(\text{GSp}_n(\mathbb{A}))$ spans by $\theta_n(\varphi, f)$ with $\varphi \in S(X(\mathbb{A}^n)$ and $f \in \sigma$. We call $\theta_n(\sigma)$ the global $\theta$-lift of $\sigma$ to $\text{GSp}(2n)$. In the case of $m = 2$, these $\theta$-lifts are weak endoscopic lifts or D-critical representations under some situations as follows. For our later use and the sake of simplicity, we assume the central character of $\sigma$ is trivial.

1) In the case that $d_X$ is a square of a rational number, $X/\mathbb{Q}$ is isometric to a quaternion algebra $B/\mathbb{Q}$ defined over $\mathbb{Q}$. Define $\rho(h_1, h_2)x = h_1^{-1}xh_2$ for $x \in B(R), h_i \in B(R)^{\times}$, where $R$ denote $\mathbb{Q}$, $\mathbb{Q}_v$ or $\mathbb{A}$. Then $\rho$ gives isomorphisms
\[
\begin{aligned}
i_\rho : \left\{ \begin{array}{l}
B(R)^* \times B(R)^* / \Delta(R^*) \simeq \text{GSO}_X(R), \\
\{ (h_1, h_2) \in B(R)^* \times B(R)^* \mid N_{B/R}(h_1) = N_{B/R}(h_2) \}/\Delta(R^*) \simeq \text{SO}_X(R),
\end{array} \right.
\end{aligned}
\]  

where \( \Delta(R^*) \) denotes the diagonal embedding into \( B(R)^* \times B(R)^* \). We identify \( \sigma \in \text{Irr}(\text{PGSO}_X(\mathbb{Q}_v)) \) with a pair \((\sigma_1, \sigma_2, v)\) of \( \text{Irr}(\text{PB}(\mathbb{Q}_v)^*) \) through \( i_\rho \). Then, \( \sigma \) is identified with \( \sigma_1 \boxtimes \sigma_2 \) for a pair \((\sigma_1, \sigma_2)\) of irreducible automorphic representations of \( \text{PGSO}_B(\mathbb{A}) \). Then, \( \Pi = \Theta_2(\sigma_1 \boxtimes \sigma_2) \) is irreducible and factors as \( \otimes_v \theta_2(\sigma_{1V} \boxtimes \sigma_{2V}) \). For an irreducible cuspidal automorphic representation \( \tau \) of \( B(\mathbb{A})^* \), we will let \( \tau^\Pi \) denote the Jacquet–Langlands transfer to \( \text{GL}_2(\mathbb{A}) \). Let \( S_\sigma \) be the set of places \( v \) for which \( \sigma_{1V}^\Pi \boxtimes \sigma_{2V}^\Pi \) is ramified. Then, \( S_\Pi = S_\sigma \), and

\[
L_{S_\sigma}(s, \Pi; \text{spin}) = L_{S_\sigma}(s, \sigma_1)L_{S_\sigma}(s, \sigma_2), \quad L_{S_\sigma}(s, \Pi; r_5) = \zeta_{S_\sigma}(s)L_{S_\sigma}(s, \sigma_1 \times \sigma_2),
\]

where \( r_5 \) indicates the 5-dimensional representation of \( \text{GSp}_4(\mathbb{C}) \) as in Section 2 of [26]. If both of \( \sigma_1 \) and \( \sigma_2 \) are cuspidal and \( \sigma_1 \neq \sigma_2 \), then \( \Pi \) is cuspidal, and thus \( \Pi \) is a weak endoscopic lift of \((\sigma_1^\Pi, \sigma_2^\Pi)\). If \( B_\mathbb{Q} \) is a definite quaternion algebra, then \( \Pi \) is the so-called Yoshida lift of \( \sigma = (\sigma_1, \sigma_2) \), and \( \Pi_{\text{sc}} \) is holomorphic. Otherwise, \( \Pi \) is not holomorphic. In particular, if \( B_{\mathbb{Q}} \simeq M_2(\mathbb{Q}) \), then \( \Pi \) is globally generic, i.e., every \( F \in \Pi \) has a nontrivial global Whittaker function. Let \( c_1, c_2 \in \mathbb{Q}^\times \). A global Whittaker function of an automorphic form \( F \) on \( \text{GSp}_4(\mathbb{A}) \) with respect to \( \psi_{c_1, c_2} \) is defined by

\[
W_{F, \psi_{c_1, c_2}}(g) = \int_{(\mathbb{Q}, \mathbb{A})^4} \psi(-(c_1 t + c_2 s_4) F \left( \begin{array}{cc}
1 & t \\
1 & 1
\end{array} \right) \left( \begin{array}{cc}
s_1 & s_2 \\
-1 & 1
\end{array} \right) \left( \begin{array}{cc}
s_4 & 1 \\
1 & 1
\end{array} \right) g) \, dt \, ds_1 \, ds_2 \, ds_4,
\]

and factors as \( \otimes_v W_{F, \psi_{c_1, c_2}} \). We call \( W_{F, \psi_{1, 1}} \) the standard Whittaker function and abbreviate as \( W_{F, \psi} \). Let \( B = M_2(\mathbb{Q}) \). Let

\[
e = \left[ \begin{array}{c}
1 \\
1
\end{array} \right], \quad \alpha = \left[ \begin{array}{cc}
1 & 1 \\
-1 & 1
\end{array} \right] \in M_2(\mathbb{Q}).
\]

The pointwise stabilizer subgroup \( Z_{(e, \alpha)}(R) \subset SO_8(R) \) of \( e, \alpha \) is isomorphic to

\[
\left\{ \left[ \begin{array}{cc}
s & 1 \\
1 & 1
\end{array} \right] \mid s \in R \right\}
\]

via \( i_\rho \). Let \( \beta_{1, \psi} = \otimes_v \beta_{1, \psi} \), \( \beta_{2, \psi} = \otimes_v \beta_{2, \psi} \) be the Whittaker functions of \( f_1, f_2 \) with respect to \( \psi \). Then, the \( v \)-component of the global standard Whittaker function of \( F = \theta_2(\varphi, f_1 \boxtimes f_2) \) on \( \text{Sp}_4(\mathbb{Q}_v) \) is

\[
W_{F, \psi}(g) = \int_{Z_{(e, \alpha)}(\mathbb{Q}_v) \text{SO}_8(\mathbb{Q}_v)} r_v^2(g, \psi_v(h_1, h_2)) \psi_v(e_1, \alpha) \beta_{1, \psi}(h_1) \beta_{2, \psi}(h_2) \, dh_1 \, dh_2.
\]

2) In the case that \( d_X \) is not a square of a rational number, \( X_\mathbb{Q} \) is isometric to

\[
X_{B, d_X} = X_B := \{ b \in B_{\mathbb{Q}} \otimes \mathbb{Q}(\sqrt{d_X}) \mid b^c = -b \}
\]

for a quaternion algebra \( B_{\mathbb{Q}} \), where \( \iota \) denotes the main involution of \( B \), and \( c \) is the generator of \( \text{Gal}(\mathbb{Q}(\sqrt{d_X})/\mathbb{Q}) \). Put \( L = \mathbb{Q}(\sqrt{d_X}) \). Let \( R \) be \( \mathbb{Q}, \mathbb{Q}_A \) or \( \mathbb{Q}_v \). But assume that \( L_v \neq \mathbb{Q}_v^2 \). For \( x \in X \), \( \tau \in R^* \), \( h \in B(RL)^* \), define \( \rho'(t, h)x = t^{-1}h^cxh \). Then, \( \rho' \) gives isomorphisms.
\[
\begin{aligned}
i_{\rho'}^* : \left\{ \begin{array}{l}
(t, b) \in R^* \times B(LR)^x \bigg/ \{(N_{LR/R}(s), s) \mid s \in LR^x \} \simeq \text{GSO}_X(R), \\
(t, b) \bigg/ \{(N_{LR/R}(s), s) \mid s \in LR^x \} \simeq \text{SO}_X(R).
\end{array} \right.
\end{aligned}
\]

(2.19)

We identify a \( \sigma_v \in \text{Irr}(\text{PGSO}_X(\mathbb{Q}_v)) \) with one of \( \text{Irr}(\text{PB}(L_v)) \) through \( i_{\rho'}^* \). If \( L_v \simeq \mathbb{Q}_v^2 \), then \( \text{GL}_2(L_w) \times \text{GL}_2(L_{w_2}) \simeq \text{GL}_2(\mathbb{Q}_v)^2 \), and \( \sigma_v \) is identified with a pair of elements of \( \text{Irr}(\text{PB}(Q_v)) \). Let \( \sigma \) be an irreducible cuspidal automorphic representation of \( \text{PB}(\mathbb{A}) \), which is identified with an irreducible representation of \( \text{PGSO}_X(\mathbb{A}) \). Contrary to the previous case, \( \Theta_2(\sigma) \) is not irreducible in some cases. Anyway, every irreducible constituent \( \tau \) of \( \Theta_2(\sigma) \) factors as \( \bigotimes_v \tau_v \), and

\[
L_{S_\ell}(s, \tau; \text{spin}) = L_{S_\ell}(s, \sigma), \quad L_{S_\ell}(s, \tau; r_s) = L_{S_\ell}(s, \chi_{\ell}L_{S_\ell}(s, \tau, \chi_{\ell}; \text{Asai}),
\]

where \( \chi_{\ell} \) is the quadratic character associated to the extension \( L/\mathbb{Q} \), and the last \( L \)-function is the \( \chi_{\ell} \)-twist of Asai’s \( L \)-function (see [1] for the definition). Suppose that \( d_{X} > 0 \) and each \( \sigma_{b,G_{\mathbb{Q}_v}} \) is holomorphic discrete series representation with lowest weight 2 or more. Employing the main result of Blasius [3], we find that \( \sigma_{b,G_{\mathbb{Q}_v}} \) is tempered. Thus, in this case, every constituent of \( \Theta_2(\sigma) \) is a D-critical representation in the sense of [31]. If \( B_{/\mathbb{Q}} \) is a definite quaternion algebra, then each irreducible constituent of \( \Theta_2(\sigma) \) is holomorphic. If \( B_{/\mathbb{Q}} \simeq M_2(\mathbb{Q}) \), then an irreducible constituent of \( \Theta_2(\sigma) \) is globally generic. Let \( B_{/\mathbb{Q}} = M_2(\mathbb{Q}) \). Define \( \psi_L(z) = \bigotimes_v \psi_v(\text{Trace}_{L_v/\mathbb{Q}_v}(z)) \), where \( w \) denotes a place of \( L \) lying over \( v \). Let \( e, \alpha \in \chi_{\ell}M_2(d_\ell, \mathbb{Q}_v) \) be the same as above. Then the pointwise stabilizer subgroup \( Z(e, \alpha)(\mathbb{A}) \subset \text{SO}_{X_\ell}(\mathbb{A}) \) is isomorphic to

\[
\left\{ \left( \begin{array}{c}
1, \quad 1 \\
1, \quad 1
\end{array} \right) \mid s \in \sqrt{d_{X}\mathbb{A}} \right\}
\]

(2.20)

via \( i_{\rho'}^* \). Let \( f \in \sigma, \varphi = \bigotimes_v \varphi_v \in \mathcal{S}(X(\mathbb{A})^1) \), and \( F = \theta_2(\varphi, f) \). Let \( \beta_{\psi} = \bigotimes_v \beta_{w} \) be the global Whittaker function of \( f \) associated to \( \psi_L \). If \( L_v = L_{w_1} \times L_{w_2} \simeq \mathbb{Q}_v^2 \), then \( W_{F, \psi_v} \) is similar to (2.17). If \( L_v/\mathbb{Q}_v \) does not split, then

\[
W_{F, \psi_v}(g) = \int_{Z(e, \alpha)(\mathbb{Q}_v) \backslash \text{SO}_{X_\ell}(\mathbb{Q}_v)} r_{\rho'}^2(g, i_{\rho'}(t, b))\psi_v(e, \alpha)\beta_{w}(b) \, dt \, db.
\]

(2.21)

The next lemma is needed to prove Theorem A.

**Lemma 2.4.** Let \( L \) be a quadratic field. Let \( \sigma \) be an irreducible cuspidal automorphic representation of \( \text{PGL}_2(\mathbb{A}) \). If \( \sigma \) is not a base change lift of an irreducible cuspidal automorphic representation of \( \text{GL}_2(\mathbb{A}) \), then every irreducible constituent of \( \Theta_2(\sigma) \) is not a weak endoscopic lift.

**Proof.** Let \( \tau \) be a constituent of \( \Theta_2(\sigma) \). On the authority of Shahidi [25], Asai’s \( L \)-function of \( \sigma \) does not vanish at \( s = 1 \). Hence \( L_{S_\ell}(s, \tau, \chi_{\ell}; r_5) \), the \( \chi_{\ell} \)-twist of \( L_{S_\ell}(s, \tau; r_5) \), has at least a simple pole at \( s = 1 \). Assume that \( \tau \) is a weak endoscopic lift. Then, \( L_{S_\ell}(s, \tau, \chi_{\ell}; r_5) \) is equal to \( L_{S_\ell}(s, \chi_{\ell}L_{S_\ell}(s, \sigma_1 \times \chi_{\ell}\sigma_2) \) for a cuspidal pair \( (\sigma_1, \sigma_2) \), and hence

\[
\text{ord}_{s=1} L_{S_\ell}(s, \tau, \chi_{\ell}; r_5) = \begin{cases} -1 & \text{if } \sigma_1 = \chi_{\ell}\sigma_2, \\ 0 & \text{otherwise}. \end{cases}
\]

Hence the assertion. \( \square \)
2.3. Degenerate Whittaker functions

Let $R$ be a commutative ring. For $1 \leq r \leq 2$, let $P_r(R) = N_r(R)M_r(R) \subset \GL_2(R)$ with

$$N_{P_r}(R) = \left\{ \begin{bmatrix} 1_r & v & t w \\ 1_{2-r} & w & 1_r \\ 1_r & 1_{2-r} & -t^{-r} u \\ 1_{2-r} & \end{bmatrix} \mid v = t^r v \in M_r(R), u, w \in M_r, 2-r(R) \right\},$$

$$M_{P_r}(R) = \left\{ \begin{bmatrix} z & a \det(g)^z & b \\ c & d \end{bmatrix} \mid g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \GL_2 \times \GL_2, z \in \GL_2 \right\} \cong \GL_2 \times \GL_2,$$

where we understand $\GL_0 = \GL_1$, $\GL_2 = \GL_2$. We write $P_1 = Q$ (resp. $P_2 = P$) and call it Klingen (resp. Siegel) parabolic subgroup. Let $e_Q, e_P$ denote the natural embedding of $\GL_2 \times \GL_2$ into $M_{P_r}$. If $E$ is a noncuspidal automorphic form on $\GL_2(\A)$, then, for $\bullet = P$ or $Q$,

$$\Phi_{\bullet} (E, g, z) := \vol(N_{\bullet}(\Q) \backslash N_{\bullet}(\A))^{-1} \int_{N_{\bullet}(\Q) \backslash N_{\bullet}(\A)} E\left(ne_{\bullet}(g, z)\right) \, dn$$

(2.22)

is a nontrivial automorphic form on $\GL_2(\A) \times \GL_1(\A)$. Let $a \in \Q^\times$. We define $\psi_{(a)}(*) = \psi(az)$. If a function $W_{\psi_{(a)}}^\bullet$ on $\GL_2(\A)$ (resp. $\GL_2(\Q)$) satisfies

$$W_{\psi_{(a)}}^\bullet \left( \begin{bmatrix} 1 & u \\ 1 & 1 \\ -u & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * \\ 1 & * & z \\ 1 & 1 & \end{bmatrix} g \right) = W_{\psi_{(a)}}^\bullet (g) \times \begin{cases} \psi(au), & (\bullet = P), \\ \psi(az), & (\bullet = Q), \end{cases}$$

(2.23)

then we say $W_{\psi_{(a)}}^\bullet$ is a $\bullet$-degenerate global (resp. local) Whittaker function.

3. Automorphic forms on $\GL_2(\A)$

Let $\Pi_{f_i}, \Pi_{g_j}$ be the irreducible cuspidal automorphic representations associated to $f_i, g_j$ (cf. Proposition 2.2). The idea of our proof of Theorem A is as follows. We will show that a $D$-critical representation associated to the Hilbert modular form $\pi(\lambda)$ of $\Q(\sqrt{2})$, and the $(\frac{2}{8})$-twist of a Saito--Kurokawa representation associated to $\rho_1$ has a $\Gamma(2, 4, 8)$-fixed vector. Because the $2$-component of this $D$-critical representation, and that of this $(\frac{2}{8})$-twist of the Saito--Kurokawa representation are given by local $\theta$-lifts from $\SO(4)$, we will do it by constructing local Whittaker functions, or local degenerate Whittaker functions defined in 2.3 of these local $\theta$-lifts. If it is done, then each of these representation has an automorphic form related to a Siegel modular form belonging to $S_2(\Gamma(2, 4, 8))$. From the eigenvalues of $\Pi_{f_i}, \Pi_{g_j}$ computed in [7], one concludes $\Pi_{g_1}$ is this $D$-critical representation and $\Pi_{g_4}$ is this $(\frac{2}{8})$-twist of the Saito--Kurokawa representation. In this way, the conjecture is verified.
3.1. D-critical representation, proof for $L(s, \Pi_{g_1}; \text{spin})$

Let $L$ be a quadratic field with the ring of integers $\mathcal{O}$. Let $\delta_L$ be the discriminant of $L$. For an integral ideal $m$ of a Dedekind ring $R$, let

$$\tilde{\Gamma}_0^{(n)}(m) = \left\{ g = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix} \in \text{GSp}_{2n}(R) \mid C_g \in \mathcal{M}_n(m) \right\},$$

$$\Gamma_0^{(n)}(m) = \tilde{\Gamma}_0^{(n)}(m) \cap \text{Sp}_{2n}(R).$$

First, we show the following proposition.

**Proposition 3.1.** Let $p$ be a prime which does not split in $L/\mathbb{Q}$, and $\mathfrak{p}$ denote the unique prime ideal of $L$ lying over $p$. Let $\pi$ be an irreducible cuspidal automorphic representation of $\text{PGl}_2(L, \mathbb{A})$ of level $n$. Then, there is an automorphic form $F \in \Theta_2(\pi)$ such that

$$Q(g)F = \chi_{L, p}(\text{det}(A_g))F, \quad g \in \Gamma_0^{(2)}(p^N\mathbb{Z}_p),$$

(3.1)

where $\chi_L$ is the quadratic character of $\mathbb{A}^\times$ associated to the extension $L/\mathbb{Q}$, and

$$N = \begin{cases} \frac{\text{ord}_p(n)}{2} + \text{ord}_p(\delta_L) & \text{if } p \text{ is ramified and } \text{ord}_p(n) \text{ is even,} \\
\frac{\text{ord}_p(n)+1}{2} + \text{ord}_p(\delta_L) & \text{if } p \text{ is ramified and } \text{ord}_p(n) \text{ is odd,} \\
\text{ord}_p(n) & \text{otherwise.} \end{cases}$$

**Proof.** For a $\psi = \bigotimes_{v} \psi_v \in S(X_{M_2}(\mathbb{A})^2)$ and an $f \in \pi$, each component $W_{F, \psi_v}$ of the global standard Whittaker function of $F = \theta_{g, \psi, f}$ is given by (2.17) or (2.21). Therefore, it suffices to construct a nontrivial $W_{F, \psi}$ which is right $\Gamma_0(p^N\mathbb{Z}_p)$-semi invariant as in (3.1). We will give a proof for the first case with $L = \mathbb{Q}(\sqrt{2})$ and $p = 2$. The other cases are easier and omitted. For an ideal $m \subset \delta_1\mathfrak{o}_L$ of $\mathfrak{o}_L$, let

$$\tilde{\Gamma}_0'(m) = \left[ \begin{array}{cc} \mathfrak{o}_p & \delta_L^{-1} \mathfrak{o}_p \\ m & \mathfrak{o}_p \end{array} \right] \cap \text{GL}_2(L_p).$$

In the case $\text{ord}_p(n) = 0$, the proof is easy and omitted. Suppose that $\text{ord}_p(n)$ is a positive (even) integer. Then, $\pi_p$ is a ramified principal series representation or a supercuspidal representation. The conductor of the additive character $\psi_{L_p} = \psi_p \circ \text{Trace}_{L_p/\mathbb{Q}_p}$ is $p^{-3}$. Using the local newform theory for $\text{GL}(2)$, we find that $\pi_p$ has a right $\tilde{\Gamma}_0'(\delta_1 p^{-n})$-invariant local Whittaker function $\beta_p$ associated to $\psi_{L_p}$ such that

$$\beta_p\left( \begin{array}{cc} 1 & t \\ m & 1 \end{array} \right) = \begin{cases} \psi_{L_p}(z) & \text{if } t \in \mathfrak{o}_p^\times, \\
0 & \text{otherwise,} \end{cases}$$

(3.2)

$$\mathcal{Q}\left( \begin{array}{cc} p^N & -1 \\ 1 & 1 \end{array} \right) \beta_p = \pm \beta_p.$$

(3.3)

For an integral ideal $m$ of a Dedekind ring $R$, let $R_0(m) = \left[ \begin{array}{c} c \end{array} \right] \in M_2(R) \mid c \in m$ be the so-called Eichler order of $M_2(R)$ of level $m$. We set

$$\phi(x_1, x_2) = \text{ch}(x_1; R_0(p^N) \cap X_{M_2(\mathbb{Q}_p)}) \text{ch}(x_2; R_0(p^N) \cap X_{M_2(\mathbb{Q}_p)})$$
where \( \text{ch} \) indicates the characteristic function. Put \( \mathbb{K}_p = \mathbb{I}_\rho'/(\mathbb{Q}_p^\times \times \Gamma_0^{(1)}(p^N)) \cap \text{SO}_X(\mathbb{Q}_p) \). If \( g \in \Gamma_0^{(2)}(p^N) \) and \( h \in \mathbb{K}_p \), then

\[
 r^2_p(g, h)\phi = \chi_{L, p}(\det A_g) r^2_p(1, h)\phi. \tag{3.4}
\]

From (2.21),

\[
 W_{F, \psi_p}(g) = \text{vol}(\mathbb{K}_p) \int_{Z_{(e, \alpha)}(\mathbb{Q}_p) \backslash \text{SO}_X(\mathbb{Q}_p)/\mathbb{K}_p} r^2_p(g, h)\phi(e, \alpha) \beta_p(h) \, dh,
\]}

where \( h \) indicates the projection of \( h \in \text{GL}_2(L_p) \) to \( \text{SO}_X(\mathbb{Q}_p) \) (see (2.20) for the definition of \( Z_{(e, \alpha)} \)). Then, we are going to see \( W_{F, \psi_p}(1) \neq 0 \). Using the Iwasawa decomposition of \( \text{GL}_2(L_p) \), we can take the following complete system of representatives for \( Z_{(e, \alpha)}(\mathbb{Q}_p) \backslash \text{SO}_X(\mathbb{Q}_p)/\mathbb{K}_p \):

\[
 \left( 2^m, \begin{bmatrix} 1 & s \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^m & 1 \\ 1 & 0 \end{bmatrix} \right), \quad \left( \frac{2^m}{2}, \begin{bmatrix} 1 & s \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^m & 1 \\ 1 & 2^N \end{bmatrix} \right)
\]

where \( s \in \mathbb{Q}_2, m \in \mathbb{Z} \) and \( l \in \mathbb{O}_2 \) modulo \( 2^N \). We will observe the contributions of these types to the integral (3.5). We will denote \( \rho'(t, h)(e, \alpha) = \left( \frac{\alpha_1}{c_1} \frac{b_1}{d_1}, \frac{\alpha_2}{c_2} \frac{b_2}{d_2} \right) \). For the former types, we calculate

\[
 \rho'(t, h)(e, \alpha) = \left( \begin{bmatrix} 2^{-m} l & 2^{-m} s \\ -2^{-m} l & -2^{-m} s \end{bmatrix}, \begin{bmatrix} 1 + 2^{-m+1}s \frac{1}{l} & 2^{-m+1}s \frac{1}{l} \\ -(l + l') - 2^{-m+1}s l' s & 1 - 2^{-m+1}s l' s \end{bmatrix} \right)
\]

where \( c \) is the generator of \( \text{Gal}(L/\mathbb{Q}) \). Suppose \( \rho'(t, h)(e, \alpha) \in \text{supp}(\phi) \). Observing \( b_1 \), we find \( m \leq 0 \). If \( m < 0 \), then

\[
 \rho' \left( \begin{bmatrix} 1 & \frac{1}{4} \\ 1 & 1 \end{bmatrix} h \right)(e, \alpha) \in \text{supp}(\phi).
\]

Because \( \beta_2(\frac{1}{4} \frac{1}{4} h) = -\beta_2(h) \), we can ignore the contribution if \( m < 0 \). Therefore, we can assume \( m = 0 \). Then, observing \( c_1 \), we find \( l \in \mathbb{P}^N \). Observing \( b_2 \), we find \( s \in 2^{-1} \mathbb{Z}_2 \). We see that, if \( m = 0 \), \( l \in \mathbb{P}^N \) and \( s \in 2^{-1} \mathbb{Z}_2 \), then \( \rho'(t, h)(e, \alpha) \in \text{supp}(\phi) \). Now, recall that \( \beta_p \) is a local new vector, which is right \( \Gamma_0(\delta_1 n) \)-invariant. Hence, if \( c \in \mathbb{P}^{-1} \delta_1 n \backslash \delta_1 n \), then

\[
 \theta\left( \begin{bmatrix} 1 & \frac{1}{4} \\ 1 & 1 \end{bmatrix} \right) \beta_2 = -\beta_2.
\]

Using this property, we conclude that the sum of the contributions of the former types are none. For the latter types, we calculate

\[
 \rho'(t, h)(e, \alpha) = \left( \begin{bmatrix} 0 & 0 \\ 2^{-m} & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -2^{N+1-m}s & 1 \end{bmatrix} \right).
\]

Suppose \( \rho'(t, h)(e, \alpha) \in \text{supp}(\phi) \). Using (3.2) and (3.3), we can assume \( m = 0 \). Observing \( c_2 \), we find that \( s \in 2^{-1} \mathbb{Z}_2 \). Then, using (3.2) again, we see that the total contribution of the latter types is non-trivial. This completes the proof. \( \square \)
Let \( \zeta_8 = \left( \frac{1+i}{\sqrt{2}} \right) \) (resp. \( K = \mathbb{Q}(\zeta_8) \)) with the ring of integers \( \mathfrak{o} \) (resp. \( \mathfrak{O} \)). Let \( p \) (resp. \( \mathfrak{p} \)) be the unique (ramified) prime ideal of \( \mathfrak{o} \) (resp. \( \mathfrak{O} \)) lying over the prime ideal 2 of \( \mathbb{Q} \). Next, we observe the irreducible cuspidal automorphic representation \( \pi(\lambda) \) of \( GL_2(L_\lambda) \) obtained from the größencharacter \( \lambda \) of \( K_\lambda^* \) on p. 870 of [7]. The definition of \( \lambda \) is as follows. For the two archimedean places \( \infty_1, \infty_2 \) of \( K \), \( \lambda_{\infty_1}(z) = |z|^3/3^2 \), \( \lambda_{\infty_2}(z) = |z|/z \in \mathbb{C}^\times \). Thus, the lowest weights of the archimedean components of \( \pi(\lambda) \) are 4, 2, respectively. The conductor of \( \lambda \) is \( \mathfrak{p}^4 = (2) \), and

\[
(\mathcal{O}/\mathfrak{p}^4)^\times = \langle \zeta_8 \bmod 2 \rangle \oplus \langle 1 + \sqrt{2} \rangle \bmod 2 \rangle \simeq \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.
\]

Then, \( \lambda_{\mathfrak{p}} \) is defined by

\[
\lambda_{\mathfrak{p}}(\zeta_8 \bmod 2) = 1, \quad \lambda_{\mathfrak{p}}((1 + \sqrt{2}) \bmod 2) = -1.
\]

We define the quasi-character \( \mu \) on \( L_\mathfrak{p}^\times \) with conductor \( p^3 \) by

\[
\mu((1 + \sqrt{2}) \bmod p^3) = i,
\]

where \( (\mathfrak{o}/p^3)^\times = ((1 + \sqrt{2}) \bmod p^3) \simeq \mathbb{Z}/4\mathbb{Z} \). Then, it holds \( \lambda_{\mathfrak{o}} = \mu \circ N_{K/L} \). Let \( \chi_{K/L} \) be the quadratic character of \( L_\lambda^* \) associated to the extension \( K/L \). The central character of \( \pi(\lambda) \) is \( \lambda_{|L_\lambda^*}|K/L \). Because both of \( \lambda_{\infty_1} \chi_{K/L, \infty_1} \) and \( \lambda_{\mathfrak{o}} \chi_{K/L, \mathfrak{o}} = \mu \circ N_{K/L, \mathfrak{o}} \chi_{K/L, \mathfrak{o}} \) are trivial, so is the central character of \( \pi(\lambda) \). Employing Theorem 4.6(iii) of [9], we find that \( \pi(\lambda)_{\mathfrak{o}} \) is the principal series representation

\[
\pi(\mu, \mu \chi_{K/L, \mathfrak{o}}) \simeq \pi(\mu, \overline{\mu})
\]

of level \( p^6 \).

Finally, we prove the conjecture. One can construct an automorphic form \( F \in \Theta_2(\pi(\lambda)) \) satisfying \( q(u)F = F \) for \( u \in \text{Sp}_4(\mathbb{Z}_p) \) at \( p \neq 2 \), and (3.1) at 2. The local standard Whittaker function \( W_{F, \psi, \lambda} \) is right \( \Gamma_0^{(2)}(2^6\mathbb{Z}_2) \)-semi invariant and \( W_{F, \psi, \lambda}(1) \neq 0 \). Let \( g_0 = \text{diag}(2^3, 2^3, 2^{-2}, 1) \in \text{GSp}_4(\mathbb{Q}) \), and \( F'(g) = F(g_0 g g_0^{-1}) = F(g g_0^{-1}) \). Let

\[
L'(g_0) := g_0^{-1} \Gamma_0^{(2)}(2^6\mathbb{Z}_2) g_0 = \begin{bmatrix}
\mathbb{Z}_2 & 2^2\mathbb{Z}_2 & 2^6\mathbb{Z}_2 & 2^5\mathbb{Z}_2 \\
2^{-2}\mathbb{Z}_2 & \mathbb{Z}_2 & 2^5\mathbb{Z}_2 & 2^3\mathbb{Z}_2 \\
\mathbb{Z}_2 & 2^2\mathbb{Z}_2 & \mathbb{Z}_2 & 2^{1-2}\mathbb{Z}_2 \\
2\mathbb{Z}_2 & 2^3\mathbb{Z}_2 & 2^2\mathbb{Z}_2 & \mathbb{Z}_2
\end{bmatrix} \cap \text{Sp}_4(\mathbb{Q}_2).
\]

Then, \( F' \) is right \( \Gamma' \)-semi invariant, and so is \( W_{F', \psi, \lambda} \). Note that \( \Gamma'(2, 4, 8)_2 = \Gamma^{(2)}_0(8\mathbb{Z}_2) \subset \Gamma' \). Because

\[
q\left(\begin{bmatrix}
1 & s_1 & s_2 \\
1 & s_2 \\
1 & 1
\end{bmatrix}\right) W_{F', \psi, \lambda}(1) = W_{F', \psi, \lambda}(1) \neq 0
\]

for \( s_1, s_2 \in \mathbb{Q}_2 \).

\[
\int_{\Gamma'(2,4,8)_2} q(u)W_{F', \psi, \lambda}(1) \, du \neq 0.
\]

Hence, there is an irreducible globally generic constituent of \( \Theta_2(\pi(\lambda)) \), which has a right \( \Gamma'(2, 4, 8)_2 \times \prod_{p \neq 2} \text{Sp}_4(\mathbb{Z}_p) \)-invariant vector. We denote this representation by \( \Pi_{\text{gen}} \).
Theorem 3.2. The irreducible cuspidal automorphic representation \( \Pi_{g_1} \) is a D-critical representation associated to \( \pi(\lambda) \). The conjecture is true.

Proof. First, employing the result of local \( \theta \)-correspondence for \( Sp_4(\mathbb{R}) \) and \( O_{2,2}(\mathbb{R}) \) due to Przebinda [20], we find that \( \Pi_{g_1}^\text{gen} \) is the large discrete series representation with Blattner parameter \((3, -1)\), a cohomological weight. Next, we claim that \( \Pi_{g_1}^\text{gen} \) is not a weak endoscopic lift, nor a CAP representation. Recall that the lowest weights of the archimedean components of \( \pi(\lambda) \) are \((4, 2)\). Hence, \( \pi(\lambda) \) is not a base change lift. From Lemma 2.4, \( \Pi_{g_1}^\text{gen} \) is not a weak endoscopic lift. On the authority of Piatetski-Shapiro [18], and Soudry [26], every partial spinor \( L \)-function of a CAP representation is, up to finitely many Euler factors, in the form of \( L(s - \frac{1}{2}, \chi)L(s + \frac{1}{2}, \chi)L(s - \frac{1}{2}, \chi')L(s + \frac{1}{2}, \chi') \), \( L(s - \frac{1}{2}, \mu)L(s + \frac{1}{2}, \mu) \), or \( L(s - \frac{1}{2}, \chi)L(s + \frac{1}{2}, \chi)L(s, \sigma_1) \). Here \( \chi, \chi' \) are some quadratic character of \( \mathbb{A}^\text{gen} \), \( \mu \) is a quasi-character of \( \mathbb{A}^\ast \) such that \( \lambda \) is invariant but not distinguished in \( \Pi_{g_1}^\text{gen} \). Hence, \( \pi(\lambda) \) is not \( \pi(\lambda) \)-distinguished in \( \Pi_{g_1}^\text{gen} \). 

Thus, \( \Pi_{g_1}^\text{hol} \) contributes to \( H^3(\text{Gr}_1(S_{\Gamma(2,4,8)}, \mathbb{C})) \simeq S_2(\Gamma(2, 4, 8)) \), i.e., \( \Pi_{g_1}^\text{hol} \) is one of the 11 irreducible representations \( \Pi_{g_1}, \ldots, \Pi_{g_4} \). Observing some \( L \)-factors of them calculated in [7], one can conclude that \( \Pi_{g_1}^\text{hol} = \Pi_{g_3} \). This completes the proof. \( \square \)

Remark 1. Using the definition of \( \mu \), one can show that \( \pi(\lambda) \) is invariant but not distinguished in the sense of Roberts [22]. Employing Theorem 8.5 of [22], we find that the set of D-critical representations associated to \( \pi(\lambda) \) consists of four irreducible representations \( \Pi_{g_1}^\text{gen} = \Pi_1, \Pi_2, \Pi_3, \Pi_4 \). They are all given by a \( \theta \)-lift from \( GSO(4) \). Further, \( \Pi_{2,\infty} \simeq \Pi_{3,\infty} \) (resp. \( \Pi_{1,\infty} \simeq \Pi_{4,\infty} \)) is the holomorphic (resp. large) discrete series representation with Blattner parameter \((3, 3)\) (resp. \((3, -1)\)), and \( \Pi_{1,p} \simeq \Pi_{2,p}, \Pi_{3,p} \simeq \Pi_{4,p} \) at every nonarchimedean place. Noting this fact, one can show the above theorem.

3.2. Saito–Kurokawa representation, proof for \( L(s, \Pi_{g_4}; \text{spin}) \)

First, we will recall some known results on Saito–Kurokawa representation. For a square free integer \( a \), let \( \chi^{(a)} \) denote the quadratic character of \( \mathbb{A}^\ast \) associated to the extension \( \mathbb{Q}(\sqrt{a})/\mathbb{Q} \). For an irreducible cuspidal automorphic representation \( \tau \) of \( GSp_{2n}(\mathbb{A}) \), we will abbreviate \( \chi^{(a)} \tau \) as \( \tau^{(a)} \). Let \( B/Q \) be a quaternion algebra. Let \( \sigma \) be an irreducible cuspidal automorphic representation of \( PB(\mathbb{A})^{\ast} \).

Suppose that \( \sigma_{g_1}^\text{hol} \) is the holomorphic discrete series representation of lowest weight 4. Let \( 1_{B(\mathbb{A})}^{\ast} = 1 \) denote the trivial representation of \( B(\mathbb{A}) \). For a \( \pm 1 \)–valued character \( \chi \) of \( \mathbb{Q}^\ast \setminus \mathbb{A}^\ast \), we denote by \( \chi \sigma \) the representation of \( PB(\mathbb{A})^{\ast} \) sending \( h \in B(\mathbb{A})^{\ast} \) to \( \chi(N_{B/Q}(h))\sigma(h) \). We will abbreviate \( \chi_{1_{B(\mathbb{A})}^{\ast}} \) as \( \chi \). If \( B/Q \) is not split, then \( \theta_2(\chi \boxtimes \sigma) \) is cuspidal. It is easy to show that \( \theta_2(\chi \boxtimes \sigma) \) is not vanishing, if and only if \( L(\frac{1}{2}, \chi \sigma) \neq 0 \), by using a result of Waldspurger [29]. On the other hand, if \( B/Q \) is split, then \( \theta_2(\chi \boxtimes \sigma) \) is non-vanishing and noncuspoid. Indeed, one can construct an \( f \in \theta_2(\chi \boxtimes \sigma) \) so that the \( P \)-degenerate Whittaker function \( W_{f, \psi} \) is nontrivial as is explained below (hence, \( \Phi_f(f) \) defined in (2.22) is nontrivial). We will recall the result of Cogdell and Piatetski-Shapiro [4] and Schmidt [24]. Let \( \pi \) be an irreducible cuspidal automorphic representation of \( PGL_2(\mathbb{A}) \). The global cuspidal Saito–Kurokawa packet \( SK_0(\pi) \) is defined as the set of irreducible cuspidal automorphic representations of \( PGSp_4(\mathbb{A}) \) whose spinor \( L \)-functions are equal to \( \xi(s - \frac{1}{2})\xi(s + \frac{1}{2})L(s, \pi) \), up to finitely many Euler factors. Let \( D_{\psi} \) be the unique division quaternion algebra over \( \mathbb{Q}_\psi \). When \( \pi_\psi \) is square-integrable, let \( \pi'_\psi \) denote the Jacquet–Langlands transfer to \( D_{\psi}^{\ast} \). The local Saito–Kurokawa packet is the following set:
\[
\text{SK}(\pi_\nu) = \begin{cases} 
\{\theta_2(1_{\chi} \otimes \pi_\nu)\}, & \text{if } \pi_\nu \text{ is square-integrable,} \\
\{\theta_2(1_{\chi} \otimes \pi_\nu)\}, & \text{otherwise.}
\end{cases}
\]

At a nonarchimedean place \( \nu = p \), as is explained on pp. 230–233 of [24], \( \theta_2(1_p \otimes \pi_p) \) is the local Saito–Kurokawa representation that is the unique irreducible quotient of the Siegel parabolically induced representation \( \chi_p^{1/2} \pi_p \times | -1/2 \) (cf. [24, 23]). For a \((\pm 1)\)-valued character \( \chi_p \), \( \theta_2(\chi_p \otimes \pi_p) \) is the \( \chi_p \)-twist of the local Saito–Kurokawa representation \( \theta_2(1_p \otimes \chi_p \pi_p) \).

Next, we will observe the global cuspidal Saito–Kurokawa packet of \( \rho_1 \), and that of \( \rho_1(-2) \). For a moment, let

\[
B/I = \mathbb{Q} + QI + QJ + QIJ, \quad I^2 = J^2 = -1, \quad IJ = -JI.
\]

This quaternion algebra splits outside of \( \{\infty, 2\} \). As is seen in Section 4 of [17], \( \rho_1 \) has the Jacquet–Langlands transfer to \( PB(A)^\times \). Denote it by \( \rho_1' \). In [17], the Siegel modular form \( F_1 \) is constructed by the Yoshida lift of \((1, \rho_1')\). This implies

\[
L\left(\frac{1}{2}, \rho_1\right) \neq 0, \quad \varepsilon\left(\frac{1}{2}, \rho_1\right) = \varepsilon\left(\frac{1}{2}, \rho_{1,2}, \psi_2\right) = 1.
\]

The 2-component \( \rho_{1,2}' \) is the finite dimensional representation of \( B_2^\times \simeq D_2^\times \) described as follows. We fix the maximal order \( \mathcal{R} = \mathbb{Z}_2 + \mathbb{Z}_2I + \mathbb{Z}_2J + \mathbb{Z}_2(1+i+J+I) \subset B_2 \). Let \( \sigma \in B_2 \) be an uniformizer. Let \( \mathcal{R}(2) = \mathbb{Z}_2 + \sigma^2 \mathcal{R} \). As a complete system of representatives \( U \) of \( \mathcal{R}^\times / \mathcal{R}(2)^\times \), we can take \( \{1, I, J, \frac{1+i+J+I}{2}\} \). Let \( W = C I + C J + C I J \). Then, we obtain a finite dimensional representation \( \tau_2 \) of \( B_2^\times \) from the automorphism of \( W \) defined by \( u^{-1}wu \). Because \( B_2^\times = B_2^\times \mathcal{R}(2)^\times A^\times \), from this representation, one can obtain an automorphic representation \( \tau \) of \( PB_2^\times \). One can construct a right \( I_0(1)^8 \)-invariant vector in \( \Theta_1(\tau \otimes \tau) \) (see also Proposition 3.8). This means \( \rho_1' = \tau \), because the space of elliptic cusp form of weight 4 of level 8 is 1-dimensional. Hence \( \tau_2 \) is irreducible and equivalent to \( \rho_{1,2}' \).

**Lemma 3.3.** The root number of \( \rho_1(-2) \) is \(-1\).

**Proof.** Because \( \rho_{1,2}' \) is unramified for \( p \neq 2 \) and \( \rho_{1,\infty} \) is the holomorphic discrete series representation of lowest weight 4, it suffices to show that \( \varepsilon\left(\frac{1}{2}, \rho_{1,2}' \right), \psi_2\right) = -1 \). We will see the \( \varepsilon\)-factor of the base change lift \( \rho_{1,p}^{BC} \) to \( GL_2(\mathbb{Q}(\sqrt{-2}))_p \) with \( p = \sqrt{-2} \). Let \( L = \mathbb{Q}(\sqrt{-2}) \). Let \( \psi_L = \psi \circ \text{Trace}_{L/\mathbb{Q}} \). We identify \( L \simeq \mathbb{Q}(I + J) \subset B/I \) for the above \( B/I \). Then \( \mathcal{R}(2) \cap L_p \) is the maximal order of \( L_p \). Thus, every character (constituent) of the restriction \( \rho_{1,2}' |_{L_p^\times} \) is unramified. Because \( (I + J)^{-1}(I + J)(I + J) = I + J \in W \), the trivial character of \( L_p^\times \) appears in this restriction. Applying Lemma 14 of [10], we have

\[
-1 = -\varepsilon\left(\frac{1}{2}, \rho_{1,2}' \right)
= \varepsilon\left(\frac{1}{2}, \rho_{1,2}' \right) \psi_L(p) \psi_2(h_2)
= \varepsilon\left(\frac{1}{2}, \rho_{1,2}' \right) \psi_2(h_2)
\]

\(\square\)
From Lemma 3.3, it follows that \( L(s, \rho_1^{(-2)}) = -L(1 - s, \rho_1^{(-2)}) \), and hence

\[
L \left( \frac{1}{2}, \rho_1^{(-2)} \right) = 0, \quad \varepsilon \left( \frac{1}{2}, \rho_1^{(-2)} \right) = \varepsilon \left( \frac{1}{2}, \rho_1^{(-2)} \right). \psi_2 = -1.
\]

Employing the main lifting theorem of [24], and Theorem 3.1 of [4], we conclude

\[
SK_0(\rho_1) = \left\{ \bigotimes_{v \neq \infty, 2} \theta_2(1 \boxtimes \rho_1^{(-2)}), \bigotimes_{v \neq \infty, 2} \theta_2(1 \boxtimes \rho_1^{(-2)}) \right\},
\]

\[
SK_0(\rho_1^{(-2)}) = \left\{ \theta_2(1 \boxtimes \rho_1^{(-2)}), \bigotimes_{v \neq \infty, 2} \theta_2(1 \boxtimes \rho_1^{(-2)}) \right\}.
\]

Note that \( \theta_2(1 \boxtimes \rho_1^{(-2)}) |_{\text{Sp}(4)} \) is the holomorphic discrete series representation with Blattner parameter \((3, 3)\). Therefore, we guess that the latter constituent of \( SK_0(\rho_1^{(-2)}) \) is \( \chi(\rho_1^{(-2)}) \Pi_{\text{Sp}} \). We want to show that \( \theta_2(\chi(\rho_1^{(-2)}) \boxtimes \rho_1) \) has a right \( L^\infty(4, 8) \)-invariant vector for every \( p \). The local \( \theta \)-lift \( \theta_2(\chi(\rho_1^{(-2)}) \boxtimes \rho_1) = \chi(\rho_1^{(-2)}) \theta_2(1 \boxtimes \rho_1^{(-2)}) \) does not have a local Whittaker function. But it has a local \( P \)-degenerate Whittaker function \( W_{\psi}^p \) as follows. Let \( e' = [1] \). Let \( Z(e, e') \subset \text{SO}_X \) be the pointwise stabilizer subgroup of \( e, e' \), which is isomorphic to

\[
\left\{ \left[ \begin{array}{cc} 1 & s \\
1 & 1 \end{array} \right] \left| s \in Q_v \right. \right\}
\]

via \( i_\rho \). Then, \( W_{\psi}^p(g) \) of \( \theta_2(\chi(\rho_1^{(-2)}) \boxtimes \rho_1) \) is

\[
\int_{Z(e, e')(Q_v) \setminus \text{SO}_M(Q_v)} r_v^1(g, i_\rho(h_1, h_2)) \psi_v(e, e') \chi_{\rho_1^{(-2)}}(\text{det}(h_1)) \beta_v(h_2) dh_1 dh_2 \quad (3.9)
\]

where \( \beta_v \) is a Whittaker function of \( \rho_1 \) with respect to \( \psi_v \). It is easy to construct a right \( \text{Sp}_4(\mathbb{Z}_p) \)-invariant \( W_{\psi}^p \) for a nonarchimedean place \( p \neq 2 \). We will construct a right \( L^\infty(4, 8) \)-invariant \( P \)-degenerate Whittaker function of \( \theta(\chi_{\rho_1^{(-2)}) \boxtimes \rho_1) \). From \( \rho_1^{(-2)} \), we take the right \( L^\infty(4, 8) \)-invariant local Whittaker function \( \beta_2 \) with respect to \( \psi_2 \) such that \( \beta_2(1) = 1 \). We define

\[
\phi'(x_1, x_2) = \chi_{\rho_1^{(-2)}(b_1) ch(x_2; M_2(\mathbb{Z}_2)) \times \left\{ \begin{array}{ll} 1 & \text{if ord}_2(a_1) \geq 0, \text{ord}_2(b_1) = 0, \text{ord}_2(c_1), \text{ord}_2(d_1) \geq 3, \\
0 & \text{otherwise,}
\end{array} \right\}
\]

where we write \( x_1 = [a_1 b_1 c_1 d_1] \in M_2(\mathbb{Q}_2) \). Let

\[
\Gamma'' = \left[ \begin{array}{cccc}
1 + 2^3 \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \\
2^3 \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \\
2^3 \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \\
2^3 \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \\
\end{array} \right] \cap \text{Sp}_4(\mathbb{Z}_2).
\]

Then, \( \phi' \) is right \( \Gamma'' \)-invariant. One can calculate \((3.9)\) is not zero at \( g = 1 \), directly. Let \( g_0' = \text{diag}(2^4, 2^3, 2^{-1}, 1) \). Then
There is a right \( g_0^{-1} \Gamma'' g_0 \)-invariant \( W_{\chi(-2)}^p \) such that \( W_{\chi(-2),2}^p(1) \neq 0 \). Then, an integral similar to (3.7) gives a nontrivial right \( \Gamma(2,4,8)_2 \)-invariant local \( P \)-degenerate Whittaker function of \( \theta_2(\chi(-2)) \). Consequently,

**Theorem 3.4.** The irreducible cuspidal automorphic representation \( \Pi_{g_4} \) is the \( \chi(-2) \)-twist of the irreducible (holomorphic) constituent of \( SK_0(\rho_1) \). The conjecture is true.

Finally, we give a remark. Observing the eigenvalues of \( g_4 \) in the table of Section 8 of [7], we find that \( \Pi_{g_4} \) does not satisfy the generalized Ramanujan conjecture. Indeed

\[
|\alpha_{p1}| = |\alpha_{p2}| = p^{\frac{3}{2}}, \quad |\alpha_{p3}| = p, \quad |\alpha_{p4}| = p^2
\]

for \( p = 3, 5, 7, 11, 13, 17, 19 \), if we write the Hecke polynomial of \( \Pi_{g_4, p} \) as \( \prod_{i=1}^{4}(X - \alpha_{pi}) \). Then, one can see that \( \Pi_{g_4} \) is a twist of a Saito–Kurokawa representation with the following proposition.

**Proposition 3.5.** For a Siegel modular 3-fold \( S_\Gamma \), if an irreducible cuspidal automorphic representation \( \Pi \) contributes to \( H^{3,0}(G_1^W(S_\Gamma, \mathbb{C})) \) and does not satisfy the Ramanujan conjecture, then \( \Pi \) is a twist of a Saito–Kurokawa representation.

**Proof.** As stated by Theorem I of Weissauer [31], there is a \( GL_4(\mathbb{Q}_2) \)-valued Galois representation \( \rho_{\Pi} \) of \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \) such that

\[
L_{S_\Pi}(s - \frac{3}{2}, \Pi; \text{spin}) = L_{S_{\Pi}}(s, \rho_{\Pi}).
\]

Assume that \( \Pi \) is not a CAP representation. Then \( \rho_{\Pi} \) is pure of weight 3, the eigenvalues of \( \rho_{\Pi}(\text{Frob}_p) \) has absolute value \( p^{3/2} \), and hence \( \Pi \) does not satisfy the Ramanujan conjecture. This is a contradiction. Hence \( \Pi \) is a CAP representation, i.e., an irreducible cuspidal automorphic representation associated to a parabolically induced representation. As stated by Theorem A of Soudry [26], every CAP representation associated to a Borel or Klingen parabolically induced representation is a constituent of a global \( \theta \)-lift of an irreducible automorphic representation \( \sigma_T \) of \( G_0(T) \) for a quadratic field \( T \). It is not hard to see that the local \( \theta \)-lift to \( Sp_4(\mathbb{R}) \) of \( \sigma_{T,\infty} \) is not a holomorphic discrete series representation with Blattner parameter \( (3, 3) \). Hence \( \Pi \) is a CAP representation associated to a Siegel parabolically induced representation. On the authority of Piatetski-Shapiro [18], such a representation is a twist of a Saito–Kurokawa representation.

\[ \square \]

3.3. Weak endoscopic lift

Let \( f_5 \) be the 6-tuple product of Igusa theta constants defined in [7], and \( \chi_{f_5} \) be the character of \( \Gamma(2) \) obtained from \( f_5 \) through the Igusa transformation formula (cf. Lemmas 5.2, 5.3 in [7]). Let \( \Pi_{f_5} \) be the irreducible cuspidal automorphic representation of \( GSp_4(\mathbb{A}) \) associated to \( f_5 \) in Proposition 2.2. Our aim is to prove

**Theorem 3.6.** An irreducible cuspidal automorphic representation which is weakly equivalent to \( \Pi_{f_5} \) contributes to \( H^{2,1}(G_3^W(S_{\ker(\chi_{f_5}), \mathbb{C}})) \).

First, we recall that $\Pi_{f_5}$ is a weak endoscopic lift of the pair $(\pi(\mu), \pi(\mu^3))$ of the following CM-elliptic cusp forms. Let $E/Q$ be the CM-elliptic curve defined by the equation $y^2 = x^3 - \mu$. Let $\mu$ be the größencharacter of $\mathbb{Q}(i)_A^\times$ such that $L(s - \frac{1}{2}, \mu) = L(s, E/Q)$. At $v = \infty$, $\mu_\infty(2) = |z|^2/2, z \in \mathbb{C}^\times$. Thus, the lowest weights of the holomorphic discrete series representations $\pi(\mu)_\infty, \pi(\mu^3)_\infty$ are $2, 4$, respectively. Let $\sigma = \mathbb{Z}[i]$. Then $p \subset \sigma$ be the prime ideal lying over $2$. The conductor of $\pi(\mu)$ is $3$. Noting the strong multiplicity theorem for $GL_2$, we find that both of $\pi(\mu), \pi(\mu^3)$ are supercuspidal. This completes the proof. □

**Lemma 3.7.** The 2-components $\pi(\mu)_2, \pi(\mu^3)_2$ are equivalent and supercuspidal.

**Proof.** From the definition, $\mu_p$ is $\{\pm 1, \pm i\}$-valued on $\mathbb{Q}_p^\times$. Thus $\mu_p = \overline{\mu}_p^3$ on $\mathbb{Q}_p^\times$. Noting that the central character of $\pi(\mu)$ is trivial, we have

$$
\pi(\mu)_2 = \pi(\overline{\mu}_p^3)_2 = \pi(\mu^3)_2.
$$

There is no quasi-character $\xi$ of $\mathbb{Q}_p^\times$ such that $\xi \circ \mathbb{N}_{\mathbb{Q}(i)/\mathbb{Q}} = \mu$. Employing Lemma 4.6 of [9], we find that $\pi(\mu)_2$ is supercuspidal. This completes the proof. □

Employing this lemma and the Jacquet–Langlands theory, we find that both of $\pi(\mu), \pi(\mu^3)$ have the Jacquet–Langlands transfers $\pi(\mu)'_2, \pi(\mu^3)'_2$ to $PB(\mathbb{A})$ for the definite quaternion algebra $B/Q$ defined in (3.8). In [17], we really construct a Siegel modular form lying in $\Pi_{f_5}$ by the Yoshida lift $\Theta_2(\pi(\mu)' \boxtimes \pi(\mu^3)')$. Thus, $\Pi_{f_5} = \Theta_2(\pi(\mu)' \boxtimes \pi(\mu^3)')$. Further, employing Theorem 8.5 of [22], we find that the set of all weak endoscopic lifts of $(\pi(\mu), \pi(\mu^3))$ is

$$
\{\Theta_2(\pi(\mu) \boxtimes \pi(\mu^3)), \Theta_2(\pi(\mu)' \boxtimes \pi(\mu^3)')\}.
$$

Therefore, we guess that the irreducible cuspidal automorphic representation of $GSp_4(\mathbb{A})$ as in Theorem 3.6 is $\Theta_2(\pi(\mu) \boxtimes \pi(\mu^3))$, which is globally generic.

Next, in order to show the theorem, we will observe the local $\theta$-lift $\theta_2(\pi(\mu)_2 \boxtimes \pi(\mu^3)_2) = \Theta_2(\pi(\mu)_2 \boxtimes \pi(\mu)_2)$, which is the 2-component of $\Theta_2(\pi(\mu) \boxtimes \pi(\mu))$. For the sake of generality, let $B/Q$ be a general quaternion algebra and consider $\Theta_2(\pi(\mu) \boxtimes \pi(\mu))$ for an irreducible cuspidal automorphic representation $\pi$ of $PB(\mathbb{A})$.

**Proposition 3.8.** Let $\sigma$ be an irreducible cuspidal automorphic representation of $PB(\mathbb{A})$. Let $\Phi_Q$ be the operator defined in Section 2.3. Then, $\Phi_Q(\Theta_2(\pi(\mu) \boxtimes \pi(\mu)))|_{GL(2)} = \sigma^{Ll}$.

**Proof.** For a $\varphi \in S(M_2(\mathbb{A}))$, put $\varphi_0(x) = \varphi(0, x) \in S(M_2(\mathbb{A}))$. Take $f \in \sigma$, and put $F = \theta_2(\varphi, f)$. We calculate $\Phi_Q(F)|_{GL(2)} = \theta_1(\varphi_0, f \boxtimes f)$ in $\sigma^{Ll}$. We abbreviate $W^1_{F, \varphi}(e_Q(g, 1))$ as $W^1(g)$ for $g \in SL(2, \mathbb{A})$. Then

$$
W^1(1) = \int \int_{Z_1(\mathbb{A}) \backslash SO_B(\mathbb{A})} \int_{Z_1(\mathbb{Q}) \backslash Z_1(\mathbb{A})} r^1(g, i_p(h_1, h_2))\varphi_0(1)(\int \overline{f}(bh_1)f(bh_2)\,db) \,dh_1 \,dh_2,
$$

where $Z_1$ denotes the stabilizer subgroup of $1 \in B(\mathbb{Q})$, which is isomorphic to $\{(b, b) \mid b \in B(\mathbb{A})^\times\}$ via $i_p$. Obviously, the integral in the parenthesis is nontrivial, and so is $W^1(1)$. Thus $\theta_1(\varphi_0, f \boxtimes f)$ is nontrivial. Because $\theta_1(\varphi_0, f \boxtimes f)$ is right $GL_2(\mathbb{Z}_p)$-invariant for almost all $p$, it is easy to see that $\Phi_Q(F)|_{GL(2)} \in \sigma^{Ll}$. Noting the strong multiplicity theorem for $GL(2)$, we find $\Phi_Q(F)|_{GL(2)} \in \sigma^{Ll}$. Hence the assertion. □

**Remark 2.** This proof implies that $\Phi_Q(\Theta_2(\sigma_1 \boxtimes \sigma_2)) = 0$ if $\sigma_1 \neq \sigma_2$. 
Remark 3. If $\pi_p$ is a supercuspidal representation, then $\theta_2(\pi_p \boxtimes \pi_p)$ (resp. $\theta_2(\pi'_p \boxtimes \pi'_p)$) is the constituent $\tau(S, \pi_p)$ (resp. $\tau(T, \pi_p)$) of the parabolically induced representation $1 \rtimes \pi_p$ (see [23] for the meanings of these symbols).

From this proof, there are a pair of $\phi_1 \in S(B(\mathbb{A}))$ and $f_0 \in \sigma$ such that $\theta_1(\phi_1, f_0 \boxtimes f_0)$ is a new-form of $\sigma^{\pm}$. In particular, if we set a $\varphi \in S(B(\mathbb{A})^2)$ so that $\varphi_0 = \phi_1$, then $A_2(\varphi, f \boxtimes f)$ is nontrivial. For example, set $\varphi(x_1, x_2) = \phi_1(x_2)\varphi_\infty(x_1) \otimes_p \text{ch}(x_1; \mathcal{R}_p)$, where $\mathcal{R}$ is a maximal order of $B(\mathbb{R})$ and $\varphi_\infty$ is an arbitrary Schwartz–Bruhat function on $B(\mathbb{R})$ such that $\varphi_\infty(0) \neq 0$. Then, $\theta_2(\varphi, f_0 \boxtimes f_0)$ is right $\text{Kl}_p(\text{ord}_p(N))$-invariant if $B_p$ is split, and $\text{Kl}'_p(\text{ord}_p(N))$-invariant otherwise, where $N$ is the level of $\sigma^{\pm}$, and

$$
\text{Kl}_p(n) := \begin{bmatrix}
\mathbb{Z}_p & \mathbb{Z}_p^n & \mathbb{Z}_p & \mathbb{Z}_p \\
\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\
\mathbb{Z}_p & \mathbb{Z}_p^n & \mathbb{Z}_p & \mathbb{Z}_p \\
\mathbb{Z}_p^n & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\
\mathbb{Z}_p^n & \mathbb{Z}_p^n & \mathbb{Z}_p & \mathbb{Z}_p \\
\mathbb{Z}_p^n & \mathbb{Z}_p^n & \mathbb{Z}_p^n & \mathbb{Z}_p \\
\mathbb{Z}_p^n & \mathbb{Z}_p^n & \mathbb{Z}_p^n & \mathbb{Z}_p 
\end{bmatrix} \cap \text{GSp}_4(\mathbb{Z}_p),
$$

$$
\text{Kl}'_p(n) := \begin{bmatrix}
\mathbb{Z}_p & \mathbb{Z}_p^n & \mathbb{Z}_p & \mathbb{Z}_p \\
\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\
\mathbb{Z}_p & \mathbb{Z}_p^n & \mathbb{Z}_p & \mathbb{Z}_p \\
\mathbb{Z}_p^n & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\
\mathbb{Z}_p^n & \mathbb{Z}_p^n & \mathbb{Z}_p & \mathbb{Z}_p \\
\mathbb{Z}_p^n & \mathbb{Z}_p^n & \mathbb{Z}_p^n & \mathbb{Z}_p \\
\mathbb{Z}_p^n & \mathbb{Z}_p^n & \mathbb{Z}_p^n & \mathbb{Z}_p 
\end{bmatrix} \cap \text{GSp}_4(\mathbb{Z}_p)
$$

for an integer $n$. van Geemen and van Straten [7] conjectured that, up to the Euler factors at 2,

$$
L(s, \Pi_{\phi}; \text{spin}) = L(s, \chi_i(\pi(\mu)))L(s, \chi_i(\pi(\mu^2)))
$$

for $4 \leq i \leq 6$, where $\chi_4 = \chi^{(-2)}$, $\chi_5 = 1$, $\chi_6 = \chi^{(2)}$.

Corollary 3.9. The above conjecture is true.

Proof. It is possible to show the level of $\pi(\mu)$ (resp. $\chi^{(\pm 2)}(\mu)$) is $2^5$ (resp. $2^6$) (cf. Proposition 4.8 of [17]). From the above argument, the local $\theta$-lift $\theta_2(\pi(\mu)_2 \boxtimes \pi(\mu)_2)$ (resp. $\theta_2(\chi^{(\pm 2)}(\mu)_1 \boxtimes \chi^{(\pm 2)}(\mu)_1)$) has a local right $\text{Kl}'_2(5)$ (resp. $\text{Kl}'_2(6)$)-invariant $\mathbb{Q}$-degenerate Whittaker function. Now, noting that

$$
\text{Kl}'_2(6) \cong \begin{bmatrix}
\mathbb{Z}_2 & 2^7\mathbb{Z}_2 & 2^5\mathbb{Z}_2 & 2^4\mathbb{Z}_2 \\
2^{-1}\mathbb{Z}_2 & \mathbb{Z}_2 & 2^4\mathbb{Z}_2 & 2^2\mathbb{Z}_2 \\
2^{-1}\mathbb{Z}_2 & 2^2\mathbb{Z}_2 & \mathbb{Z}_2 & 2^{-1}\mathbb{Z}_2 \\
2^2\mathbb{Z}_2 & 2^3\mathbb{Z}_2 & 2^7\mathbb{Z}_2 & \mathbb{Z}_2 
\end{bmatrix} \cap \text{GSp}_4(\mathbb{Q}_2),
$$

one can show that the local $\theta$-lift has a right $\Gamma(4, 8)_2$-invariant vector and verify the conjecture in the same manner as in 3.1.

Finally, we will prove the theorem. Put

$$
f'_5(Z) := \frac{\theta(1, 1, 0, 0, 0)(Z)\theta(1, 1, 0, 0, 0)(Z)}{\theta(0, 0, 0, 1)(Z)}.
$$

From $f'_5$, a character of $\Gamma(2)$ is obtained through the Igusa transformation formula. Using Proposition 6.2 of [7], we check that this character coincide with $\chi_{f'_5}$. For our computation, we put
For a positive integer \( \kappa \) and a congruence subgroup \( \Gamma_1 \subset \text{GL}_2(\mathbb{Q}) \), let \( S^{(1)}_{\kappa}(\Gamma_1) \) denote the space of elliptic cusp forms of weight \( \kappa \) with respect to \( \Gamma_1 \). Identifying this space with a subspace of automorphic forms on \( \text{GL}_2(\mathbb{A}) \), we define the subspace

\[
S^{(1)}_{\kappa}(\Gamma_1)^{\otimes 2, \text{dis}} = \left\{ (f_1, f_2) \in S^{(1)}_{\kappa}(\Gamma_1)^{\otimes 2} \mid \int_{Z(\mathbb{A})/\text{GL}_2(\mathbb{Q}) \backslash \text{GL}_2(\mathbb{A})} \overline{f}_1(g) f_2(g) \, dg \neq 0 \right\}
\]

of automorphic forms on \( \text{GL}_2(\mathbb{A})^{\otimes 2} \). Composing Remark 2 and the proof of Theorem 2 of Oda [15], we can obtain the following lemma.

**Lemma 3.10.** Let \( \kappa \) be a positive integer. Let \( \Gamma_1 \) be a congruence subgroup of \( \text{GL}_2(\mathbb{Q}) \). Suppose that a \( \varphi \in \bigotimes_{p < \infty} S(M_2(\mathbb{Q}_p)) \) satisfies \( \varphi(q(h_1, h_2)x) = \varphi(x) \) for any \( h_1, h_2 \in \Gamma_1, \mathbb{A} \). Then, there is a \( \varphi_\infty \in S(M_2(\mathbb{R})) \) such that \( \theta_1(\varphi_\infty \times \varphi, f) \neq 0 \) for a certain \( f \in S^{(1)}_{\kappa}(\Gamma_1)^{\otimes 2, \text{dis}} \).

Applying this lemma to the above \( \bigotimes_{p < \infty} \varphi_0^{\otimes 2} \), we find that there is \( \phi^{\prime\prime} \) such that \( \phi_p^{\prime\prime} = \phi_p^{\prime\prime} \) for all \( p < \infty \) and \( \theta_1(\phi^{\prime\prime}, f) \) is not trivial for a certain \( f \in S^{(1)}_{\kappa}(\Gamma_1)^{\otimes 2, \text{dis}} \). However, \( S^{(1)}_{\kappa}(\Gamma_1)^{\otimes 2, \text{dis}} \) is 1-dimensional, generated by a newform \( f_{\text{new}} \) of \( \pi(\mu) \). Thus

\[
\theta_1(\phi^{\prime\prime}, f_{\text{new}} \otimes f_{\text{new}}) \neq 0.
\]

From the above argument, \( \Theta_2(\pi(\mu) \boxtimes \pi(\mu)) \) has a right \( \ker(\chi_{f_5}^\prime)_{\mathbb{A}} \)-invariant vector. Thus \( \Theta_2(\pi(\mu) \boxtimes \pi(\mu^2)) \) also has a right \( \ker(\chi_{f_5}^\prime)_{\mathbb{A}} \)-invariant vector, and Theorem 3.6 follows immediately.

### 4. Hermitian modular forms

Let \( K = \mathbb{Q}(\sqrt{-d}) \) be an imaginary quadratic field. For a Hermitian space \( W \) over \( K \), let \( U_W(K) \) denote the unitary group acting on \( W \) and \( GU_W(K) \) the similitude one. In particular, we write

\[
GU_{n,n}(K) = \left\{ g \in \text{GL}_{2n}(K) \mid g \eta_n^t g = v(g) \eta_n, \ v(g) \in \mathbb{Q}^\times \right\}
\]

and the 2\( n \)-dimensional split Hermitian space as \( W_{n,n} \). Let \( B_{\mathbb{Q}} \) be a definite quaternion algebra such that \( B_{\mathbb{Q}} \otimes \mathbb{Q} \cong M_2(K) \). We set the 6-dimensional positive quadratic space \( V = K + B_{\mathbb{Q}} \). Then, \( \text{PGSO}_V(\mathbb{Q}) \cong \text{PGU}_{W_B}(K) \) for a certain 4-dimensional Hermitian space \( W_B \) (cf. Section 11 of [12]). Let \( r_{U_{W_{n,n}} \otimes W_B} \) be the global Weil representation of \( U_{W_{n,n}} \otimes W_B \) associated to the trivial character of \( \mathbb{A}^n \) and the additive character \( \psi_K = \psi \circ \text{Trace}_{K/\mathbb{Q}} \) (cf. [8,30]). We get the Weil representation \( r_{U,n} \) of \( \{(g,h) \in GU_{n,n} \times GU_W \mid v(g) = v(h)\} \) by restricting \( r_{U_{W_{n,n}} \otimes W_B} \). For a \( \varphi \in S(W_B(K)^n) \), we define

\[
\theta_{U,n}(\varphi)(g,h) = \sum_{y \in W_B(K)^n} r_{U,n}(g,h) \varphi(y).
\]
For an automorphic form $f$ on $\text{GU}_4(K_\lambda)$, define

$$\theta_{U,n}(\varphi, f)(g) = \int_{U_W(K)\backslash U_W(K_\lambda)} \theta_{U,n}(\varphi, hh_1)f(hh') \, dh,$$

where $h'$ is chosen so that $\nu(g) = \nu(h')$ and $dh$ is a right Haar measure on $U_W(K)\backslash U_W(K_\lambda)$. Because $W_\lambda$ is positive definite, this integral converges absolutely, and $\theta_{U,n}(\varphi, f)$ is an automorphic form on $U_W(K_\lambda)$. For an irreducible cuspidal automorphic representation $\sigma$ of $\text{GU}_4(K_\lambda)$, let $\Theta_{U,n}(\sigma)$ denote the space spanned by $\theta_{U,n}(\varphi, f)$ with $f \in \sigma$ and $\varphi \in \mathcal{S}(W_\lambda(K_\lambda))$. In the case $n = 2$, imitating the method in Section 4 of [27], it is possible to show that

$$\Theta_{U,2}(\sigma)_w \simeq \sigma_w,$$

if $\sigma_w$, $K_w/Q_v$ and $B_v$ are all unramified, where $w$ is a place of $K$ lying over a place $v$ of $\mathbb{Q}$. We will identify irreducible cuspidal automorphic representations of $\text{PGSO}_4(\mathbb{A})$ and those of $\text{PGU}_W(K_\lambda)$ via the isomorphism. Then, consider global $\theta$-lifts of $\sigma$ to $\text{GSp}_4(\mathbb{A})$. Let $\sigma'$ be an irreducible constituent of $\sigma|_{\text{SO}_4}$. Assume $\Theta_{2}(\sigma') \neq 0$. Let $\Pi'$ be an irreducible constituent of $\Theta_{2}(\sigma')$. Using [14], we calculate

$$L_{S_{\sigma'}}(s, \sigma') = \xi_{S_{\sigma'}}(s)L_{S_{\sigma'}}(s, \Pi', \left(\frac{-d}{*}\right); r_5),$$

(4.1)

where $L_{S_{\sigma'}}(s, \sigma')$ is the standard Langlands $L$-function of $\sigma'$ (of degree 6) and $L_{S_{\sigma'}}(s, \Pi', \chi_K; r_5)$ is the $(\frac{-d}{*})$-twist of $L_{S_{\sigma}}(s, \Pi'; r_5)$ (note $S_{\sigma} = S_{\Pi}$). Assume $\Theta_{U,2}(\sigma) \neq 0$. Let $\tau'$ be an irreducible constituent of $\Theta_{U,2}(\sigma)$. Using the description of $L$-functions of unramified $\tau'_w \in \text{Irr}(\text{GU}_2(K_w))$ in Section 3 of [11], we calculate

$$L_{S_{\sigma'}}(s, \tau'; \xi^2) = L_{S_{\sigma'}}(s, \sigma').$$

Now (1.2) is shown. We will show the existence of $\tilde{F}$ of Theorem B.

**Proposition 4.1.** Let $K$, $B/Q$, $V$ and $W_\lambda$ be as above. Let $\sigma$ be an irreducible automorphic representation of $\text{PGSO}_4(\mathbb{A}) \simeq \text{PGU}_W(K_\lambda)$. If $\Theta_{2}(\sigma)$ is cuspidal and nontrivial, then $\Theta_{U,2}(\sigma) \neq 0$.

**Proof.** Since $\Theta_{2}(\sigma) \neq 0$, there is an automorphic form $f \in \text{Ind}_{\text{GSO}_4}^{\text{GU}_4} \sigma$ and $\phi \in \mathcal{S}(V(\mathbb{A})^2)$ such that

$$F(g) := \int_{O_V(\mathbb{Q})\backslash O_V(\mathbb{A})} \theta_{2}(\phi)(g, hh_0)f(hh_0) \, dh$$

is nontrivial, where $h_0 \in \text{GO}_V(\mathbb{A})$ is chosen so that $\nu(g) = \nu(h_0)$. Since $V$ is positive definite, $F$ is a cusp form on $\text{GSp}_4(\mathbb{A})$ is related to a (holomorphic) Siegel modular form. Since $F$ is a cusp form, $F_T(1) \neq 0$ for a positive $T = t^2T$. Take $x_1, x_2 \in V$ so that $(x_1, x_2) = T$. Let $Z(x_1, x_2)(\mathbb{Q}) \subset O_V(\mathbb{Q})$ be the pointwise stabilizer subgroup of $(x_1, x_2)$. Then,

$$F_T(1) = \int_{Z(x_1, x_2)(\mathbb{Q})\backslash O_V(\mathbb{A})} r^2(1, h)\phi(x_1, x_2)f(h) \, dh.$$
Hence,
\[ \int_{Z_{(x_1,x_2)}(\mathbb{Q}) \setminus Z_{(x_1,x_2)}(\mathbb{A})} f(zh) \, dz \neq 0. \]

Because \( Z_{(x_1,x_2)}(\mathbb{Q}) \cong \mathbb{O}_4(\mathbb{Q}) \), there is a subgroup \( U_x(K) \cong U_2(K) \) of \( Z_{(x_1,x_2)}(\mathbb{Q}) \) such that
\[ \int_{U_x(K) \setminus U_x(K_\mathbb{A})} f(zh) \, dz \neq 0. \]

Now then, we will consider \( \Theta_{U,2}(\sigma) \). Let \( \langle *, * \rangle \) denote the Hermite form of \( W_B \). Notice that \( U_x \) stabilizes a pair \((y_1, y_2) \in W_B(K)^2\). Put \( Y = \left\{ \tfrac{y_1, y_1}{(y_1, y_1)}, \tfrac{y_2, y_2}{(y_2, y_2)} \right\} \), which is positive definite. Then, for a \( \varphi \in S(W_B(K)^2) \), the Fourier coefficient of \( \theta_{U,2}(\varphi, f)(g) \) at \( Y \)
\[ \int_{U_x(K) \setminus U_x(K_\mathbb{A})} r_{U,2}(g, h) \varphi(y_1, y_2) f(h) \, dh \]
\[ = \text{vol}(U_x(K) \setminus U_x(K_\mathbb{A}))^{-1} \int_{U_x(K_\mathbb{A}) \setminus U_x(K_\mathbb{A})} r_{U,2}(g, h) \varphi(y_1, y_2) \left( \int_{U_x(K) \setminus U_x(K_\mathbb{A})} f(zh) \, dz \right) \, dh, \]
where \( dh \) indicates the Haar measure of \( U_x(K) \setminus U_x(K_\mathbb{A}) \) associated to \( dh \). Since the integral in the parenthesis is nontrivial, it is possible to choose \( \varphi \) so that this value does not vanish at \( g = 1 \) (cf. concluding remarks in [28]). Hence the assertion. \( \square \)

Finally, we will show the last assertion of the theorem, observing the \( L \)-function \( L_{S_1}(s, \tau; \mathbb{A}^2) \) for an irreducible, noncuspidal, automorphic representation \( \tau \) of \( \text{GU}_{2,2}(K_\mathbb{A}) \). Let \( K^1 = \{ z \in K^x \mid N_{K/Q}(z) = 1 \} \). Let \( P_1(K) = N_1(K)M_1(K) \) with
\[
N_1(K) = \left\{ \begin{bmatrix} 1 & v & w \\ v & 1 & \bar{w} \\ w & \bar{v} & 1 \end{bmatrix} \left| \begin{array}{c} v \in \mathbb{Q}, \ u, w \in K \end{array} \right. \right\}, \\
M_1(K) = \left\{ \begin{bmatrix} tz & \bar{z}^\alpha \\ \bar{z}^\gamma & \bar{t}^{-1}zv(g_1) \end{bmatrix} \left| g_1 = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \text{GU}_{1,1}(K), \ z \in K^1, \ t \in \mathbb{Q}^x \right. \right\}.
\]

The modular character \( \delta_{P_1} \) of \( P_1(K_\mathbb{A}) \) is given by \( \delta_{P_1}(nm) = |v(g)|^{-4|t|^6} \). We embed \( \text{GU}_{1,1}(K) \times K^1 \times \mathbb{Q}^x \) into \( M_1(K) \), naturally. For a triple of irreducible automorphic representations \( \pi, \mu, \xi \) of \( \text{GU}_{1,1}(K_\mathbb{A}) \times K^1_{\mathbb{A}} \times \mathbb{A}^x \), let \( \pi \otimes \mu \otimes \xi \) denote the representation of \( P_1(K_\mathbb{A}) \) sending \( nm = n(g_1, z, t) \) to \( \pi(g_1) \mu(z) \xi(t) \). Hermitian modular forms of \( \text{SU}_{2,2}(K) \) are related to automorphic forms on \( \text{GU}_{2,2}(K_\mathbb{A}) \) with a manner similar to that in Section 2.1. We will identify them. A Hermitian modular form is noncuspidal, if and only if
\[
\Phi_U(F)(g, t, z; h) := \text{vol}(N_1(k) \setminus N_1(\mathbb{A}))^{-1} \int_{N_1(K) \setminus N_1(K_\mathbb{A})} F(n(g_1, t, z)h) \, dn.
\]
is not a zero function of \((g_1, t, z)\) at some \(h \in GU_2(K_\lambda)\), where \(\Phi_U\) is equal to the Siegel operator in [13], essentially. Hence, if a noncuspidal \(\tau\) is generated by a Hermitian modular form, then \(\tau\) is a constituent of an induced representation from \(\pi \otimes \mu \otimes \xi\). In this case, there is an automorphic form \(f \in \tau\), such that

\[
\Phi_U(f)(nmh) = \left|\psi(g_1)\right|^{-2} |t|^3 \pi(g_1) \mu(z) \xi(t) \Phi_U(f)(h).
\]

Further, if the central character of \(\pi_1\) is trivial, with regarding \(\pi_1\) as an irreducible automorphic representation of \(PGL_2(K_\lambda) \cong SO_{2,1}(K_\lambda) \simeq PGU_1,1(K_\lambda)\), we write

\[
L_S(s, \tau; \wedge^2) = L_S\left( s - \frac{1}{2}, \sigma_1 \right) L_S\left( s - \frac{1}{2}, \sigma_1, \xi \right) L_S(s, \mu).
\]  

(4.2)

Now, apply the above argument to our case. Since every automorphic form of \(\Theta_{U,2}(\sigma)\) is related to a Hermitian modular form of weight 4, the weight of \(\xi\) is 4 – 3 = 1, if \(\Theta_{U,2}(\sigma)\) is noncuspidal. Since the central character of \(\sigma\) is trivial, so is that of \(\Theta_{U,1}(\sigma)\). Then, obviously, (4.2) does not satisfy the Ramanujan conjecture. The last assertion of the theorem follows, immediately. This completes the proof.

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References