ON COMPLEXES EQUIVALENT TO $\mathbb{S}^3$-BUNDLES OVER $\mathbb{S}^4$

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Introduction

$\mathbb{S}^3$-bundles over $\mathbb{S}^4$ have played an important role in topology and geometry since Milnor showed that the total spaces of such bundles with Euler class ± 1 are manifolds homeomorphic to $\mathbb{S}^7$ but not always diffeomorphic to it. In 1974, Gromoll and Meyer exhibited one of these spheres (a generator in the group of homotopy 7-spheres) as a double coset manifold i.e. a quotient of $\mathrm{Sp}(2)$ hence showing that it admits a metric of nonnegative curvature (cf. [6]). Until recently, this was the only exotic sphere known to admit a metric of nonnegative sectional curvature. Then in [7], K. Grove and W. Ziller constructed metrics of nonnegative curvature on the total space of $\mathbb{S}^3$-bundles over $\mathbb{S}^4$. They also asked for a classification of these bundles up to homotopy equivalence, homeomorphism and diffeomorphism. These questions have been addressed in many papers such as [12], [11], [15] and more recently in [3]. In this paper we attempt to fill the gap in the previous papers; we consider the problem of determining when a given CW complex is homotopy equivalent to such a bundle. The problem was motivated by [7]: the Berger space, $\mathrm{Sp}(2)/\mathrm{Sp}(1)$, is a 7-manifold that has the cohomology ring of an $\mathbb{S}^3$-bundle over $\mathbb{S}^4$, but does it admit the structure of such a bundle? The fact that it cannot be a principal $\mathbb{S}^3$-bundle over $\mathbb{S}^4$ is straightforward and is proved in [7].

Let X be a simply connected CW complex with integral cohomology groups given by

$$\begin{aligned} H^i(X) &= \mathbb{Z} \quad \text{if } i = 0, 7 \\ &= \mathbb{Z}_n \quad \text{if } i = 4 \end{aligned} \tag{1}$$

where n is some fixed non-zero integer. We say that X is oriented with fundamental class $[X]$ if a generator $[X] \in H_7(X)$ is specified. For oriented X we define the *linking form* as,

$$\begin{aligned} b: & H^4(X) \otimes H^4(X) \to \mathbb{Z}_n \\ & x \otimes y \longmapsto \langle \beta^{-1}(x), y \cap [X] \rangle \end{aligned}$$

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where $\beta : H^3(X, \mathbb{Z}_n) \to H^4(X)$ is the Bockstein isomorphism. The homomorphism $\cap[X] : H^4(X) \to H_3(X)$ is obtained by capping with the fundamental class and $\langle , \rangle : H^3(X, \mathbb{Z}_n) \otimes H_3(X) \to \mathbb{Z}_n$ is the Kronecker pairing. We now state our main theorem:

Theorem 1. *Let X be a simply connected CW complex as above. Then X is homotopy equivalent to an $\mathbb{S}^3$-bundle over $\mathbb{S}^4$ if and only if the following two conditions hold:*

(I) *The secondary cohomology operation Θ is trivial, where*

$$\Theta : H^4(X, \mathbb{F}_2) \to H^7(X, \mathbb{F}_2)$$

corresponds to the relation $\mathrm{Sq}^2\,\mathrm{Sq}^2 = \mathrm{Sq}^3\,\mathrm{Sq}^1$ in the mod 2 Steenrod algebra.

(II) *The linking form $b : H^4(X) \otimes H^4(X) \to \mathbb{Z}_n$ is equivalent to a standard form for some choice of orientation on X i.e. there exists an isomorphism $\psi : \mathbb{Z}_n \to H^4(X)$ such that $b(\psi(x), \psi(y)) = xy$.*

Using the method outlined on page 32 of [8], it is easy to show that if X smoothable, then Θ is trivial and we have

Corollary 2. *Let M be a simply connected 7-manifold with integral cohomology groups given by*

$$\begin{aligned} H^i(M) &= \mathbb{Z} \quad \text{if } i = 0, 7 \\ &= \mathbb{Z}_n \quad \text{if } i = 4. \end{aligned}$$

Then M is homotopy equivalent to an $\mathbb{S}^3$-bundle over $\mathbb{S}^4$ if and only if its linking form is equivalent to a standard form for some choice of orientation on M.

The previous corollary can be strengthened considerably using the methods of C. T. C. Wall (cf. [14]); see also [3]. From the discussion in Section 5 we have

Theorem 3. *Let M be a simply connected manifold with integral cohomology as in (1). Then M is PL-homeomorphic to an $\mathbb{S}^3$-bundle over $\mathbb{S}^4$ if and only if its linking form is equivalent to a standard form.*

A word or two must be said about the preceding theorem. Our original result only covered the case when $n > 2$ was twice an odd number. Shortly after our original preprint was circulated, D. Crowley and C. Escher extended the result showing it was true for all n (cf. [3]) by using the Eells–Kuiper invariant (cf. [5]). The proof of our original result as well as the subsequent generalization by Crowley and Escher both rely heavily on the results stated in [15]. In that paper, [15], D. Wilkens classified simply connected manifolds with integral cohomology as above up to the addition of homotopy 7-spheres (and hence up to PL-homeomorphism type). However, the proofs in [15] are incomplete and although Wilkens apparently proves these results in his thesis, the proofs have never been published. For this reason, we prefer to provide an alternate argument which is similar in spirit to the approach of Wilkens, but one that is ultimately independent of his paper. The proof of Theorem 3 in Section 5 is due to Matthias Kreck and we are grateful to him for sharing it with us.

The seven dimensional Berger manifold is a curious space. It is described as the homogeneous space, $M = \mathrm{Sp}(2)/\mathrm{Sp}(1) = \mathrm{SO}(5)/\mathrm{SO}(3)$, where the embedding of $\mathrm{Sp}(1)$ in $\mathrm{Sp}(2)$ is maximal. It is an isotropy irreducible space and has the cohomology ring as in (1) with $H^4(M) = \mathbb{Z}_{10}$ (see Section 4). It admits a normal homogeneous metric of positive sectional curvature (cf. [2]). In [7] the following question was asked:

Question. Does the Berger space, $M = \mathrm{Sp}(2)/\mathrm{Sp}(1)$, admit the structure of an $\mathbb{S}^3$-bundle over $\mathbb{S}^4$?

In Section 4 it is shown that the linking form for the Berger space is equivalent to a standard form. By applying Theorem 3 we get

Corollary 4. *The Berger space, $M = \mathrm{Sp}(2)/\mathrm{Sp}(1)$, is PL-homeomorphic to an $\mathbb{S}^3$-bundle over $\mathbb{S}^4$.*

It remains open whether the Berger space is in fact diffeomorphic to such a bundle. This involves computing the Eells-Kuiper invariant, μ, for this manifold (cf. [5]). The μ invariant for a 7-manifold is computed by exhibiting the space as a spin boundary; we are unable to do this for the Berger space.

Another application of Theorem 1 is the case when $n = |H^4(X)| = p^m$ where p is a prime of the form $p = 4k + 3$. Since -1 is not a square in the ring $\mathbb{Z}_{p^m}$, any non-degenerate form, $\alpha : \mathbb{Z}_{p^m} \otimes \mathbb{Z}_{p^m} \to \mathbb{Z}_{p^m}$, is equivalent to a standard form (up to sign). This shows

Theorem 5. *Let X be a Poincaré duality complex with integral cohomology as in (1) and $H^4(X) = \mathbb{Z}_{p^m}$ where p is a prime of the form $p = 4k + 3$. Then X is homotopy equivalent to an $\mathbb{S}^3$-bundle over $\mathbb{S}^4$.*

The proof of Theorem 1 is organized as follows: In Section 1 we show that the conditions (I) and (II) are necessary for X to support the structure of an $\mathbb{S}^3$ fibration over $\mathbb{S}^4$. In Section 2 we establish sufficiency of the conditions and exhibit X as the total space of an $\mathbb{S}^3$ fibration over $\mathbb{S}^4$. In Section 3 we prove that any $\mathbb{S}^3$ fibration over $\mathbb{S}^4$ is equivalent to a linear $\mathbb{S}^3$-bundle over $\mathbb{S}^4$. In Section 4 we calculate the cohomology of the Berger space and show that its linking form is equivalent to a standard form. Finally in Section 5 we present the revised proof of Theorem 3.

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1. Necessity of conditions in Theorem 1

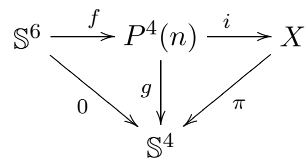
Let X be a simply connected CW complex with integral cohomology as in (1). It follows that the 4-skeleton of X can be chosen to be equivalent to the space $P^4(n)$ defined as the cofiber of the self map of degree n on $\mathbb{S}^3$. The space X is then equivalent to the cofiber of some map f given by,

$$\mathbb{S}^6 \xrightarrow{f} P^4(n) \longrightarrow X$$

Assume now that X supports the structure of the total space in a fibration:

$$\mathbb{S}^3 \longrightarrow X \longrightarrow \mathbb{S}^4 \tag{2}$$

Consider the commutative diagram:



where $\pi : X \to \mathbb{S}^4$ is the projection map and $g = \pi \circ i$. An easy argument using the Serre spectral sequence for the fibration (2) shows that $g^* : H^4(\mathbb{S}^4) \to H^4(P^4(n))$ is an epimorphism. From the theory of secondary cohomology operations (cf. [10]) it follows that the secondary operation Θ is trivial where,

$$\Theta : H^4(X, \mathbb{F}_2) \to H^7(X, \mathbb{F}_2)$$

corresponds to the relation $\mathrm{Sq}^2\,\mathrm{Sq}^2 = \mathrm{Sq}^3\,\mathrm{Sq}^1$ in the mod 2 Steenrod algebra.

It remains to check condition (II) in the theorem. Consider the Serre spectral sequence for the fibration (2) converging to $H^*(X)$. We have:

$$\begin{aligned} E_2^{p,q} = E_4^{p,q} &= H^p(\mathbb{S}^4) \otimes H^q(\mathbb{S}^3) \\ d_4(y_3) &= n y_4 \end{aligned} \tag{3}$$

where y_3 and y_4 are suitably chosen generators of $H^3(\mathbb{S}^3)$ and $H^4(\mathbb{S}^4)$ respectively. Similarly, for the spectral sequence in $\mathbb{Z}_n$-coefficients converging to $H^*(X, \mathbb{Z}_n)$, we have:

$$E_2^{p,q} = E_\infty^{p,q} = H^p(\mathbb{S}^4, \mathbb{Z}_n) \otimes H^q(\mathbb{S}^3, \mathbb{Z}_n)$$

Note that in both spectral sequences there are no extension problems, hence we may identify $H^*(X)$ or $H^*(X, \mathbb{Z}_n)$ with the fifth stage in the respective spectral sequences.

Let $[X] \in H_7(X)$ be the orientation class defined by

$$\langle y_4 \otimes y_3, [X] \rangle = 1 \tag{4}$$

It follows from (3) that the Bockstein isomorphism, $\beta : H^3(X, \mathbb{Z}_n) \to H^4(X)$ is given by

$$\beta([y_3]) = y_4 \tag{5}$$

where we henceforth use the notation $[y]$ to denote the $\bmod (n)$ reduction of an integral class y. Using (4) and (5), we get

$$\langle \beta^{-1}(y_4), y_4 \cap [X] \rangle = \langle [y_3], y_4 \cap [X] \rangle = \langle [y_3] \cup [y_4], [X] \rangle \equiv 1 \mod (n)$$

which is simply the statement that the linking form is equivalent to a standard form.

2. Construction of a $\mathbb{S}^3$ fibration

The purpose of this section is to show that any CW complex X satisfying conditions (I) and (II) of Theorem 1 is equivalent to the total space of a fibration with base $\mathbb{S}^4$ and fiber equivalent to $\mathbb{S}^3$.

Given such a complex, recall that X fits into a cofiber sequence

$$\mathbb{S}^6 \xrightarrow{f} P^4(n) \longrightarrow X$$

Let $p : P^4(n) \to \mathbb{S}^4$ be any map inducing an epimorphism in cohomology. Such a map always exists since $P^4(n)$ is a four dimensional complex. Condition (I) ensures that the composite, $p \circ f$, is null homotopic. Thus we get an extension $\tilde{\pi}$ making the following diagram commute.

$$\begin{array}{ccccc} \mathbb{S}^6 & \xrightarrow{f} & P^4(n) & \longrightarrow & X \\ & & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle \tilde{\pi}} & \\ & & \mathbb{S}^4 & & \end{array}$$

We now have the following map of cofibrations:

$$\begin{array}{ccccc} P^4(n) & \longrightarrow & X & \longrightarrow & \mathbb{S}^7 \\ \| & & \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle f(\tilde{\pi})} \\ P^4(n) & \xrightarrow{p} & \mathbb{S}^4 & \xrightarrow{[n]} & \mathbb{S}^4 \end{array} \tag{6}$$

where the map from X to $\mathbb{S}^7$ has degree 1 and $[n]$ denotes the self map of degree n. The choice of the extension $f(\tilde{\pi})$ of the map $\tilde{\pi}$ can be made so that it depends canonically on the homotopy class of $\tilde{\pi}$. The extension $\tilde{\pi}$ is not unique; the set of extensions admits a transitive action of the group $\pi_7(\mathbb{S}^4)$, which we describe below. Note that the further extension of $\tilde{\pi}$ to $f(\tilde{\pi})$ depends only on the homotopy class of $\tilde{\pi}$. Given $\alpha \in \pi_7(\mathbb{S}^4)$ we define $\alpha\tilde{\pi}$ by

$$\begin{array}{ccc} X & \xrightarrow{\text{pinch}} & \mathbb{S}^7 \vee X \\ \downarrow{\scriptstyle \alpha\tilde{\pi}} & \swarrow{\scriptstyle \alpha \vee \tilde{\pi}} & \\ \mathbb{S}^4 & & \end{array}$$

In terms of the diagram (6) it is not hard to verify that

$$\mathrm{H}(f(\alpha\tilde{\pi})) = n^2 \cdot \mathrm{H}(\alpha) + \mathrm{H}(f(\tilde{\pi})) \tag{7}$$

where $\mathrm{H}(g)$ denotes the Hopf invariant of the map g.

Let F be the homotopy fiber of $\tilde{\pi}$. We will calculate the cohomology of F using the Serre spectral sequence for the fibration,

$$\Omega\mathbb{S}^4 \to F \to X \tag{8}$$

Recall that $H^*(\Omega\mathbb{S}^4) = \langle z_{3k}, k = 0, 1, 2, \dots \rangle$. Consider the diagram of fibrations:

$$\begin{array}{ccc} \Omega\mathbb{S}^4 & = & \Omega\mathbb{S}^4 \\ \downarrow & & \downarrow \\ F & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \xrightarrow{\tilde{\pi}} & \mathbb{S}^4 \end{array} \tag{9}$$

The next proposition is an easy consequence of the naturality of the Serre spectral sequence with respect to maps of fibrations.

Proposition 2.1. *In the Serre spectral sequence for (7), we have*

$$d_4(z_{3k}) = y_4 \otimes z_{3k-3}$$

where $y_4 \in H^4(X)$ is a generator.

It follows that the classes, $nz_{3k} \in E_4^{0,3k}$, survive to the next stage. It remains to calculate $d_7(nz_{3k})$. Let G be the homotopy fiber of $f(\tilde{\pi})$. Using (6) we get a diagram of fibrations:

$$\begin{array}{ccc} \Omega\mathbb{S}^4 & \xrightarrow{\Omega[n]} & \Omega\mathbb{S}^4 \\ \downarrow & & \downarrow \\ F & \longrightarrow & G \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{S}^7 \end{array} \tag{10}$$

Notice that in the Serre spectral sequence for $\Omega\mathbb{S}^4 \to G \to \mathbb{S}^7$, we have the identity $d_7(z_6) = \mathrm{H}(f(\tilde{\pi})) \cdot y_7$ where $y_7 \in H^7(\mathbb{S}^7)$ is a generator. Moreover since $(\Omega[n])^*(z_6) = n^2 z_6$, using (10) we have,

Proposition 2.2. *The Hopf invariant, $\mathrm{H}(f(\tilde{\pi}))$, is a multiple of n i.e. $\mathrm{H}(f(\tilde{\pi})) = \lambda n$, and in the Serre spectral sequence for (8), we have*

$$d_7(nz_{3k}) = \lambda y_7 \otimes z_{3k-6}$$

From Propositions 2.1 and 2.2, we deduce that

$$\begin{aligned} H^i(F) &= \mathbb{Z} \cdot y_3 \quad i = 3, \quad y_3 = nz_3 \\ &= \mathbb{Z}_\lambda \cdot y_{7+3k} \quad i = 7 + 3k, \quad k = 0, 1, 2, \dots \end{aligned}$$

Using the universal coefficients theorem, the homology is:

$$\begin{aligned} H_i(F) &= \mathbb{Z}\cdot x_3 \quad i=3,\\ &= \mathbb{Z}_\lambda \cdot x_{6+3k} \quad i = 6+3k, \quad k=0,1,2,\dots \end{aligned}$$

where x_3 and y_3 are dual to each other. Now considering the Serre spectral sequence for the fibration, $F \to X \to \mathbb{S}^4$ converging to $H_*(X)$ we have:

$$\begin{aligned} E^2_{p,q} = E^4_{p,q} &= H_p(\mathbb{S}^4)\otimes H_q(F)\\ d_4(x_4) &= nx_3 \end{aligned}$$

where $x_4 \in H_4(\mathbb{S}^4)$ is a suitably chosen generator. Since $H_6(X) = 0$, the map, $d_4 : E^4_{4,3} \to E^4_{0,6} = \mathbb{Z}_\lambda$ must be an epimorphism. Hence the class $\lambda x_4 \otimes x_3 \in E^\infty_{4,3}$ represents an orientation $[X] \in H_7(X)$.

In the dual picture for the cohomology Serre spectral sequence converging to $H^*(X)$, we have:

$$\begin{aligned} E^{p,q}_2 = E^{p,q}_4 &= H^p(\mathbb{S}^4)\otimes H^q(F)\\ d_4(y_3) &= ny_4 \end{aligned} \tag{11}$$

where $y_4 \in H^4(\mathbb{S}^4)$ is the class dual to x_4. The classes y_3 and y_4 are permanent cycles in the spectral sequence converging to $H^*(X, \mathbb{Z}_n)$.

From the definition of $[X]$ we have,

$$\langle [y_3], y_4 \cap [X]\rangle = \langle [y_3]\cup[y_4], [X]\rangle \equiv \lambda \mod (n) \tag{12}$$

As in the previous section, (11) and (12) imply

$$\langle \beta^{-1}(y_4), y_4 \cap [X]\rangle \equiv \lambda \mod (n)$$

Since the linking form is assumed to be equivalent to a standard form, it follows that $\lambda \equiv \pm\tau^2 \mod (n)$ for some $\tau \in (\mathbb{Z}_n)^*$. Let m be an integer so that $m \equiv \tau^{-1} \mod (n)$. Define $\pi : X \to \mathbb{S}^4$ as the composite, $\pi = [m]\circ\tilde{\pi}$. We now have a commutative diagram:

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{S}^7 \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow {\scriptstyle f(\pi)=[m]\,\circ\, f(\tilde{\pi})} \\ \mathbb{S}^4 & \xrightarrow{[n]} & \mathbb{S}^4 \end{array}$$

Note that $\mathrm{H}(f(\pi)) \equiv \pm n \mod (n^2)$. Using (7) we may further assume $\mathrm{H}(f(\pi)) = \pm n$. It follows now from Proposition 2.2 that the homotopy fiber of π is equivalent to $\mathbb{S}^3$ since it is a simply connected homology 3-sphere. We have therefore succeeded in writing X as the total space of an $\mathbb{S}^3$ fibration over $\mathbb{S}^4$ as required.

3. From fibrations to bundles

In this section we show that any $\mathbb{S}^3$ fibration over $\mathbb{S}^4$ is equivalent to an $\mathbb{S}^3$-bundle over $\mathbb{S}^4$. The argument is fairly standard, but we outline it for completeness.

Let ξ be an $\mathbb{S}^3$ fibration over $\mathbb{S}^4$. Restricting ξ to the hemispheres, D_+ and D_-, we get trivial fibrations, ξ_+ and ξ_-, respectively. The fibration ξ may then be obtained by gluing ξ_+ and ξ_- along their common boundary, $\mathbb{S}^3$, by a map lying in a homotopy class,

$$\mathbb{S}^3 \to SG(3)$$

where $SG(3)$ denotes the monoid of orientation preserving self maps of $\mathbb{S}^3$.

Recall that (linear) $\mathbb{S}^3$ bundles are classified by homotopy classes of maps, $\mathbb{S}^3 \to \mathrm{SO}(4)$. The classical J homomorphism identifies $\mathrm{SO}(4)$ with the submonoid of $SG(3)$ of linear actions. Therefore to show that any $\mathbb{S}^3$ fibration over $\mathbb{S}^4$ is equivalent to an $\mathbb{S}^3$-bundle over $\mathbb{S}^4$, it suffices to show that the map,

$$J_* : \pi_3(\mathrm{SO}(4)) \to \pi_3(SG(3))$$

is an epimorphism.

For a fixed basepoint of $\mathbb{S}^3$, one has an evaluation map that evaluates the effect of a self map of $\mathbb{S}^3$ on the basepoint. It is easy to see that this map has a section and hence it follows that we have a map of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_3(\mathrm{SO}(3)) & \longrightarrow & \pi_3(\mathrm{SO}(4)) & \xrightarrow{\mathrm{ev}_*} & \pi_3(\mathbb{S}^3) & \longrightarrow & 0 \\
 & & \big\downarrow J_* & & \big\downarrow J_* & & \big\| & & \\
0 & \longrightarrow & \pi_3(SG_*(3)) & \longrightarrow & \pi_3(SG(3)) & \xrightarrow{\mathrm{ev}_*} & \pi_3(\mathbb{S}^3) & \longrightarrow & 0
\end{array}$$

where $SG_*(3)$ are basepoint preserving elements of $SG(3)$.

It suffices to show that $J_* : \pi_3(\mathrm{SO}(3)) \to \pi_3(SG_*(3))$ is an epimorphism. Then the result will follow by the 5-lemma. Consider the stabilization of J:

$$\begin{array}{ccc}
\pi_3(\mathrm{SO}(3)) & \xrightarrow{J_*} & \pi_3(SG_*(3)) \\
{\times 2}\big\downarrow & & \big\downarrow{\times 2} \\
\pi_3(\mathrm{SO}) & \xrightarrow{J^s_*} & \pi^s_3(S^0)
\end{array}$$

It is well known that $\pi_3(SO(3)) = \mathbb{Z} = \pi_3(SO)$. Furthermore, it is also known that $\pi_3(SG_*(3)) = \mathbb{Z}_{12}$ and $\pi^s_3(S^0) = \mathbb{Z}_{24}$. By [1], J^s_* is an epimorphism and hence J_* is an epimorphism as well. This completes the proof of Theorem 1.

4. The Berger space

We briefly recall the construction of the Berger space. Consider $\mathbb{R}^5$ represented as the space of 3×3 traceless, symmetric matrices. Then the conjugation action of $\mathrm{SO}(3)$ on this space affords a (maximal) representation into SO(5). The quotient space, $M^7 = \mathrm{SO}(5)/\mathrm{SO}(3)$ may also be written as $\mathrm{Sp}(2)/\mathrm{Sp}(1)$ for a maximal embedding of Sp(1) into Sp(2). Berger showed in [2] that this space admits a normal homogeneous metric of positive sectional curvature. In [7] it was shown that this space cannot be a principal $\mathbb{S}^3$-bundle over $\mathbb{S}^4$. We shall address the question of whether it is equivalent to an $\mathbb{S}^3$ fiber bundle over $\mathbb{S}^4$.

The cohomology of this space is well known. However, we outline the calculation as we will need the setup to compute the linking form. In terms of the standard maximal tori we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Sp}(1) & \xrightarrow{\psi} & \mathrm{Sp}(2) \\ \uparrow & & \uparrow \\ S^1 & \xrightarrow{\psi_{1,3}} & S^1 \times S^1 \end{array} \tag{13}$$

where $\psi_{1,m}(z) = (z, z^m)$.

Let $B\psi : B_{\mathrm{Sp}(1)} \to B_{\mathrm{Sp}(2)}$ be the map on the level of classifying spaces. It follows from (13) that in cohomology we have:

$$\begin{aligned} B\psi^*(p_1) &= 10p_1 \\ B\psi^*(p_2) &= 9p_1^2 \end{aligned} \tag{14}$$

where $H^*(B_{\mathrm{Sp}(2)}) = \mathbb{Z}[p_1, p_2]$ and $H^*(B_{\mathrm{Sp}(1)}) = \mathbb{Z}[p_1]$.

The homogeneous space, $M = \mathrm{Sp}(2)/\psi(\mathrm{Sp}(1))$ is the concrete description of the Berger space. To calculate its cohomology, consider the fibration

$$\mathrm{Sp}(2) \to M \to B_{\mathrm{Sp}(1)} \tag{15}$$

We have a pullback diagram:

$$\begin{array}{ccc} \mathrm{Sp}(2) & = & \mathrm{Sp}(2) \\ \downarrow & & \downarrow \\ M & \longrightarrow & E_{\mathrm{Sp}(2)} \\ \downarrow & & \downarrow \\ B_{\mathrm{Sp}(1)} & \xrightarrow{B\psi} & B_{\mathrm{Sp}(2)} \end{array}$$

Recall that $H^*(\mathrm{Sp}(2)) = \mathrm{E}(y_3, y_7)$, where y_3 and y_7 transgress to p_1 and p_2 respectively in the Serre spectral sequence for the universal fibration. Using (14) and the pullback diagram above, we have:

Proposition 4.1. *In the Serre spectral sequence for (15) converging to $H^*(M)$, we have,*

$$\begin{aligned} d_4(y_3) &= 10p_1 \\ d_8(y_7) &= 9p_1^2 \end{aligned}$$

It follows immediately from Proposition 4.1 that

$$\begin{aligned} H^i(M) &= \mathbb{Z} \quad i = 0, 7 \\ &= \mathbb{Z}_{10} \quad i = 4 \end{aligned} \tag{16}$$

We would like to know whether M is homotopy equivalent to an $\mathbb{S}^3$-bundle over $\mathbb{S}^4$. By Corollary 2, it suffices to compute the linking form for M.

Let $S_{1,3} \subset \mathrm{Sp}(2)$ denote the ψ-image of the standard maximal torus in $\mathrm{Sp}(1)$. We then have a fibration:

$$\mathbb{S}^2 \to \mathrm{Sp}(2)/S_{1,3} \to M \tag{17}$$

An easy spectral sequence argument then yields

Proposition 4.2. *$H^*(M)$ maps isomorphically onto $H^*(\mathrm{Sp}(2)/S_{1,3})$ in degrees 0,4 and 7. The corresponding maps in homology are isomorphisms as well in degrees 0,3 and 7.*

Fix an orientation $[M] \in H_7(M)$. We identify $[M]$ with a class $[M]$ in $H_7(\mathrm{Sp}(2)/S_{1,3})$ using Proposition 4.2. It is clear that the linking form on M is equivalent to the form,

$$\begin{aligned} \alpha : & H^4(\mathrm{Sp}(2)/S_{1,3}) \otimes H^4(\mathrm{Sp}(2)/S_{1,3}) \to \mathbb{Z}_{10} \\ & x \otimes y \longmapsto \langle \beta^{-1}(x), y \cap [M] \rangle \end{aligned}$$

It will be easier to calculate α on $\mathrm{Sp}(2)/S_{1,3}$ than the linking form on M.

Let $\Delta = \mathrm{Sp}(1) \times \mathrm{Sp}(1) \subset \mathrm{Sp}(2)$, be the standard diagonal embedding of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$. Consider the fibration:

$$\Delta/S_{1,3} \to \mathrm{Sp}(2)/S_{1,3} \to \mathrm{Sp}(2)/\Delta \tag{18}$$

The homogeneous spaces $\Delta/S_{1,3}$ and $\mathrm{Sp}(2)/\Delta$ can be identified with the spaces $\mathbb{S}^2 \times \mathbb{S}^3$ and $\mathbb{S}^4$ respectively. Hence their cohomologies have the structure of exterior algebras:

$$H^*(\Delta/S_{1,3}) = \mathrm{E}(y_2, y_3), \qquad H^*(\mathrm{Sp}(2)/\Delta) = \mathrm{E}(y_4),$$

where y_4 is chosen so that in the Serre spectral sequence for (18), we have

$$d_4(y_3) = 10 y_4 \tag{19}$$

From Proposition 4.2, this is the only non-trivial differential. Since there are no extension problems, we identify $H^*(\mathrm{Sp}(2)/S_{1,3})$ with the E_∞ term. We can do the same for the Serre spectral sequence in $\mathbb{Z}_{10}$ coefficients, converging to $H^*(\mathrm{Sp}(2)/S_{1,3}, \mathbb{Z}_{10})$. Note that $y_4 \otimes y_3 \in H^7(\mathrm{Sp}(2)/S_{1,3})$ is a generator and hence,

$$\langle [y_3], y_4 \cap [M] \rangle = \langle [y_3] \cup [y_4], [M] \rangle = \langle [y_3 \otimes y_4], [M] \rangle \equiv \pm 1 \mod (10) \tag{20}$$

As in Section 1, we deduce from (19) and (20) that $\alpha(y_4, y_4) \equiv \pm 1 \mod (10)$. So if $[M]$ is chosen suitably, then the linking form for the Berger space is equivalent to a standard form.

5. Proof of Theorem 3

Given a simply connected 7-manifold, M, with integral cohomology as in (1), M can be written as a boundary of an 8-dimensional spin manifold W. Using standard techniques from surgery theory for killing low dimensional homotopy groups (cf. [9]), we may assume that W is 3-connected. In the proof of Theorem 1 in [13], Wall gives an argument for the following statement. Let W be a 3-connected, compact, smooth 8-manifold whose boundary is 2-connected. For a given basis of $H_4(W)$ there is a handlebody decomposition of W as the 8-disk D^8 with only 4-handles attached, where the homology classes represented by the 4-handles correspond to the chosen basis elements.

For an abstract handlebody N, Wall constructs invariants as follows. Notice that $H_4(N)$ be a free abelian group of rank r where r is the number of handles of N. Since N is 3-connected, we can identify $\pi_4(N)$ with $H_4(N)$ by the Hurewicz theorem. Any element of $\pi_4(N)$ can be represented by an imbedding that is unique up to diffeotopy. The normal bundle of this imbedding is classified by an element of $\pi_3(\mathrm{SO}(4))$. Thus we get a map $\alpha : H_4(N) \to \pi_3(\mathrm{SO}(4))$. Wall shows in [13] that the intersection form on $H_4(N)$ and the map α form a complete set of invariants for the handlebody N. Moreover, these invariants can take any values that satisfy the relations given by Lemma 2 in [13]. It follows from the above classification that a handlebody with just one 4-handle is simply a disk bundle over $\mathbb{S}^4$ corresponding to the value of the invariant α on some generator of $H_4(N)$.

With these results in mind we will prove Theorem 3 by showing that up to connected sum with a homotopy 7-sphere, M is the boundary of a handlebody with a single 4-handle.

Proof of Theorem 3. Consider a piece of the long exact sequence of the pair (W, M).

$$0 \longrightarrow H_4(W) \xrightarrow{\lambda} H_4(W, M) \xrightarrow{\partial} H_3(M) \longrightarrow 0$$

Notice that $H_4(W, M)$ can be identified with $\mathrm{Hom}(H_4(W), \mathbb{Z})$ by Poincaré Duality and the Universal Coefficients theorem. Under this identification, the map λ corresponds to the intersection form on W. The form λ in turn induces a form b on the cokernel $H_3(M)$ via the boundary map ∂ with values in $\mathbb{Q}/\mathbb{Z}$ defined as

$$b(\partial(x), \partial(y)) = x(\lambda_{\mathbb{Q}}^{-1}(y)) \mod \mathbb{Z},$$

where $\lambda_{\mathbb{Q}}$ is the extension of λ to $H_4(W, \mathbb{Q})$. It is easy to see that the form b induced on $H_3(M)$ is well defined and corresponds to the linking form on M by Poincaré Duality.

By assumption the linking form b on M is standard. This implies, from [14] and [4], that the form λ is stably equivalent under orthogonal direct sum to the form,

$$\lambda_n : \mathbb{Z} \longrightarrow \mathbb{Z}^*$$
$$1 \longmapsto \lambda_n(1) : m \mapsto mn$$

namely, there exist unimodular forms f_1 and f_2 such that

$$\lambda \oplus f_1 = \lambda_n \oplus f_2 \tag{21}$$

We note that any unimodular form can be realized as the intersection form of a 3-connected 8-manifold with boundary a homotopy 7-sphere (see for example [13]). Conversly, a handlebody with a unimodular intersection form bounds a homotopy 7-sphere.

Let F_1 be a 3-connected 8-manifold realizing f_1 and we realize $\lambda \oplus f_1$ as the intersection form of $W \natural F_1$ (here $\natural$ denotes boundary connected sum). We now apply the result mentioned at the beginning of this section by choosing bases for λ_n and f_2 and realizing their union as a handlebody decompositon of $W \natural F_1$. Since the basis element of λ_n is orthogonal to the basis elements of f_2, it follows from [13] that this handlebody splits as $W' \natural F_2$ where the intersection form of W' is λ_n and that of F_2 is f_2. Note that W' is a handlebody with a single 4-handle. Restricting our attention to the boundary we get the equality

$$M \# \partial F_1 = \partial W' \# \partial F_2$$

which completes the proof. □

Remark. It seems reasonable to expect that the methods outlined here can be used to prove the analogs of Theorem 1 and Theorem 3 in higher dimensions.

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