RANK TWO FUNDAMENTAL GROUPS OF POSITIVELY CURVED MANIFOLDS

KARSTEN GROVE* AND KRISHNAN SHANKAR

Abstract. This paper deals with the construction of previously unknown fundamental groups for positively curved manifolds.

Introduction

It is well known that any finite group is the fundamental group of some non-negatively curved manifold. The only proposed obstruction in positive curvature goes back to S. S. Chern (cf. [Ch, p. 167]): is every abelian subgroup of the fundamental group cyclic? This was recently answered in the negative in [Sh] by observing that there are positively curved manifolds that admit a free, isometric SO(3) action. In particular, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \subset \mathrm{SO}(3)$ is the fundamental group of some positively curved manifold. However, the approach of [Sh] fails for $\mathbb{Z}_p \oplus \mathbb{Z}_p$ where p is an odd prime. Nevertheless, we will show that the obstruction proposed by Chern is false for groups of odd order as well, by establishing

Main Theorem. *The Aloff-Wallach space $N_{k,l} = \mathrm{SU}(3)/S^1_{k,l}$ admits a free, isometric $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ action if and only if $H^4(N_{k,l}, \mathbb{Z}) \cong \mathbb{Z}/(k^2 + l^2 + kl)$ has 3-torsion.*

It is a pleasure to thank Stephan Stolz for leading us to [Bo] in our search for non-toral elementary abelian p-groups.

1. Torus actions on manifolds of positive curvature

Let M^n be a manifold of positive sectional curvature. If M is even dimensional then $\pi_1(M)$ is 0 or $\mathbb{Z}_2$ according as M is orientable or not (Synge's theorem). So we only concern ourselves with quotients of odd dimensional manifolds.

To fix notation, recall that if a group G acts on a manifold M then the *isotropy group* of a point $x \in M$ is $G_x := \{g \in G : g \cdot x = x\} \subset G$. If $G_x = G$, then x is said to be a fixed point of G. If $G_x = \{1\}$ for all x, then G is said to act freely. The following lemma is due to Berger (cf. [Be]).

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Lemma 1.1. *Any circle acting isometrically on an even dimensional manifold of positive curvature has a fixed point.*

The analogous statement in odd dimensions can be found in [Su] (cf. [Ro]).

Lemma 1.2. *If a torus* $\mathrm{T}^2 = S^1 \times S^1$ *acts isometrically on a manifold* M^{2n+1} *of positive curvature, then there exists* $x \in M$ *for which* T^2_x *contains a circle.*

The lemmas imply, in particular, that a connected Lie group G acting freely and isometrically on a manifold of positive curvature must have $\mathrm{rank}(G) \leq 1$. However, we are concerned with free actions of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ or more generally rank 2 groups i.e. finite groups for which the maximal rank of any elementary abelian p-subgroup is 2, where p is any prime. The following proposition cuts down our search considerably. Let $\mathbf{I}(M)$ denote the isometry group of the Riemannian manifold M.

Proposition 1.3. *Let* M *be a manifold of positive curvature. If* $\mathbb{Z}_p \oplus \mathbb{Z}_p$ *is contained in a torus of* $\mathbf{I}(M)$*, then it cannot act freely on* M*.*

Proof: Without loss of generality, we may assume that $\mathbb{Z}_p \oplus \mathbb{Z}_p$ lies in some 2-torus T. We also assume that M is odd dimensional because of Synge's theorem. By the previous lemma, there exists $x \in M$ such that T_x contains a circle and the orbit $\mathrm{T}(x)$ must be a circle or a fixed point. Then $\mathbb{Z}_p \oplus \mathbb{Z}_p$ acts freely on the orbit $\mathrm{T}(x)$ which is a contradiction. □

Note that the considerations above show the following: if a compact group $G \subset \mathbf{I}(M)$ acts freely on a positively curved manifold M, then its intersection with any torus of $\mathbf{I}(M)$ must be a cyclic group or a circle.

2. Free actions of elementary abelian 3-groups

Our search for free actions of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ now narrows down to subgroups of the isometry group that do not lie in tori. Let G be a compact, connected Lie group. The following theorem was proved in [Bo].

Theorem 2.1 (Borel)**.** *Let* p *be a prime number. Then every subgroup of* G *isomorphic to* $\mathbb{Z}_p \oplus \mathbb{Z}_p$ *is contained in a torus if and only if* $\pi_1(G)$ *does not have* p*-torsion.*

The Aloff-Wallach spaces are the homogeneous spaces $N_{k,l} := \mathrm{SU}(3)/S^1_{k,l}$, where $S^1_{k,l} = \{\mathrm{diag}(z^k, z^l, \bar{z}^{k+l}) : z \in \mathrm{U}(1), \gcd(k,l) = 1\} \subset \mathrm{SU}(3)$ (cf. [AW]). If $kl(k+l) \neq 0$, then the quotient space admits a homogeneous metric of positive curvature. The space $N_{1,1}$ is the only one that admits a normal homogeneous metric of positive curvature (cf. [Wi]). However, the action of SU(3) is not always effective. Let ω be a primitive third root of unity. Then the matrix $\mathrm{diag}(\omega, \omega, \omega)$, which generates the center of SU(3), lies in $S^1_{k,l}$ precisely when $3 \nmid kl(k+l)$. This can be seen as follows: let $z^k = z^l = \bar{z}^{k+l} = \omega$. Since $\gcd(k,l) = 1$, there exist integers a and b such that $ak+bl=1$. Then, $z = z^{ak+bl} = \omega^{a+b}$ and z is 1, ω or ω^2. If $3 \mid kl(k+l)$, then $z^k = z^l = \bar{z}^{k+l} = 1$ which is a contradiction. If $3 \nmid kl(k+l)$, then k, l and $-k-l$ are all congruent to ϵ mod (3), where ϵ is 1 or 2; take $z = \omega^\epsilon$ to see that $Z = Z(\mathrm{SU}(3)) \subset S^1_{k,l}$. We have proved

Proposition 2.2. SU(3) *acts ineffectively on the Aloff-Wallach space* $N_{k,l}$ *if and only if* $3 \nmid kl(k+l)$.

When $3 \nmid kl(k+l)$, the effective group that acts is $\mathrm{P\,SU}(3) = \mathrm{SU}(3)/Z$. In this case, the effective representation of the homogeneous space $N_{k,l}$ is $\mathrm{P\,SU}(3)/(S^1_{k,l}/Z)$. To apply Theorem 2.1., we need to find the largest connected effective group that acts on $N_{k,l}$, namely $\mathbf{I}_0(N_{k,l})$, the identity component of the isometry group of $N_{k,l}$. Note, however, that the following proposition and Theorem 2.1. together imply that $\pi_1(\mathbf{I}_0(N_{k,l}))$ has 3-torsion.

Proposition 2.3. *If* $3 \nmid kl(k+l)$, *the group* $\Gamma_0 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \subset P\mathrm{SU}(3) \subset \mathbf{I}_0(N_{k,l})$ *acts freely on* $N_{k,l}$.

Proof: The construction of Γ_0, explicitly given in [Bo], is as follows: Let $\{e_1, e_2, e_3\}$ be the standard basis in $\mathbb{C}^3$. Let ω be a primitive third root of unity. Consider the following transformations,

$$u \cdot e_i = \omega^i \cdot e_i \qquad v \cdot e_i = e_{i+1 \bmod (3)}$$

The eigenvalues of u and v are $\{1, \omega, \omega^2\}$, the third roots of unity. It is clear that they lie in SU(3). As matrices they look like,

$$u = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad v = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Let $\Gamma = < u, v >$ be the group generated by u and v. Then Γ is a non-abelian group of order 27 and the commutator $[u,v] = uvu^{-1}v^{-1}$ generates the center of SU(3). Then $\Gamma/[\Gamma,\Gamma] = \Gamma/Z(\mathrm{SU}(3)) = \Gamma_0 \subset \mathrm{P\,SU}(3)$.

Since Γ_0 is a quotient of Γ by its center, it acts freely on $N_{k,l}$ if and only if any $\gamma \in \Gamma$ conjugate to some $h \in S^1_{k,l}$ must lie in the center. Note that every non-central element of Γ has the same set of eigenvalues, namely $\{1, \omega, \omega^2\}$. So if γ is a non-central element that is conjugate to some $h = \mathrm{diag}(z^k, z^l, \bar{z}^{k+l}) \in S^1_{k,l}$, then h and γ have the same eigenvalues. Without loss of generality let $z^k = 1$ and $z^l = \omega$. Since $\gcd(k,l) = 1$, there exist integers a and b such that $ak+bl = 1$. Then we have $z = z^{ak+bl} = z^{bl} = \omega^b$. So z is 1, ω or ω^2; if $z = 1$, then $h = \gamma = \mathrm{id}$ is central. Otherwise $z^k = 1$ implies $3 \mid k$ which contradicts our assumption that $3 \nmid kl(k+l)$. Hence $\Gamma_0 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$ acts freely on $N_{k,l}$. □

When $3 \mid kl(k+l)$, $\mathbf{I}_0(N_{k,l}) = \mathrm{SU}(3) \times_{\Delta Z} S^1 = \mathrm{U}(3)$ (cf. [On, p. 146-147], [Sh2]), where $S^1 = N(S^1_{k,l})/S^1_{k,l}$ and the group of components, $\mathbf{I}/\mathbf{I}_0$, is isomorphic to $\mathbb{Z}_2$ (cf. [Sh2], see also [WZ, p. 240-241, Theorem 3.1]). Hence, any subgroup of $\mathbf{I}(N_{k,l})$ isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ must lie in $\mathbf{I}_0(N_{k,l})$. By

Theorem 2.1. it must lie in a torus and by Proposition 1.3. it cannot act freely.

Finally, we note that $H^4(N_{k,l}, \mathbb{Z})$ is a finite cyclic group of order k^2+l^2+kl (cf. [AW]). For relatively prime k and l, $3 \nmid kl(k+l)$ is equivalent to $3 \mid (k^2+l^2+kl)$, which completes the proof of the main theorem.

3. Remarks

1. The SU(5) action on the Berger space $\mathrm{SU}(5)/(\mathrm{Sp}(2) \times S^1)$ is also ineffective. However, the resulting $\mathbb{Z}_5 \oplus \mathbb{Z}_5$ action is not free.

2. The normal homogeneous Aloff-Wallach space $N_{1,1}$ also admits a free, isometric SO(3) action (cf. [Sh]) that commutes with the action of Γ. It is easy to see that for any subgroup $\Lambda \subset \mathrm{SO}(3)$ without elements of order 3, the group $\Gamma_0 \times \Lambda$ acts freely on $N_{1,1}$. In particular we have free, isometric actions of $\mathbb{Z}_6 \oplus \mathbb{Z}_{6q}$, $6 \nmid q$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_{3r}$, $3 \nmid r$, on $N_{1,1}$.

3. In [GZ], it is shown that the Eschenburg spaces (cf. [Es]) $M_p := \{\mathrm{diag}(z, z, z^p)\} \backslash \mathrm{SU}(3)/\{\mathrm{diag}(1, 1, \bar{z}^{p+2})\}$ admit free, isometric actions by $\mathbb{Z}_2 \oplus \mathbb{Z}_{2q}$ whenever p and q are odd and $\gcd(p+1, q) = 1$. Note that $M_1 = N_{1,1}$.

4. Note that from the calculation of the isometry group of $N_{k,l}$ (cf. [Sh2]), Theorem 2.1. and Proposition 1.3., it follows that $\mathbb{Z}_p \oplus \mathbb{Z}_p$ cannot act freely on $N_{k,l}$ for odd primes $p > 3$. This supports a stable version of Chern's conjecture for a given dimension (cf. [Ro]).

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Department of Mathematics, University of Maryland, College Park, MD 20742
E-mail address: kng@math.umd.edu

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109
E-mail address: shankar@math.lsa.umich.edu