PERRON-FROBENIUS EIGENVALUES, SNOWFLAKE GROUPS, AND ISOPERIMETRIC SPECTRA

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ABSTRACT. To each non-negative integer matrix $P$ and positive rational number $r$, we associate a finite, aspherical 2-complex $X_{r,P}$ and calculate the Dehn function of its fundamental group $G_{r,P}$ in terms of $r$ and the Perron-Frobenius eigenvalue of $P$. In particular we construct groups with Dehn function $\delta(x) = x^s$, where $s \in \mathbb{Q} \cap [2, \infty)$ is arbitrary. We study the higher-dimensional isoperimetric behaviour of multiple HNN extensions of $G_{r,P}$ and prove that the spectrum of exponents of second order Dehn functions also contains $\mathbb{Q} \cap [3/2, \infty)$.

INTRODUCTION

The quest to understand the complexity of word problems has been at the heart of combinatorial group theory since its inception. When one attacks the word problem for a finitely presented group $\Gamma$ directly, the most natural measure of complexity is the Dehn function $\delta(x)$ which bounds the number of defining relations that one must apply to a word $w \Rightarrow 1$ to reduce it to the empty word; the bound is a function of word-length $|w|$. (Modulo coarse Lipschitz equivalence $\simeq$, the Dehn function of a finitely presented group does not depend on the choice of presentation.)

Progress in the last ten years has led to a fairly complete understanding of which functions arise as Dehn functions of finitely presented groups. The most comprehensive information comes from [11] where, modulo issues associated to the $P = NP$ question, Birget, Rips and Sapir essentially provide a characterisation of the Dehn functions greater than $x^4$. In particular they show that the following isoperimetric spectrum is dense in the range $[4, \infty)$.

$$\text{IP} = \{ \alpha \in [1, \infty) \mid f(x) = x^\alpha \text{ is } \simeq \text{ a Dehn function} \}.$$ 

Gromov proved that $\text{IP} \cap (1, 2)$ is empty and that word hyperbolic groups can be characterised as those which have linear Dehn functions. In [3] Brady and Bridson completed the understanding of the coarse structure of $\text{IP}$ by providing a dense set of exponents in $\text{IP} \cap [2, \infty)$. What remains unknown is the fine structure of $\text{IP} \cap (2, 4)$. In particular, it has remained unknown whether $\mathbb{Q} \cap (2, 4) \subset \text{IP}$.

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What Brady and Bridson actually do in [3] is associate to each pair of positive integers $p \geq q$ a finite aspherical 2-complex whose fundamental group $G_{p,q}$ has Dehn function $x^{\log_2 2p/q}$. These complexes are obtained by attaching a pair of annuli to a torus, the attaching maps being chosen so as to ensure the existence of a family of discs in the universal cover that display a certain snowflake geometry (cf. figure 4 below). In the present article we present a more sophisticated version of the snowflake construction that yields a much larger class of isoperimetric exponents.

**Theorem A.** Let $P$ be an irreducible non-negative integer matrix with Perron-Frobenius eigenvalue $\lambda > 1$, and let $r$ be a rational number greater than every row sum of $P$. Then there is a finitely presented group $G_{r,P}$ with Dehn function $\delta(x) \simeq x^{2 \log_3(r)}$.

By taking $P$ to be the $1 \times 1$ matrix $(2^{2q})$ and $r = 2^r$ we obtain:

**Corollary A.** $\mathbb{Q} \cap (2, \infty) \subset \text{IP}$.

The influential work of M. Gromov [8], [9] embedded the word problem in the broader context of filling problems for Riemannian manifolds and combinatorial complexes. For example, Gromov’s Filling Theorem [4] states that given a compact Riemannian manifold $M$, the smallest function bounding the area of least-area discs in $M$ as a function of their boundary length is coarsely Lipschitz equivalent to the Dehn function of $\pi_1 M$. In the geometric context, it is natural to extend questions about the size of optimal fillings to higher-dimensional spheres, exploring higher-dimensional isoperimetric functions that bound the volume of optimal ball-fillings of spheres mapped into the manifold (or complex). Correspondingly, one defines higher-order Dehn functions $\delta^{(k)}$ for finitely presented groups $\Gamma$ that have a classifying space with a compact $(k+1)$-skeleton (see section 7). The $\simeq$ class of $\delta^{(k)}$ is a quasi-isometry invariant of $\Gamma$.

For each positive integer $k$ one has the higher-order isoperimetric spectrum

$$\text{IP}^{(k)} = \{ \alpha \in [1, \infty) \mid f(x) = x^\alpha \text{ is } \simeq \text{ a } k\text{-th order Dehn function} \}.$$ 

We do not yet have as detailed a knowledge of the structure of these sets as we do of $\text{IP} = \text{IP}^{(1)}$. Indeed our knowledge is remarkably sparse even for $\text{IP}^{(2)}$: the results of [1], [14], [13] provide infinite sets of exponents in the range $[3/2, 2)$ and provide evidence for the existence of exponents in the range $[2, \infty)$; the snowflake construction of [3] provides a dense set of exponents in the interval $[3/2, 2)$; and in [5] it is was proved that $2, 3 \in \text{IP}^{(2)}$ (see also [7]). Gromov and others have investigated the isoperimetric behaviour of lattices [9].

Our second theorem relieves the dearth of knowledge about the coarse structure of $\text{IP}^{(k)}, k \geq 2$.

**Theorem B.** Let $P$ be an irreducible non-negative integer matrix with Perron-Frobenius eigenvalue $\lambda > 1$, and let $r$ be an integer greater than every row sum of $P$. Then for every $k \geq 2$ there is a group $\Sigma^{k-1}G_{r,P}$ of type $F_{k+1}$ with order $k$ Dehn function
\(\delta^{(k)}(x) \simeq x^{2\log_\lambda(r)}\). There are also groups \(\Sigma^{k-1}\mathbb{Z}^2\) of type \(F_{k+1}\) with order \(k\) Dehn function \(\delta^{(k)}(x) \simeq x^2\).

By taking \(P\) to be the \(1 \times 1\) matrix \((2^{2q})\) and \(r = 2^p\) we see that \(\mathbb{Q} \cap [2, \infty) \subset \text{IP}^{(k)}\). And the results in section 6 of [1] imply that \(2 - \frac{1}{\alpha} \in \text{IP}^{(2)}\) for each of the exponents \(\alpha = 2 \log_\lambda r\) produced in Theorem A.

[Discuss the theorem about products with \(\mathbb{Z}\).]

**Corollary B.** \(\mathbb{Q} \cap [1 + 1/k, \infty) \subset \text{IP}^{(k)}\).

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**Figure 1.** Isoperimetric exponents of \(\Sigma^{k-1}G_{r,p} \times \mathbb{Z}^l\). Colors correspond to fixed values of \(k\).

The main aim of Brady and Bridson’s initial construction of snowflake groups [3] was to prove that the closure of \(\text{IP}^{(1)}\) is \(\{1\} \cup [2, \infty)\). Corollary B shows that the closure of \(\text{IP}^{(2)}\) contains \(\{1\} \cup [3/2, \infty)\), but we do not know if it is equal to it. More generally, we anticipate that higher-dimensional suspensions of \(G_{r,p}\) will have \(k\)-th order Dehn functions of the form \(x^\alpha\) where the exponents \(\alpha\) form a dense set in \([1 + 1/k, \infty)\) (including all rationals in this range). This might lead one to speculate that \(\overline{\text{IP}}^{(k)} = \{1\} \cup [1 + 1/k, \infty)\) for all positive integers \(k\), but we have no promising heuristics on which to base a conjecture to this effect. Moreover, the fact that there exist second order Dehn functions of the form \(x \log x\) and \(x^2/\log x\) (see [1], [13]) would seem to argue against the existence of the gap \((1, 1 + 1/k)\) in \(\text{IP}^{(k)}\).

This article is organised as follows. In section 1 we outline the construction of the snowflake groups \(G_{r,p}\) and their HNN extensions \(H_{r,p}\), deferring a detailed account to section 4. In section 2 we gather some pertinent definitions and the elements of Perron-Frobenius theory that we require. The groups \(G_{r,p}\) are fundamental groups of graphs of groups; in section 3 we analyze the geometry of the vertex groups in these decompositions. The snowflake geometry of \(G_{r,p}\) is described in section 4 and this is analyzed in further detail in section 5 to prove Theorem A. In section 6 we turn our attention to higher order Dehn functions and prove Theorem B.
1. An outline of the basic construction

The snowflake groups $G_{r, p}$ in Theorem A are the fundamental groups of aspherical 2-complexes $X_{r, p}$ assembled from a finite collection of tori and annuli. With respect to a fixed framing on the tori, the attaching maps of the annuli are all powers of the slopes $\{1/0, 0/1, 1/1\}$. From this perspective, it is perhaps surprising that one can encode the range of isoperimetric exponents stated in Theorem A.

More algebraically, $G_{r, p}$ is the fundamental group of a graph of groups with vertex groups $\mathbb{Z}^2$ and edge groups $\mathbb{Z}$. The rational number $r$ encodes the multiplicity of the attaching maps of the annuli while the positive integer matrix $P$ encodes a prescription for the number and orientation of the tubes connecting each of the tori to its neighbours (and to itself).

It is natural to describe the assembly of $X_{r, p}$ in two stages. At the first stage, one groups the basic tori into connected families which form the vertices of a coarser graph-of-spaces decomposition $\mathcal{G}$ of $X_{r, p}$ than the one sketched above. If $P$ is an $R \times R$ matrix, then there are $R$ vertex spaces in this coarser decomposition, corresponding to the rows of $P$; the edge spaces are all annuli, and the underlying graph of the decomposition is the directed graph whose transition matrix is equal to $P$.

If the sum of the entries in a given row of $P$ is $m$ then the corresponding vertex space consists of a chain of $m - 1$ framed tori, the $(j + 1)$st being attached to the $j$th $T_j$ by an annulus, one end of which wraps once around a coordinate circle in $T_j$ and the other end of which wraps once around the circle of slope $1/1$ (“diagonal”) in $T_{j+1}$; it is convenient to then collapse each of the connecting annuli to a circle. A detailed study of these vertex spaces $V_m$ will be undertaken in section 3. There is a distinguished element that plays an important role in this study, namely the diagonal element $c$ in the first of the $m - 1$ tori.

If the $(i, j)$ entry of $P$ is $M$ then in $\mathcal{G}$ there are $M$ cylinders running from the vertex corresponding to row $i$ to the vertex corresponding to row $j$. The rational number $r$ determines the attaching maps of the ends of these cylinders: if $r = p/q$ then each cylinder wraps $p$ times around a coordinate circle on one of the tori at vertex $i$ and $q$ times around the diagonal (slope $1/1$ circle) on one of the tori at vertex $j$.

The “coarser” decomposition of $X_{r, p}$ is the one that we focus on throughout this article. The (less informative) fact that $X_{r, p}$ is a union of tori connected along annuli attached with slopes in $\{0/0, 0/1, 0/1\}$, is recovered by simply forgetting the grouping of the basic tori into vertex spaces $V_m$.

This completes our sketch of the construction of the groups $G_{r, p}$. A key feature of the construction is that for each positive integer $d$ there is an injective endomorphism $\phi_d: G_{r, p} \to G_{r, p}$ whose restriction to the fundamental group of each basic torus is $x \mapsto x^d$.

For each integer $r$, we define $H_{r, p}$ to be the HNN extension of $G_{r, p}$ that is associated to the endomorphism $\phi_r$ and has two stable letters. In other words, if one realizes
the endomorphism $\phi_r$ by the natural cellular map $\Phi_r : X_{r,P} \to X_{r,P}$, then $H_{r,P}$ is the fundamental group of the union $Y_{r,P}$ of two copies of the following mapping torus, identified along the images of $X \times \{0\}$:

$$X_{r,P} \times [0,1] / [(x, 1) \sim (\Phi_r(x), 0)].$$

Note that the 3-complex $Y_{r,P}$ is aspherical, and hence may be used to calculate the second order Dehn function of $H_{r,P}$.

**The strategy for proving Theorems A and B.** The key geometric idea behind Theorem A is that efficient van Kampen diagrams for the groups $G_{r,P}$ exhibit the snowflake geometry illustrated in figure 4. The essential features of such diagrams are these: the diagram is composed of polygonal subdiagrams joined across strips so that the dual to the decomposition is a tree $T$; each of the polygonal subdiagrams is a van Kampen diagram in one of the vertex groups $V_m$ typically it is an $(m + 1)$-gon with a base labelled by a power of the distinguished $c \in V_m$ and $m$ other faces labelled by powers of the coordinate circles in the chain of tori defining $V_m$; the strips in the diagram correspond to the connecting annuli whose pattern of existence is encoded in $P$; by construction, the number of 1-cells is altered by a factor of $r$ as one passes from one side of the strip to the other.

The most important class of diagrams are those that are as symmetric as possible, having the property that as one moves from the circumcentre of the dual tree to the boundary of the diagram, the joining strips are all oriented in such a way that the length of the side strip increases by a factor of $r$ as one journeys towards the boundary. The labels on the outer sides of the strips are powers of the diagonal elements in various vertex groups $V_m$, and a crucial feature of our construction is that the cyclic subgroups $\langle c \rangle \subset G_{r,P}$ are distorted in a precisely understood manner, with distortion function $\simeq x^\alpha$ where $\alpha = \log_\lambda(r)$. (This fact is at the heart of our calculations and it is where the Perron-Frobenius theory enters – see section 4.)

If the tree $T$ has radius $d$, then arguing by induction on $d$ in a suitable class of diagrams, one calculates the the length of the boundary to be $\sim d^{k/\alpha}$ if the central polygon has base $\sim d^k$. One has a precise understanding of the quadratic Dehn functions of the vertex groups $V_m$, and this leads to an area estimate of $\sim d^{2k}$ on these diagrams of diameter $\sim d^k$. Thus we obtain a family of diagrams with area $\sim d^{2k}$ and perimeter $\sim d^{k/\alpha}$, and an elementary manipulation of logs provides the required lower bound $x \mapsto x^{2\log_\lambda(r)}$ on the Dehn function of $G_{r,P}$. The complementary upper bound is established in section 5; the main points are a calculation of the distortion function for $V_m \subset G_{r,P}$ and an improvement of the Shuffling Lemma from [3].

A key feature in our construction of $G_{r,P}$ is that, when $r$ is an integer, the snowflake diagrams we were just discussing admit a precise scaling by a factor of $r$. This means that one can stack a sequence of scaled copies of these diagrams to form 3-dimensional
balls embedded in the universal cover of the 3-complex $Y_{r,P}$ (see figures 7, 8). Such sequences of balls provide a sharp lower bounds on the second order Dehn functions of the groups of $H_{r,P}$, and the simple relationship between $G_{r,P}$ and $H_{r,P}$ means that the complementary upper bounds can be deduced easily as in [14]. In fact, the scaling phenomenon continues into arbitrary dimensions to provide lower bounds on the higher order Dehn function of iterated HNN extensions of $G_{r,P}$. What prevents us from extending Theorem B to arbitrary dimensions at the moment is a lack of technology for obtaining upper bounds.

**An explicit example.** We conclude our sketch of our basic constructions with an explicit example. Strictly speaking, this example differs slightly from the construction given in the next section, but it illustrates the essential features. (The precise difference is in the choice of tree used to define the group $V_{2q}$.) The example that we present here has Dehn function $n^{2p/q}$, where $p > q$ are positive integers ($q$ may be even).

Let $P$ be the $1 \times 1$ matrix with entry $2^q$ and let $r = 2^p$. Then $G_{r,P}$ is the fundamental group of a graph of groups $G$ with one vertex group and $2^q$ infinite cyclic edge groups. The single vertex group $V_{2q}$ is the fundamental group of a tree-of-groups that we shall describe in a moment. $V_{2q}$ has generators $\gamma_1, \ldots, \gamma_{2q}$. The product of these generators, $c = \gamma_1, \ldots, \gamma_{2q}$ plays a special role.

The $i$-th edge group of $G$ has two monomorphisms to the vertex group $V_{2q}$; one maps the generator to $c$ and the other maps the generator to $\gamma_i$. Thus we have a relative presentation

$$G_{p/q} = G_{r,P} = \langle V_{2q}, s_1, \ldots, s_{2q} \mid s_i \gamma_i^{2p} s_i^{-1} = c, \ (i = 1, \ldots, 2q) \rangle.$$

It remains to elucidate the structure of the group $V_{2q}$. This is the fundamental group of a tree of groups in which each of the vertex groups is isomorphic to $\mathbb{Z}^2$ and each of the edge groups is infinite cyclic. The underlying tree is a rooted binary tree of radius $q - 2$. A basis $\{a, b\}$ is fixed for each vertex group, the generator of the incident edge group on the side of the root maps to the diagonal element $ab$, and the generators of the edge groups on the side away from the root map to $a$ and $b$ respectively.

The generators $\gamma_1, \ldots, \gamma_{2q}$ mentioned above are the generators of the vertex groups at the leaves of the tree. The distinguished element $c$ is the diagonal of the $\mathbb{Z}^2$ at the root of the tree.

Theorem B tells us that the Dehn function of $G_{p/q}$ is $n^\alpha$ where $\alpha = 2 \log_{2q} 2^q = 2p/q$. Thus, for example, if we take $p = 5$ and $q = 2$, then we obtain a group with Dehn function $n^{5/2}$. In this case, the binary tree described above has radius 3, and the above description of $V_{2q}$ yields the presentation (indexing the generators of the vertex groups in the obvious manner):

$$\langle a_0, b_0, a_{i,j}, b_{i,j} \ (i = 1, 2; \ j = 1, \ldots, 2^i) \mid [a_0, b_0] = 1 = [a_{i,j}, b_{i,j}] \rangle;$$
\[ a_0 = a_{1,1}b_{1,1}, \quad b_0 = a_{1,2}b_{1,2}, \quad a_{1,1} = a_{2,1}b_{2,1}, \quad b_{1,1} = a_{2,2}b_{2,2}, \quad a_{1,2} = a_{2,3}b_{2,3}, \quad b_{1,2} = a_{2,4}b_{2,4}. \]

Eliminating the superfluous generators \( a_0, \ldots, b_{1,2} \) and relabelling the leaf-group generators \( \gamma_1, \ldots, \gamma_8 \), as in our account of the general \( G_{p/q} \), we get
\[
V_4 = \langle \gamma_1, \ldots, \gamma_8 \mid \theta \in \mathcal{C}_4 \rangle.
\]

where \( \mathcal{C}_4 \) is the set following set of commutators
\[
[\gamma_1, \gamma_2] = [\gamma_3, \gamma_4], \quad [\gamma_5, \gamma_6], \quad [\gamma_7, \gamma_8], \quad [\gamma_1\gamma_2, \gamma_3\gamma_4], \quad [\gamma_5\gamma_6, \gamma_7\gamma_8], \quad [\gamma_1\gamma_2\gamma_3\gamma_4, \gamma_5\gamma_6\gamma_7\gamma_8].
\]

Thus we obtain the explicit presentation
\[
G_{5/4} = \langle \gamma_1, \ldots, \gamma_8, s_1, \ldots, s_8 \mid \mathcal{C}_4; \ s_1\gamma_i^5s_i^{-1} = \gamma_1 \ldots \gamma_8 (i = 1, \ldots, 8) \rangle.
\]

**Remark 1.1.** We have just described a 16-generator, 15-relator presentation of \( G_{5/4} \). The corresponding presentation for \( G_{p/q} \) has \( 2^{q+2} \) generators and \( 2^{q+2} - 1 \) relations.

## 2. Preliminaries

In the first part of this section we recall the basic definitions associated to Dehn functions and the second part we gather those elements of Perron-Frobenius theory that will be needed in the sequel.

**First order Dehn functions.** Given a finitely presented group \( G = \langle \mathcal{A} \mid \mathcal{R} \rangle \) and a word \( w \) in the generators \( \mathcal{A}^{\pm 1} \) that represents \( 1 \in G \), one defines
\[
\text{Area}(w) = \min\{N \in \mathbb{N}^+ \mid \exists \text{ equality } w = \prod_{j=1}^N u_jr_ju_j^{-1} \text{ freely, where } r_j \in \mathcal{R}^{\pm 1} \}.
\]

The \textit{Dehn function} \( \delta(x) \) of the finite presentation \( \langle \mathcal{A} \mid \mathcal{R} \rangle \) is defined by
\[
\delta(x) = \max\{\text{Area}(w) \mid w \in \ker(F(\mathcal{A}) \to G), \ |w| \leq x \}
\]
where \(|w|\) denotes the length of the word \( w \). It is straightforward to show that the Dehn functions of any two finite presentations of the same group are \( \simeq \) equivalent in the following sense (and modulo this equivalence relation it therefore makes sense to talk of “the” Dehn function of a finitely presented group).

Given two functions \( f, g: [0, \infty) \to [0, \infty) \) we define \( f \preceq g \) if there exists a positive constant \( C \) such that
\[
f(x) \leq C g(Cx) + Cx
\]
for all \( x \geq 0 \). If \( f \preceq g \) and \( g \preceq f \) then \( f \) and \( g \) are said to be \( \simeq \) equivalent, denoted \( f \simeq g \).

**Remark 2.1.** In order to establish the relation \( f \preceq g \) between non-decreasing functions, it suffices to consider relatively sparse sequences of integers. For if \( (n_i) \) is an increasing sequence of integers for which there is a constant \( C > 0 \) such that \( n_0 = 0 \) and \( n_{i+1} \leq Cn_i \) for all \( i \), and if \( f(n_i) \leq g(n_i) \) for all \( i \), then \( f \preceq g \).
Indeed, given \( x \in [0, \infty) \) there is an index \( i \) such that \( n_i \leq x \leq n_{i+1} \), whence \( f(x) \leq f(n_{i+1}) \leq g(n_{i+1}) \leq g(Cn_i) \leq g(Cx) \).

We refer to [4] for general facts about Dehn functions, in particular the interpretation of \( \text{Area}(w) \) in terms of van Kampen diagrams over \( \langle A \mid R \rangle \). Recall that a van Kampen for \( w \) is a labelled, contractible, planar 2-complex with a basepoint and boundary label \( w \). Associated to such a diagram \( D \) one has a cellular map \( \tilde{D} \) from \( D \) to the universal cover \( \tilde{K} \) of the standard 2-complex of \( \langle A \mid R \rangle \), respecting labels and basepoint. The diagram is said to be \textit{embedded} if this map in injective.

\textbf{Remark 2.2.} If the presentation \( \langle A \mid R \rangle \) is aspherical and the diagram \( D \) is embedded, then \( D \) has the smallest area among all diagrams with the same boundary label.

To see this, note that if \( \Delta \) is a diagram with the same boundary circuit as \( \tilde{D} \), then \( \tilde{D} - \Delta \) defines a 2-cycle in \( \tilde{K} \), which must be zero since \( H_2(\tilde{K}; \mathbb{Z}) = 0 \) and there are no 3-cells. Thus each 2-cell in the image of \( \tilde{D} \) must also occur in the image of \( \tilde{g} \). And since \( \tilde{D} \) is an embedding, the number of 2-cells in the image (hence domain) of \( \Delta \) is at least \( \text{Area}(D) \).

\textbf{Higher order Dehn functions.} If \( M \) is a compact \( k \)-dimensional manifold and \( X \) a CW complex, an \textit{admissible map} is a continuous map \( f : M \to X^{(k)} \subset X \) such that \( f^{-1}(X^{(k-1)}) = M_0 \) is a codimension-zero submanifold and \( M - M_0 \) is a disjoint union of open \( k \)-dimensional balls, mapped by \( f \) homeomorphically onto \( k \)-cells of \( X \).

If \( f : M \to X \) is admissible we define the \textit{volume} of \( f \), denoted \( \text{Vol}^k(f) \), to be the number of open \( k \)-balls in \( M \) mapping to \( k \)-cells of \( X \). This notion is useful because of the abundance of admissible maps:

\textbf{Lemma 2.3.} Let \( M \) be a compact manifold (smooth or PL) of dimension \( k \) and let \( X \) be a CW complex. Then every continuous map \( f : M \to X \) is homotopic to an admissible map. If \( f(\partial M) \subset X^{(k-1)} \) then the homotopy may be taken rel \( \partial M \).

\textbf{Proof.} We prove the lemma in the smooth case; analogous methods apply in the PL category (cf. the transversality theorem of [6]).

First arrange that \( f(M) \subset X^{(k)} \) using cellular approximation. Then let \( C \subset X^{(k)} \) be a set consisting of one point in the interior of each \( k \)-cell of \( X \). By Sard’s theorem we can choose each point of \( C \) to be a regular value of \( f \). Here we are considering \( X^{(k)} - X^{(k-1)} \) as a smooth manifold and restricting \( f \) to the preimage of this open set. The preimage \( f^{-1}(C) \) is now a 0-dimensional submanifold of \( M \) (i.e. a finite set of points) and \( f \) is a local diffeomorphism at each of these points, by the inverse function theorem. Choose an open ball neighborhood of each point of \( C \) which is evenly covered by \( f \). By stretching each such ball over the \( k \)-cell containing it, and pushing its complement into \( X^{(k-1)} \), we can arrange that the preimage of each open \( k \)-cell is a union of open \( k \)-balls in \( M \), mapping homeomorphically by \( f \) onto the \( k \)-cell, and that the rest of \( M \) maps into \( X^{(k-1)} \). \(\square\)
Given a group $G$ of type $F_{k+1}$, fix an aspherical CW complex $X$ with fundamental group $G$ and finite $(k+1)$-skeleton. Let $	ilde{X}$ be the universal cover of $X$. If $f: S^k \to \tilde{X}$ is an admissible map, define the filling volume of $f$ to be the minimal volume of an extension of $f$ to $B^{k+1}$:

$$F\text{Vol}(f) = \min\{\text{Vol}^{k+1}(g) \mid g: B^{k+1} \to \tilde{X}, \ g|_{\partial B^{k+1}} = f\}.$$ 

Note that the maps $g$ must be admissible for volume to be defined. Such extensions exist by Lemma 2.3, since $\pi_k(\tilde{X})$ is trivial. Next we define the order $k$ Dehn function of $X$ to be

$$\delta^{(k)}(x) = \max\{F\text{Vol}(f) \mid f: S^k \to \tilde{X}, \ \text{Vol}^k(f) \leq x\}.$$ 

Again, the maps $f$ are assumed to be admissible. We will sometimes write $\delta^{(k)}(x)$ as $\delta^G_k(x)$ (recall that $G$ is the fundamental group of $X$).

**Remarks 2.4.** (1) In these definitions one could equally well use $X$ in place of $\tilde{X}$, since maps $S^k \to X$ (or $B^{k+1} \to X$) and their lifts to $\tilde{X}$ have the same volume. There are reasons to prefer $\tilde{X}$, however, as we shall see in the next definition below.

(2) It is not difficult to show that the Dehn function $\delta^{(k)}(x)$ agrees with the notion defined by Alonso et al. in [2]. A discussion along these lines is given in section 5 of [5]. Moreover it is proved in [2] that up to $\simeq$ equivalence this function depends only on $G$ (and in fact is a quasi-isometry invariant); hence we refer to it as “the” order $k$ Dehn function of $G$.

**Other Dehn functions.** The definition of $\delta^{(k)}(x)$ generalizes in a natural way to give Dehn functions modeled on manifolds other than $B^{k+1}$. For example, Gromov has defined genus $g$ filling invariants based on surfaces other than the disk [9]. Here we need to consider arbitrary manifolds of any dimension.

Let $(M, \partial M)$ be a compact manifold pair (smooth or PL) with $\dim M = k + 1$. If $f: \partial M \to \tilde{X}$ is an admissible map define

$$F\text{Vol}^M(f) = \min\{\text{Vol}^{k+1}(g) \mid g: M \to \tilde{X}, \ g|_{\partial M} = f\}$$

and

$$\delta^M(x) = \max\{F\text{Vol}^M(f) \mid f: \partial M \to \tilde{X}, \ \text{Vol}^k(f) \leq x\}.$$ 

Here it is important that we use maps into $\tilde{X}$, which is contractible, since maps $f: \partial M \to X$ need not have extensions to $M$. Note that if $(M, \partial M) = (B^{k+1}, S^k)$ then the definitions of $\delta^M(x)$ and $\delta^{(k)}(x)$ agree.

The order of $\delta^M(x)$ is $k$, the dimension of $\partial M$ (when $\partial M \neq \emptyset$). In general we do not assume that $M$ is connected or that $\partial M \neq \emptyset$, though if $M$ is closed then $\delta^M(x)$ is identically zero. Again, we will sometimes use the notation $\delta^G_k(x)$ for $\delta^M(x)$.

**Remark 2.5.** An obvious adaptation of the argument in Remark 2.2 shows that if $X$ is an aspherical $(k+1)$-dimensional CW complex, $g: M^{k+1} \to X$ is an embedding,
and \( f = g|_{\partial M} \) (with \( f \) and \( g \) admissible) then \( \text{FVol}^M(f) = \text{Vol}^{k+1}(g) \). That is, \( g \) has minimal volume among all \( M \)-fillings of \( f \).

**Perron-Frobenius Theory.** A square non-negative matrix \( P \) is said to be *irreducible* if for every \( i \) and \( j \) there exists \( k \geq 1 \) such that the \( ij \)-entry of \( P^k \) is positive. The basic properties of irreducible matrices are summarized in the Perron-Frobenius theorem below. See [12] and [10] for a more thorough treatment of this theory and its applications.

**Proposition 2.6** (Perron-Frobenius theorem). Let \( P \) be an irreducible non-negative \( R \times R \) matrix. Then \( P \) has one (up to a scalar) eigenvector with positive coordinates and no other eigenvectors with non-negative coordinates. Moreover, the corresponding eigenvalue \( \lambda \) is simple, positive, and is greater than or equal to the absolute value of all other eigenvalues. If \( m \) and \( M \) are the smallest and largest row sums of \( P \), then \( m \leq \lambda \leq M \), with equality on either side implying equality throughout.

**Lemma 2.7.** Let \( P \) be an irreducible non-negative \( R \times R \) matrix with Perron-Frobenius eigenvalue \( \lambda \). Let \( \{v_1, \ldots, v_R\} \) be a generalized eigenbasis for \( P \), with \( v_1 \) a positive eigenvector for \( \lambda \), and with corresponding inner product \( \langle \cdot, \cdot \rangle \). Then \( \langle u, v_1 \rangle > 0 \) for every non-negative vector \( u \in \mathbb{R}^R - \{0\} \).

**Proof.** Decompose \( \mathbb{R}^R \) as \( W_1 \oplus \cdots \oplus W_k \) where each \( W_i \) is a generalized eigenspace for \( P \), with \( W_1 = \langle v_1 \rangle \). Each \( W_i \) is \( P \)-invariant, as is the non-negative orthant \( \mathcal{N} \), since \( P \) is non-negative. The intersection \( (W_2 \oplus \cdots \oplus W_k) \cap \mathcal{N} \) must then be trivial, for otherwise it contains an eigenvector for \( P \) other than \( v_1 \) (or a scalar multiple), by the Brouwer fixed point theorem. Hence \( \langle u, v_1 \rangle \neq 0 \) for every \( u \in \mathcal{N} - \{0\} \). Since \( \mathcal{N} - \{0\} \) is connected and contains \( v_1, \langle u, v_1 \rangle \) is positive. \( \square \)

**Proposition 2.8** (Growth rate). Let \( P \) be an irreducible non-negative \( R \times R \) matrix with Perron-Frobenius eigenvalue \( \lambda \). Let \( \| \cdot \| \) be a norm on \( \mathbb{R}^R \). Then there are positive constants \( A_0, A_1 \) such that for every non-negative vector \( u \) in \( \mathbb{R}^R \) and every integer \( k > 0 \), \( A_0 \lambda^k \| u \| \leq \| P^k u \| \leq A_1 \lambda^k \| u \| \).

**Proof.** First, it is clear that by varying the constants, it suffices to consider any single norm \( \| \cdot \| \). Consider a generalized eigenbasis \( \{v_1, \ldots, v_R\} \) as in Lemma 2.7 (with \( v_1 \) a positive eigenvector for \( \lambda \)). Let \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) be the corresponding inner product and norm on \( \mathbb{R}^R \). Let \( \pi: \mathbb{R}^R \to \langle v_1 \rangle \) be orthogonal projection \( \pi(u) = \langle u, v_1 \rangle v_1 \).

Define \( A_0 = \inf\{\| \pi(u) \| / \| u \| \mid u \in \mathcal{N} - \{0\}\} \). Note that \( A_0 > 0 \) by Lemma 2.7 and compactness of \( \mathcal{N} - \{0\} \) modulo homothety. For every \( u \) in \( \mathcal{N} - \{0\} \) we now have \( \lambda^k A_0 \| u \| \leq \lambda^k \| \pi(u) \| = \| P^k \pi(u) \| \leq \| P^k u \| \). We also have \( \| P^k u \| \leq \lambda^k \| u \| \) since \( \lambda \) is the spectral radius of \( P \); hence \( A_1 = 1 \) will work. \( \square \)
3. The vertex groups $V_m$

In this section we define groups $V_m$ for each integer $m \geq 2$. We begin with a very brief overview of the construction of the groups $G_{r,P}$ so that the reader knows where the groups $V_m$ fit into the overall picture.

An irreducible matrix $P$ determines a directed graph (whose transition matrix is $P$). This graph is the underlying graph in a graph of groups description of the $G_{r,P}$ in Theorem A. The vertex groups in this graph of groups are precisely the groups $V_m$ which we define and study in this section.

The groups $V_m$ satisfy a number of the properties that the free abelian groups $\mathbb{Z}^m$ do, but they have geometric dimension 2. In particular, $V_m$ has generators $a_1, \ldots, a_m$ and has the following scaling property (Corollary 3.3 to Lemma 3.1): for any integer $N > 0$, the equality $a_1^N \cdots a_m^N = (a_1 \cdots a_m)^N$ holds. Moreover, this equality requires on the order of $N^2$ relations of $V_m$. This follows as a special case of Lemma 3.5, which gives careful estimates on the areas of certain words in $V_m$.

**The groups $V_m$.** Begin with $m - 1$ copies of $\mathbb{Z} \times \mathbb{Z}$, the $i$th copy having generators $\{a_i, b_i\}$. The group $V_m$ is formed by successively amalgamating these groups along infinite cyclic subgroups by adding the relations

$$b_1 = a_2 b_2, \quad b_2 = a_3 b_3, \quad \ldots, \quad b_{m-2} = a_{m-1} b_{m-1}.$$ 

Thus $V_m$ is the fundamental group of a graph of groups whose underlying graph is a segment having $m-2$ edges and $m-1$ vertices. We define two new elements: $c = a_1 b_1$ and $a_m = b_{m-1}$. Then $a_1, \ldots, a_m$ generate $V_m$ and the relation $a_1 \cdots a_m = c$ holds; see figure 2(a). The element $c$ is called the **diagonal element** of $V_m$. The additional relations $b_{m-2} = a_{m-1} a_m$, $b_{m-k} = a_{m-k+1} \cdots a_m$ are also evident from figure 2(a).

![Figure 2](image_url)

**Figure 2.** Some relations in $V_4$: $c = a_1 a_2 a_3 a_4$ and $c^3 = (a_1)^3(a_2)^3(a_3)^3(a_4)^3$

If $m = 1$ then we define $V_m$ to be the infinite cyclic group $\langle a_1 \rangle$ and we set $c = a_1$. Lemmas 3.1 and 3.5 below clearly hold in this case.
Lemma 3.1 (Shuffling Lemma). Let \( w = w(a_1, \ldots, a_m, c) \) be a word representing \( c^N \) in \( V_m \) for some integer \( N \). Let \( n_i \) be the exponent sum of \( a_i \) in \( w \), and \( n_c \) the exponent sum of \( c \) in \( w \). Then the words \( a_1^{n_1} \cdots a_m^{n_m} c^{n_c} \) and \( c^{n_c} a_m^{n_m} \cdots a_1^{n_1} \) also represent \( c^N \) in \( V_m \) and \( n_i = N - n_c \) for all \( i \).

Proof. First we prove the second statement. The abelianization \( V_m/[V_m, V_m] \cong \mathbb{Z}^m \) has \( \{a_1, \ldots, a_m\} \) as a basis and the image of \( w \) is \( a_1^{n_1+n_c} \cdots a_m^{n_m+n_c} \). Since \( c^N \) abelianizes to \( a_1^N \cdots a_m^N \), we must have \( n_i = N - n_c \) for all \( i \).

To prove the first statement it now suffices to establish the following set of equalities for any integer \( N \):

\[
(a_1 \cdots a_m)^N = a_1^N \cdots a_m^N = a_m^N \cdots a_1^N = (a_m \cdots a_1)^N.
\]

(3.2)

In fact we shall prove the following equalities, by induction on \( k \):

\[
(a_m \cdots a_{m-k+1})^N = a_{m-k+1}^N \cdots a_m^N = a_m^N \cdots a_{m-k+1}^N = (a_m \cdots a_{m-k+1})^N.
\]

The case \( k = 1 \) is evidently true. Suppose the equations hold for a given \( k \geq 1 \). By the induction hypothesis \( a_m^N \cdots a_{m-k+1}^N = a_m^N (a_m \cdots a_{m-k+1})^N \). Then since \( b_{m-k} = a_{m-k+1} \cdots a_m \) and this element commutes with \( a_{m-k} \), we have \( a_{m-k}^N (a_{m-k+1} \cdots a_m)^N = (a_{m-k} \cdots a_m)^N \). The same commutation relation also yields

\[
a_{m-k}^N (a_{m-k+1} \cdots a_m)^N = (a_{m-k+1} \cdots a_m)^N a_m^{-1}
\]

\[
= (a_m \cdots a_{m-k+1})^N a_m^{-k}
\]

\[
= a_m^N \cdots a_{m-k+1}^N a_m^N.
\]

Finally we have \( (a_m \cdots a_{m-k+1})^N a_m^{-k} = (a_m \cdots a_{m-k+1} a_m^{-k})^N \), again because \( a_{m-k} \) and \( b_{m-k} (= a_m \cdots a_{m-k+1}) \) commute. \(\square\)

Remark 3.3 (Scaling in \( V_m \)). Equation (3.2) plays a key role in this paper. It shows that the basic relation shown in figure 2(a) holds at larger scales as well. Figure 2(b) illustrates how these larger relations follow from the triangular relations \( b_{i-1} = a_i b_i \) and \( b_{i-1} = b_i a_i \).

The spaces \( X_m \). To compute area in \( V_m \) we shall use a specific aspherical 2-complex \( X_m \) with fundamental group \( V_m \). This complex is a union of \( m-1 \) tori, each triangulated with two 2-cells realizing the relations \( a_i b_i = b_{i-1} \) and \( b_i a_i = b_{i-1} \) (where \( b_0 = c \) in the case \( i = 1 \)). Thus the ith torus has standard generators given by the 1-cells \( a_i \) and \( b_i \), and its diagonal is joined to the 1-cell \( b_{i-1} \) of the previous torus. In all there is one vertex, 1-cells \( a_1, \ldots, a_{m-1}, b_0, \ldots, b_{m-1} \), and 2\((m-1)\) triangular 2-cells.

The universal cover \( \tilde{X}_m \) is a union of planes, each covering one of the tori below. Each plane contains three families of parallel lines covering the 1-cells \( a_i \), \( b_i \), and \( b_{i-1} \). The plane intersects neighboring planes along the \( b_j \)-lines for \( j \neq 0, m-1 \). These planes are the vertex spaces of \( \tilde{X}_m \) corresponding to the graph of groups decomposition.
of $V_m$ described earlier. The incidence graph of the vertex spaces is the Bass-Serre tree for this decomposition, with edges corresponding to $b_j$-lines ($j \neq 0, m - 1$).

**Remark 3.4.** Figure 2(b) shows an embedded disk in $\tilde{X}_m$ with boundary word of the form $c^N = a_1^N \cdots a_m^N$ ($N = 3$). The triangles shown are 2-cells of $\tilde{X}_m$. Each large triangular region lies in a vertex space of $\tilde{X}_m$. There are similar embedded disks with boundary word $c^N = a_m^N \cdots a_1^N$ as well. All of these disks have area $(m - 1)N^2$.

Throughout this paper we usually work with the standard generators $\{a_1, \ldots, a_m\}$ for $V_m$. However in the area computation below we allow words involving the elements $b_i$ as well.

**Lemma 3.5 (Area in $V_m$).** Let $w(a_1, \ldots, a_{m-1}, b_0, \ldots, b_{m-1})$ be a word representing the element $x^N$ for some $N$, where $x$ is a generator $a_i$ or $b_i$. Let $w$ be expressed as $w_1 \cdots w_k$ where each $w_i$ is a power of a generator. Then $N \leq |w|$ and $\text{Area}(wx^{-N}) \leq 3 \sum_{i<j} |w_i||w_j|.$

Note that if the sum included diagonal terms of the form $(3/2) \sum_i |w_i|^2$ then the area bound would simply be $(3/2)|w|^2$. This fact will be used in the proof of the upper bound of Theorem A. Also the statement $N \leq |w|$ implies that every vertex space is a totally geodesic subspace of $\tilde{X}_m$.

**Proof.** First we prove that $N \leq |w|$ and then we establish the area bound. Both proofs are by induction on the complexity of the word $w$, defined as follows. Let $p$ be a path in the 1-skeleton of $\tilde{X}_m$ whose edge labels read $w$. Since $w$ represents $x^N$, the endpoints of $p$ lie in a single vertex space. Hence the induced path $\tilde{p}$ in the Bass-Serre tree is a closed path. The complexity of $w$ is the length of $\tilde{p}$. Note that vertices of $\tilde{p}$ correspond to edges of $p$ (or letters of $w$) and edges correspond to transitions between certain pairs of generators. Thus the complexity is also the number of such transitions occurring in $w$.

If $w$ has complexity zero then $p$ lies in a plane. The statement $N \leq |w|$ amounts to saying that $x^N$ is a geodesic, which is clear. If $\tilde{p}$ has positive length then there is a non-trivial proper subpath $p' \subset p$ with endpoints on a single $b_j$-line. (These endpoints correspond to edges in $\tilde{p}$ that map to the same edge of the Bass-Serre tree, crossing and returning.) The subword $w' \subset w$ corresponding to $p'$ represents an element of the form $b_j^M$. Let $u$ be the word obtained from $w$ by substituting $b_j^M$ for $w'$. Then $u$ and $w'$ both have complexity strictly smaller than that of $w$. By the induction hypothesis, $M \leq |w'|$ and $N \leq |u| = (|w| - |w'|) + M$. Therefore $N \leq |w|$.

Next we establish the area bound when $w$ has complexity zero. Since $p$ then lies entirely within a vertex space of $\tilde{X}_m$, we may assume without loss of generality that $V_m = V_1$ and $x = b_0$, so that $w(a_1, b_0, b_1) = b_0^N$ in $V_1 = \langle a_1, b_1, b_0 \mid a_1b_1 = b_0 = b_1a_1 \rangle$. Since this group is abelian we can successively transpose adjacent subwords $w_i$ and cancel pairs of the form $xx^{-1}$, to obtain $v = a_1^ib_1^cb_0^{N-n}$ for some $n$. Each transposition
of letters contributes 2 to \( \text{Area}(wu^{-1}) \), so we have \( \text{Area}(wu^{-1}) \leq 2 \sum_{i<j} |w_i| |w_j| \). Next let \( I_a \) and \( I_b \) be the sets of indices for which \( w_i \) is a power of \( a_1 \) and \( b_1 \) respectively. Then \( \sum_{i \in I_a} |w_i| \geq |n| \) and \( \sum_{i \in I_b} |w_i| \geq |n| \), and therefore \( \sum_{i<j} |w_i| |w_j| \geq n^2 = \text{Area}(vb_0^{-N}) \). Then we have \( \text{Area}(wb_0^{-N}) \leq \text{Area}(wu^{-1}) + \text{Area}(vb_0^{-N}) \leq 3 \sum_{i<j} |w_i| |w_j| \) as desired.

Now suppose \( w \) has positive complexity. Define \( w' \subset w \) and \( u \) as before, so that \( w' \) represents \( b_j^M \), \( u \) is obtained from \( w \) by substituting \( b_j^M \) for \( w' \), and both \( u \) and \( w' \) have smaller complexity than \( w \). Note that \( w' = w_{i_0} \cdots w_{i_1} \subset w_1 \cdots w_k \) for some \( i_0 \) and \( i_1 \), and so \( u = w_1 \cdots w_{i_0-1} b_j^M w_{i_1+1} \cdots w_k \). Let \( I = \{ i_0, \ldots, i_1 \} \). Applying the induction hypothesis to \( u \) and \( w' \) we obtain

\[
\text{Area}(ux^{-N}) \leq 3 \sum_{i<j, i,j \notin I} |w_i| |w_j| + 3 \sum_{i \notin I} |w_i| M \tag{3.6}
\]

and

\[
\text{Area}(w'b_j^{-M}) \leq 3 \sum_{i<j, i,j \notin I} |w_i| |w_j|. \tag{3.7}
\]

Since \( M \leq |w'| = \sum_{j \in I} |w_j| \), inequality (3.6) becomes

\[
\text{Area}(ux^{-N}) \leq 3 \sum_{i<j} |w_i| |w_j| + 3 \left( \sum_{i \notin I} |w_i| \right) \left( \sum_{j \in I} |w_j| \right). \tag{3.8}
\]

Adding together (3.7) and (3.8) yields

\[
\text{Area}(w'b_j^{-M}) + \text{Area}(ux^{-N}) \leq 3 \sum_{i<j} |w_i| |w_j|
\]

which proves the lemma because \( \text{Area}(ux^{-N}) \leq \text{Area}(wu^{-1}) + \text{Area}(ux^{-N}) \) and \( \text{Area}(wu^{-1}) = \text{Area}(w'b_j^{-M}) \). \( \square \)

4. The groups \( G_{r,P} \) and snowflake words

The groups \( G_{r,P} \). Start with a non-negative square integer matrix \( P = (p_{ij}) \) with \( R \) rows. Let \( m_i \) be the sum of the entries in the \( i \)th row and let \( n = \sum_i m_i \), the sum of all entries. Form a directed graph \( \Gamma \) with vertices \( \{ v_1, \ldots, v_R \} \) and having \( p_{ij} \) directed edges from \( v_i \) to \( v_j \). Label the edges as \( \{ e_1, \ldots, e_n \} \) and define two functions \( \rho, \sigma: \{ 1, \ldots, n \} \to \{ 1, \ldots, R \} \) indicating the initial and terminal vertices of the edges, so that \( e_i \) is a directed edge from \( v_{\rho(i)} \) to \( v_{\sigma(i)} \) for each \( i \). These functions also indicate the row and column of the matrix entry accounting for \( e_i \). Partition the set \( \{ 1, \ldots, n \} \) as \( \bigcup_i I_i \) by setting \( I_i = \rho^{-1}(i) \). Note that \( |I_i| = m_i \).

Let \( M = \max \{ m_i \} \) and choose a rational number \( r = p/q \) with \( p > Mq > 0 \). We define a graph of groups \( G_{r,P} \) with underlying graph \( \Gamma \) as follows. The vertex group \( G_{v_i} \) at \( v_i \) will be \( V_{m_i} \), and all edge groups will be infinite cyclic. Relabel the standard
generators of these vertex groups as \{a_1, \ldots, a_n\} in such a way that the standard generating set for \(G_{v_i}\) is \(\{a_j \mid j \in I_i\}\). Let \(c_i\) be the diagonal element of the vertex group \(G_{v_i}\). Then the inclusion maps are defined by mapping the generator of the infinite cyclic group \(G_{e_i}\) to the elements \(a_i^p \in G_{v_{i(j)}}\) and \(c_{\sigma(i)} q \in G_{v_{\sigma(i)}}\).

Let \(s_i\) be the stable letter associated to the edge \(e_i\). The fundamental group \(G_{r,P}\) of \(\mathcal{G}_{r,P}\) is obtained from the presentation

\[
\langle G_{v_1}, \ldots, G_{v_R}, s_1, \ldots, s_n \mid s_i^{-1}a_i^p s_i = c_{\sigma(i)} q \text{ for all } i \rangle
\]

by adding relations \(s_i = 1\) for each edge \(e_i\) in a maximal tree in \(\Gamma\). However, we shall continue to use the generating set \(\{a_1, \ldots, a_n, s_1, \ldots, s_n\}\) for \(G_{r,P}\) even though some of these generators are trivial.

The spaces \(X_{r,P}\). We define aspherical 2-complexes \(X_{r,P}\) by forming graphs of spaces modeling \(\mathcal{G}_{r,P}\). Namely, take the disjoint union of the spaces \(X_{v_i} \approx X_{m_i}\) (one for each vertex \(v_i\)) and attach annuli \(A_i\), one for each edge \(e_i\) of the graph. The two boundary curves of \(A_i\) are attached to the paths labeled \(a_i^p\) in \(X_{v_{i(j)}}\) and \(c_{\sigma(i)} q\) in \(X_{v_{\sigma(i)}}\). The resulting 2-complex \(X_{r,P}\) has fundamental group \(G_{r,P}\) and it is aspherical because it is the total space of a graph of aspherical spaces.

The universal cover \(\tilde{X}_{r,P}\) is a union of copies of the universal covers \(\tilde{X}_{v_i}\) and infinite strips \(\mathbb{R} \times [-1,1]\) covering the annuli \(A_i\). Each strip is tiled by 2-cells whose boundary labels read \(s_i^{-1}a_i^p s_i c_{\sigma(i)}^{-q}\); the two sides \(\mathbb{R} \times \{\pm 1\}\) consist of edges labeled \(a_i\) and \(c_{\sigma(i)}\) respectively. Note that if a path crosses a strip along an edge labeled \(s_i\) and returns over \(s_i^{-1}\) then the power of \(a_i\) represented by the path is divisible by \(p\).

Snowflake words. For each group element of the form \(c_i^N\) we will define two types of words in the generators \(\{a_1, \ldots, a_n, s_1, \ldots, s_n\}\) representing that element, called positive and negative snowflake words. The structure of these words is governed by the dynamics of the matrix \(P\). Some snowflake words are close to geodesics, and these are useful in determining the large scale geometry of \(G_{r,P}\).

We define snowflake words recursively on \(|N| \in \mathbb{N}\) as follows. Let

\[
N_0 = \frac{p(M(q+2) + (M-1)(p-1))}{p-Mq}.
\]

Note for future reference that \(N_0 > p\) (this is easily verified). Let \(c\) be the diagonal element of a vertex group with standard ordered generating set \(\{a_1, \ldots, a_m\}\). A word \(w\) representing \(c^N\) is a positive snowflake word if either

(i) \(|N| \leq N_0\) and \(w = a_1^{N_1} \cdots a_m^{N_m}\), or

(ii) \(|N| > N_0\) and \(w = (s_{i_1} u_{i_1} s_{i_1}^{-1}) (a_1^{N_1}) \cdots (s_{i_m} u_{i_m} s_{i_m}^{-1}) (a_1^{N_m})\) where each \(u_j\) is a positive snowflake word representing a power of \(c_{\sigma(i_j)}\) and \(|N_j| < p\) for all \(j\).

In the second case note that each subword \((s_{i_j} u_j s_{i_j}^{-1}) (a_1^{N_j})\) represents a power of \(a_{i_j}\), and by Lemma 3.1 this power is \(N\). Then since \(|N_j| < p\), the word \((s_{i_j} u_j s_{i_j}^{-1})\)
represents either $a_{ij}^{\lfloor N/p\rfloor p}$ or $a_{ij}^{\lceil N/p\rceil p}$. Consequently, the word $u_j$ represents either $c_{\sigma(i_j)}^{\lfloor N/p\rfloor q}$ or $c_{\sigma(i_j)}^{\lceil N/p\rceil q}$.

A \textit{negative snowflake word} is defined similarly, with the ordering of the terms representing powers of $a_{ij}$ reversed. More specifically, $w$ satisfies either

(i') $|N| \leq N_0$ and $w = a_{i_1}^{N_1} \cdots a_{i_t}^{N_t}$, or

(ii') $|N| > N_0$ and $w = (a_{i_m}^{N_m})(s_{i_m} u_m s_{i_m}^{-1}) \cdots (a_{i_1}^{N_1})(s_{i_1} u_1 s_{i_1}^{-1})$ where $u_j$ is a negative snowflake word representing a power of $c_{\sigma(i_j)}$ and $|N_j| < p$ for all $j$.

As with positive snowflake words, each word $u_j$ will represent either $c_{\sigma(i_j)}^{\lfloor N/p\rfloor q}$ or $c_{\sigma(i_j)}^{\lceil N/p\rceil q}$.

To see that the recursion is well-founded note that the definition describes an iterated curve shortening process in which subwords of the form $c^N$ are replaced by the words described in case (ii) or (ii'), with appropriate powers of $c_{\sigma(i_j)}$ in place of $u_j$; see figure 3. Writing $|N| = Ap + B$ with $0 \leq B < p$, the new word representing

\begin{figure}
\centering
\includegraphics[width=\textwidth]{shortening.png}
\caption{One way of shortening $c^N$. Here \{a_1, a_2, a_3\} is the generating set for a vertex group $V_3$ with diagonal element $c$. The exponents \(N_1\) and \(N_2\) are both $N - \lfloor N/p\rfloor p$ and $N_3$ is $N - \lceil N/p\rceil p$. The short black edges are labeled \(s_1, s_2, s_3\).}
\end{figure}

$c^N$ has length at most $M((A + 1)q + 2 + B)$, which is strictly less than $|N|$ provided $|N| > N_0$. Eventually the subwords $c^N$ all have length at most $N_0$ and the shortening procedure terminates. See also figure 4 for the end result of this process. In this figure the top and bottom halves of the boundary are positive and negative snowflake words representing $c^N$, the diameter.

Note that every snowflake word has a nested structure in which various subwords are themselves snowflake words. These are the subwords $u_j$ arising at each stage. The minimal such subwords are those given by (i) and (i') and these will be called \textit{terminal subwords}. The \textit{depth} of a snowflake subword is the number of snowflake subwords of type (ii) or (ii') properly containing it, including the original snowflake word itself. Equivalently, it is the number of matching $s_j, s_j^{-1}$ pairs enclosing it. Note
that a snowflake word \( w \) contains a depth zero terminal subword if and only if \( w \) has
the form (i) or (i').

It is worth emphasizing that the curve shortening process is not canonically
determined, but allows many choices. In each “remainder” term \( a_{i_j}^{N_j} \), the exponent \( N_j \)
may be positive or negative; the two possible values for \( N_j \) are \( N - \lfloor N/p \rfloor p \) and
\( N - \lfloor N/p \rfloor p \). Figure 3 shows both possibilities occurring in a single step, for exa-
ple. For this reason, a single snowflake word may have terminal subwords of different
depths. However, Lemma 4.2 below shows that these depths will not differ sub-
stantially.

**Remark 4.1.** A special type of snowflake word plays a key role in the proof of
Theorem B. If \( r \) is an integer (that is, \( r = p/1 \)) and \( N = r^k \) for some \( k \), then the
positive (resp. negative) snowflake word representing \( c_N^r \) is unique. What happens is
that the exponents \( N_j \) in the expressions (ii) or (ii') at each stage are always zero;
there are no “remainder” terms \( a_{i_j}^{N_j} \). Each subword \( u_j \) represents \( c_{\sigma(i_j)} N/r \), and \( N/r \) is
again a power of \( r \). Furthermore, all terminal subwords will have the form \( a_{i_1} \cdots a_{i_m}
or \( a_{i_m} \cdots a_{i_1} \).

**Lemma 4.2** (Snowflake word depth). *Given \( r \) and \( P \) there are positive constants
\( B_0, B_1 \) with the following property. If a non-trivial snowflake word \( w \) containing \( c_N^r \)
contains a terminal subword of depth \( d \) then \( B_0 r^d \leq |N| \leq B_1 r^d \).

**Proof.** If \( d = 0 \) then \( w \) has the form (i) or (i') and \( 1 \leq |N| \leq N_0 \). Thus we need to
arrange that \( B_0 \leq 1 \) and \( B_1 \geq N_0 \) for the lemma to hold in this case.

If \( d > 0 \) then we will show by induction on \( d \) that
\[
N_0 r^{d-1} - p(r^{d-2} + \cdots + r + 1) \leq |N| \leq N_0 r^d + p(r^{d-1} + \cdots + r + 1). \tag{4.3}
\]
The lower bound then gives
\[
|N| \geq N_0 r^{d-1} - p \left( \frac{r^{d-1} - 1}{r - 1} \right) \geq \frac{1}{r} \left( N_0 - \frac{p}{r - 1} \right) r^d.
\]
Recall that \( N_0 > p \) and \( r \geq 2 \), which imply \( N_0 > p/(r - 1) \). Now we may find \( B_0 > 0 \)
so that \( B_0 \leq r^{-1}(N_0 - p/(r - 1)) \) and \( B_0 \leq 1 \), giving the desired bound.

The upper bound in (4.3) gives
\[
|N| \leq N_0 r^d + p \left( \frac{r^d - 1}{r - 1} \right) \leq (N_0 + p) r^d
\]
where the last inequality uses the fact that \( r - 1 \geq 1 \). Now choose \( B_1 \geq N_0 + p \) to
obtain the desired bound.

Next we prove (4.3) by induction on \( d \). If \( d = 1 \) then \( |N| > N_0 \) and \( w \) is of the form
(ii) or (ii') where some \( u_j \) has the form (i) or (i'). Then \( u_j \) represents \( c_{\sigma(i_j)} N' \) with
\( N' \leq N_0 \), and so \((s_{i_j} u_j s_{i_j}^{-1})\) represents \( a_{i_j} r^{N'} \). This implies \( |N| = |r N' + N_j| \leq r N_0 + p \).
For $d > 1$ write $w$ in the form (ii) or (ii'). Then the terminal subword has depth $d - 1$ in $u_j$ for some $j$. By the induction hypothesis $u_j$ represents $c_{\sigma(i)}N'$ where

$$N_0 r^{d-2} - p(r^{d-3} + \cdots + 1) \leq |N'| \leq N_0 r^{d-1} + p(r^{d-2} + \cdots + 1). \quad (4.4)$$

Then $(s_j u_j s_j^{-1})$ represents $a_j r_N$ and $r N' - p \leq |N| \leq r N' + p$. These bounds and (4.4) together imply (4.3).

**Proposition 4.5** (Snowflake word length). Given $r$ and $P$ there are positive constants $C_0, C_1$ with the following property. If $c$ is the diagonal element of one of the vertex groups and $w$ is a snowflake word representing $c^N$ then $C_0 |w|^\alpha \leq |N| \leq C_1 |w|^\alpha$, where $\alpha = \log_\lambda(r)$ and $\lambda$ is the Perron-Frobenius eigenvalue of $P$.

**Proof.** If $w$ is non-trivial and has the form (i) or (i') then $1 \leq |N| \leq N_0$ and $|N| \leq |w| \leq r |N|$. Then $|w|^\alpha \leq (rN_0)^\alpha$, which implies

$$(rN_0)^{-\alpha} |w|^\alpha \leq |N| \leq |w|^\alpha.$$  

Thus we need to arrange that $C_0 \leq (rN_0)^{-\alpha}$ and $C_1 \geq 1$ to cover this case.

Next assume that $w$ is of type (ii) or (ii') which implies that the depth of every terminal subword is at least one. Equivalently, $w$ contains the letters $s_j, s_j^{-1}$ for some $j$. Let $s(w)$ be the number of letters $s_j$ or $s_j^{-1}$ in $w$ (for all indices $j$). Note that a subword of $w$ containing no such letters has length at most $rN_0$. Since $s(w) \neq 0$, this implies

$$s(w) \leq |w| \leq 2(rN_0 + 1)s(w). \quad (4.6)$$

Let $\| \cdot \|_1$ denote the $\ell_1$ norm on $\mathbb{R}^R$: $\|v\|_1$ is the sum of the entries of the vector $v$. Let $x_1, \ldots, x_R$ be the standard basis vectors of $\mathbb{R}^R$. Recall from the definition of $G_{x, P}$ that the entry $p_{ij}$ of $P$ gives the number of directed edges from vertex $u_i$ to vertex $u_j$. Thus if $w$ has the form (ii) or (ii') and represents a power of $c_i$, then $p_{ij}$ is the number of subwords $u_k$ representing powers of $c_j$. Thus the number of subwords $u_k$ in the expression (ii) or (ii') is given by the row sum $m_i = \sum_j p_{ij}$. If $w$ represents a power of $c_k$ and every terminal subword has depth $d$ then a straightforward induction on $d$ shows that

$$s(w) = 2 \left( \| P^T(x_k) \|_1 + \|(P^T)^2(x_k)\|_1 + \cdots + \|(P^T)^d(x_k)\|_1 \right)$$

where $P^T$ is the transpose of $P$. The term $\|(P^T)^i(x_k)\|_1$ counts the number of matching $s_j, s_j^{-1}$ pairs (for all $j$) of depth $i$.

If we let $d_0$ and $d_1$ denote the smallest and largest depths of terminal subwords of $w$ then we obtain

$$2 \sum_{i=1}^{d_0} \|(P^T)^i(x_k)\|_1 \leq s(w) \leq 2 \sum_{i=1}^{d_1} \|(P^T)^i(x_k)\|_1.$$
Applying Proposition 2.8 with the norm $\| \cdot \|_1$ we have
\[
2A_0 \sum_{i=1}^{d_0} \lambda^i \leq s(w) \leq 2A_1 \sum_{i=1}^{d_1} \lambda^i = \frac{2A_1 \lambda}{\lambda - 1} (\lambda^{d_1} - 1)
\]
which implies
\[
2A_0 \lambda^{d_0} \leq s(w) \leq \frac{2A_1 \lambda}{\lambda - 1} \lambda^{d_1}.
\]
Hence by (4.6) we have
\[
(2A_0) \lambda^{d_0} \leq |w| \leq \left( \frac{4(rN_0 + 1)A_1 \lambda}{\lambda - 1} \right) \lambda^{d_1}.
\]
(4.7)

We complete the proof by applying Lemma 4.2 separately for the upper and lower bounds. Using $d = d_1$ we obtain
\[
|N| \geq B_0 r^{d_1} = B_0 (\lambda^{d_1})^{\log_\lambda (r)} \geq B_0 \left( \frac{4(rN_0 + 1)A_1 \lambda}{\lambda - 1} \right)^{-\log_\lambda (r)} |w|^{\log_\lambda (r)}.
\]
Now choose $C_0 > 0$ satisfying $C_0 \leq B_0 \left( \frac{4(rN_0 + 1)A_1 \lambda}{\lambda - 1} \right)^{-\alpha}$ and $C_0 \leq (rN_0)^{-\alpha}$ to obtain the desired lower bound.

Applying Lemma 4.2 with $d = d_0$ gives
\[
|N| \leq B_1 r^{d_0} = B_1 (\lambda^{d_0})^{\log_\lambda (r)} \leq B_1 (2A_0)^{-\log_\lambda (r)} |w|^{\log_\lambda (r)}
\]
so choose $C_1$ with $C_1 \geq B_1 (2A_0)^{-\alpha}$ and $C_1 \geq 1$. \qed

5. Proof of Theorem A

Throughout this section $G_{r,p}$ is fixed, with $r = p/q$ greater than all the row sums of $P$, and $\alpha = \log_\lambda (r)$, where $\lambda$ is the Perron-Frobenius eigenvalue of $P$. Unless otherwise stated, all words use the generating set \{a_1, \ldots, a_n, s_1, \ldots, s_n\} for $G_{r,p}$.

The lower bound. To establish the lower bound $\delta(x) \geq x^{2\alpha}$ we will show that $\delta(n_i) \geq (C_0 2^{4-\alpha}) n_i^{2\alpha}$ for certain integers $n_i$ tending to infinity. This is sufficient by Remark 2.1, provided the sequence $(n_i)$ grows at most exponentially.

Note also that to establish a single inequality $\delta(n) \geq A$, it is enough to exhibit an embedded disk in $\tilde{X}_{r,p}$ with boundary length $n$ and area $A$ or greater, by Remark 2.2. Here we are using the facts that $X_{r,p}$ is aspherical and 2-dimensional.

Choose a vertex group $V_m$ in $G_{r,p}$ with $m \geq 2$ and let $c$ be its diagonal element. There must be at least one vertex group of this type, for otherwise $P$ would be a permutation matrix with Perron-Frobenius eigenvalue 1. For each $i$ choose positive and negative snowflake words $w_i^+$ and $w_i^-$ representing $c^i$. Then define $w_i = w_i^+ (w_i^-)^{-1}$
and \( n_i = |w_i| \). Note that \( C_0 2^{-\alpha} |w_i|^{\alpha} \leq i \leq C_1 2^{-\alpha} |w_i|^{\alpha} \) by Proposition 4.5. It follows that the sequence \( (n_i) \) tends to infinity, and that it is exponentially bounded:
\[
\frac{n_{i+1}}{n_i} \leq \frac{(n_{i+1})^\alpha}{(n_i)^\alpha} \leq \frac{(i+1) C_1}{C_0} \frac{1}{i} \leq 2C_1 \frac{1}{C_0}
\]
for \( i \geq 1 \).

Next we find embedded disks \( \Delta_i \) in \( \tilde{X}_{r,P} \) with boundary words \( w_i \) and estimate their areas. Each \( \Delta_i \) is made of two disks \( \Delta_i^+ \) and \( \Delta_i^- \) with boundary words \( w_i^+ e^{-i} \) and \( e^i (w_i^-)^{-1} \) respectively, joined along the boundary arcs labeled \( e^{-i}, e^i \).

The disk \( \Delta_i^\pm \) is a union of embedded disks in vertex spaces \( \tilde{X}_{m_i} \) and pieces of strips joining them. Consider the curve shortening process that transforms \( e^i \) into \( w_i^\pm \). To build \( \Delta_i^\pm \) simply fill the central region shown in figure 3 with the embedded disk from figure 2(b). Then fill each strip with either \( [i/p] \) or \( \lfloor i/p \rfloor \) copies of the 2-cell with the appropriate boundary word \( s_j c_{\alpha(i)} q s_j^{-1} a_j^{-p} \), and repeat the procedure. The resulting disk is an union of embedded disks in \( \tilde{X}_{r,P} \) joined along boundary arcs, with no folding along these arcs. Since each strip separates \( \tilde{X}_{r,P} \), one can see inductively (on the number of strips crossed by \( \Delta_i^\pm \)) that \( \Delta_i^\pm \) is embedded. For the same reason, it suffices to note that no folding occurs when \( \Delta_i^+ \) and \( \Delta_i^- \) are joined together to conclude that \( \Delta_i \) is embedded. Figure 4 shows an example of a disk \( \Delta_i \) with boundary word \( w_i \).

To estimate the area of \( \Delta_i \) consider the central region in \( \Delta_i^+ \) adjacent to \( \Delta_i^- \). By Remark 3.4 this subdisk of \( \Delta_i \) has area \( (m-1)i^2 \geq i^2 \). Then since \( i \geq C_0 2^{-\alpha} n_i^{\alpha} \) (as observed above) we conclude that
\[
\text{Area}(\Delta_i) \geq (C_0^2 4^{-\alpha}) n_i^{2\alpha}
\]
and therefore \( \delta(n_i) \geq (C_0^2 4^{-\alpha}) n_i^{2\alpha} \).

The upper bound. Suppose a word \( w \) represents an element of a vertex group \( V_m \). The graph of groups structure of \( G_{r,P} \) yields a decomposition of \( w \) as \( w_1 \cdots w_k \) where each \( w_i \) is either an element of \( V_m \), or begins with \( s_j^\pm \) and ends with \( s_j^+ \) for some \( j \). These latter cases occur when the path described by \( w \) leaves the vertex space \( \tilde{X}_m \) and then returns again over a strip in \( \tilde{X}_{r,P} \).

Recall that a strip in \( \tilde{X}_{r,P} \) has sides labeled \( a_i \) and \( c_{\alpha(i)} \). The next lemma shows that a geodesic (in the generators \( \{ a_1, \ldots, a_n, s_1, \ldots, s_n \} \)) can only enter a strip from (and return to) the \( a_i \)-side.

**Lemma 5.2.** Let \( w \) be a geodesic in \( G_{r,P} \) representing an element of a vertex group \( V_m \). Then \( w \) is a product of subwords \( w_1 \cdots w_k \) where each \( w_i \) is a power of a generator \( a_j \), or begins with \( s_j \) and ends with \( s_j^{-1} \) (for some \( j \)) and represents a power of \( a_j \).

**Proof.** Let \( w' \subset w \) be an innermost word that begins with \( s_j^{-1} \) and ends with \( s_{\ell} \) (for some \( \ell \)) and whose corresponding path in \( \tilde{X}_{r,P} \) has endpoints in the same vertex space.
Figure 4. A snowflake disk based on the matrix $P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$.
have the form $s_\ell v s_\ell^{-1}$ after all. Now rearrange the subwords so that $s_\ell v s_\ell^{-1}$ occurs last. Again $w$ can be shortened by replacing $u$ with this rearranged word and then cancelling $s_\ell^{-1} s_\ell$ at the end. \qed

**Proposition 5.3.** Let $c$ be the diagonal element of one of the vertex groups in $G_{r,p}$. Then for every $N$ there is a snowflake word $w_{sf}$ and a geodesic $w_{geo}$, both representing $c^N$, with $|w_{sf}| \leq rN_0 |w_{geo}|$.

**Proof.** The proof is by induction on $|N|$. Let $w$ be a geodesic representing $c^N$. We shall apply Lemma 3.1 inductively to rearrange and modify $w$ into two words, a geodesic $w_{geo}$ and a positive snowflake word $w_{sf}$. The two constructions are identical except at the base of the induction, which involves only certain segments of length at most $rN_0$.

Let $a_{i_1}, \ldots, a_{i_m}$ be the standard generators (in order) of the vertex group $V_m$ containing $c$. If $|N| \leq N_0$ then define $w_{geo} = w$ and $w_{sf} = a_{i_1}^N \cdots a_{i_m}^N$. The desired conclusion holds in this case since $r > m$.

Suppose next that $|N| > N_0$. By Lemma 5.2 we can write $w$ as $w_1 \cdots w_k$ where each subword has the form $a_{j}^{N_j}$ or $s_j u_j s_j^{-1}$. In the latter case $s_j u_j s_j^{-1}$ represents a power of $a_j$.

By Lemma 3.1 we can permute the subwords $w_\ell$ of $w$ to arrange that those representing powers of $a_{i_1}$ come first, those representing powers of $a_{i_2}$ occur next, and so on. The resulting word is still a geodesic representing $c^N$. Note that two subwords cannot both be of the form $s_i u_j s_i^{-1}$ since they could be made adjacent, and then a cancellation of $s_i^{-1} s_i$ would be possible. Hence we can arrange for $w$ to have the form

$$w = (s_{i_1} u_1 s_{i_1}^{-1})(a_{i_1}^{N_{i_1}})(s_{i_2} u_2 s_{i_2}^{-1})(a_{i_2}^{N_{i_2}}) \cdots (s_{i_m} u_m s_{i_m}^{-1})(a_{i_m}^{N_{i_m}})$$

(5.4)

where each $s_i u_j s_i^{-1}$ represents a power of $a_{i_j}$. Next observe that $|N_j| < p$ for all $j$, since otherwise a subword of the form $s_{i_j}^{-1} a_{i_j}^{\pm p}$ could be replaced by a word of the form $a_{i_1}^{\pm q} \cdots a_{i_m}^{\pm q} s_{i_j}^{-1}$ (that is, $c_{\sigma(j)}^{-1} s_{i_j}^{-1}$ expressed in the standard generators). Here $m'$ is a row sum of $P$ and so $r > m'$, making the new word shorter than $w$.

Recall that $u_j$ represents a power of $c_{\sigma(j)}$. By Lemma 3.1 the power of $a_{i_j}$ represented by $s_i u_j s_i^{-1}$ is $N - N_j$, and so $u_j$ represents $c_{\sigma(j)}^{(N - N_j)/r}$. Recall that $N_0 > p$, hence $|N| > p > |N_j|$. Then since $r > 2$ it follows that $|(N - N_j)/r| < |N|$.

By induction $c_{\sigma(j)}^{(N - N_j)/r}$ is represented by a geodesic $(u_j)_{geo}$ and a positive snowflake word $(u_j)_{sf}$ satisfying the conclusion of the lemma. Define $w_{geo}$ and $w_{sf}$ by replacing each subword $u_j$ in (5.4) by $(u_j)_{geo}$ or $(u_j)_{sf}$ accordingly. Then the desired conclusion also holds for $w_{geo}$ and $w_{sf}$, since they agree except in the subwords $(u_j)_{geo}$ and $(u_j)_{sf}$. \qed
Corollary 5.5 (Edge group distortion). Given r and P there is a positive constant D with the following property. If c is a diagonal element and w is a word representing $c^N$ then $|N| \leq D |w|^\alpha$.

Proof. It suffices to consider the case when w is a geodesic. Apply Proposition 5.3 to obtain the geodesic $w_{\text{geo}}$ and snowflake word $w_{\text{sf}}$ representing $c^N$ with $|w_{\text{sf}}| \leq rN_0 |w_{\text{geo}}|$. Then Proposition 4.5 implies $|N| \leq C_1 |w_{\text{sf}}|^{\alpha} \leq C_1 (rN_0)^{\alpha} |w_{\text{geo}}|^{\alpha}$. \[\square\]

The statement and proof of the next proposition are similar to those of Proposition 3.2 of [3].

Proposition 5.6 (Area bound). Given r and P there is a positive constant E with the following property. If w is a word in $G_{r,P}$ representing $x^N$ for some N, where x is either a generator $a_i$ or the diagonal element of one of the vertex groups, then area$(wx^{-N}) \leq E |w|^{2\alpha}$.

The case $N = 0$ establishes the upper bound of Theorem A.

Proof. We argue by induction on $|w|$. We shall prove the statement with $E = (3/2)r^2D^2$ (D given by Corollary 5.5). Let c denote the diagonal element of the vertex group $V_m$ containing x.

Write w as $w_1 \cdots w_k$ where each $w_i$ has the form $a_j^{N_i}$ or is a word beginning in $s_j^{\pm 1}$ and ending in $s_j^{\mp 1}$. In the latter cases $w_i$ represents an element of the form $c_j^{N_i}$ or $a_j^{N_i}$. Let $I_c$ and $I_a$ be the sets of indices for which these two cases occur, and let $w'$ be the word obtained from w by replacing each subword $w_i$ of this type with the appropriate word $c_j^{N_i}$ or $a_j^{N_i}$. Then $w'$ is a word in the standard generators of $V_m$ (and the diagonal element) representing $x^N$, of length $\sum_i N_i$.

By Lemma 3.5 we have area$(w'x^{-N}) \leq 3 \sum_{i<j} N_i N_j$. To estimate each $N_i$ we use Corollary 5.5 as follows. If $i \in I_c$ then $w_i$ represents $c_j^{N_i}$ and Corollary 5.5 gives $N_i \leq D |w_i|^\alpha$. If $i \in I_a$ then $w_i = s_j^{-1}u_is_j$ for some $u_i$ representing $c_{\sigma(h_i)}^{N_i/r}$ (because $w_i$ represents $a_j^{N_i}$). Then by Corollary 5.5 we have $N_i/r \leq D(|w_i| - 2)^\alpha \leq D |w_i|^\alpha$, so $N_i \leq rD |w_i|^\alpha$. Finally if $i \not\in (I_c \cup I_a)$ then $N_i = |w_i| \leq |w_i|^\alpha$. Putting these observations together we have

$$\text{area}(w'x^{-N}) \leq 3r^2D^2 \sum_{i<j} |w_i|^\alpha |w_j|^\alpha. \quad (5.7)$$

Next we use the induction hypothesis and Corollary 5.5 to bound area$(ww'^{-1})$. First note that area$(ww'^{-1}) \leq \sum_{i \in I_c} \text{area}(w_i c^{-N_i}) + \sum_{i \in I_a} \text{area}(w_i a_j^{-N_i})$.

If $i \in I_c$ then $w_i = s_j^{-1}u_is_j$ where $u_i$ represents $a_j^{-rN_i}$. Applying the induction hypothesis to $u_i$ we have area$(u_i a_j^{-rN_i}) \leq (3/2)r^2D^2(|w_i| - 2)^{2\alpha}$. The strip
$s_{j_i}^{-1} a_{j_i} r N_i s_{j_i} c^{-N_i}$ has area $N_i/q \leq (D/q) |w_i|^\alpha \leq D |w_i|^\alpha$, by Corollary 5.5. Thus

$$\text{Area}(w_i c^{-N_i}) \leq (3/2)r^2 D^2 (|w_i| - 2)^{2\alpha} + D |w_i|^\alpha$$

$$\leq (3/2)r^2 D^2 (|w_i| - 2)^{2\alpha} + |w_i|^\alpha$$

(5.8)

The last inequality above uses the fact that for numbers $x \geq 0$ one has $(x + 2)^{2\alpha} \geq x^{\alpha} (x + 2)^{2\alpha} + 2^\alpha (x + 2)^\alpha \geq x^{2\alpha} + (x + 2)^\alpha$.

If $i \in I_a$ then $w_i = s_{j_i} u_i s_{j_i}^{-1}$ where $u_i$ represents $c_{\sigma(j_i)}^{N_i/r}$. Applying the induction hypothesis to $u_i$ we have $\text{Area}(u_i c_{j_i}^{-N_i/r}) \leq (3/2)r^2 D^2 (|w_i| - 2)^{2\alpha}$. The strip $s_{j_i} c_{j_i}^{N_i/r} s_{j_i}^{-1} a_{j_i}^{-N_i}$ has area $(N_i/r)/q \leq (D/q)(|w_i| - 2)^\alpha \leq D(|w_i| - 2)^\alpha$, by Corollary 5.5. Therefore

$$\text{Area}(w_i a_{j_i}^{-N_i}) \leq (3/2)r^2 D^2 (|w_i| - 2)^{2\alpha} + D(|w_i| - 2)^\alpha$$

$$\leq (3/2)r^2 D^2 (|w_i| - 2)^{2\alpha} + (|w_i| - 2)^\alpha$$

(5.9)

Combining (5.8) and (5.9) we then have

$$\text{Area}(w w' r^{-1}) \leq \sum_{i \in I_a \cup I_a} (3/2)r^2 D^2 |w_i|^{2\alpha} \leq \sum_i (3/2)r^2 D^2 |w_i|^{2\alpha}.$$  

(5.10)

Finally, adding (5.7) and (5.10) together gives the desired result:

$$\text{Area}(w x^{-N}) \leq (3/2)r^2 D^2 \left( \sum_i |w_i|^{\alpha} \right)^2$$

$$\leq (3/2)r^2 D^2 \left( \sum_i |w_i|^{\alpha} \right)^{2\alpha} = (3/2)r^2 D^2 |w|^{2\alpha}. \quad \square$$

6. SUSPENSION AND SNOWFLAKE BALLS

Throughout this section $P$ denotes a non-negative $R \times R$ integer matrix with Perron-Frobenius eigenvalue $\lambda$, and $r$ is an integer which is strictly greater than the largest row sum of $P$. In this section, we give an explicit description of the suspended snowflake groups $\Sigma G_{r,P}$ and the 3-dimensional $K(\Sigma G_{r,P}, 1)$ spaces $X^3_{r,P}$. Then we describe snowflake balls $B^3_{r,P}$ which embed in the universal cover of $X^3_{r,P}$ and estimate their boundary areas. We show how to iterate this suspension procedure to obtain groups $\Sigma^k G_{r,P}$ and $(k + 2)$-dimensional spaces $X^{k+2}_{r,P}$. Lastly we define higher-dimensional snowflake balls and estimate their boundary volumes.

**Remark 6.1.** In order to realize the exponents $1 + 1/k$ (the endpoints of the intervals in figure 1, which are omitted otherwise) we add the free abelian group $\mathbb{Z}^2$ to the class
of snowflake groups $G_{r,p}$. We endow $\mathbb{Z}^2$ with snowflake structure as follows

$$\mathbb{Z}^2 = \langle a_1, a_2, c \mid a_1a_2 = c = a_2a_1 \rangle$$

and use the corresponding presentation 2-complex $X$ in place of $X_{r,p}$. There is no matrix $P$ associated to the group $\mathbb{Z}^2$, and so the only condition that we impose on the integer $r$ is that $r \geq 2$. Since there are no stable letters $s_i$, we define the snowflake words to be the commutators $w_i = [a_i^r, a_i^s]$ and define the snowflake disks $B_i^2 = \Delta_r$, to be the unique embedded disks in $X$ with boundary $w_i$.

In the discussions that follow, whenever we talk about snowflake groups $G_{r,p}$, we shall always include $\mathbb{Z}^2$, and whenever we use the complexes $X_{r,p}$ we shall always include the presentation 2-complex $X$ for $\mathbb{Z}^2$ described above.

**The groups** $\Sigma G_{r,p}$. Let $\phi: G_{r,p} \to G_{r,p}$ be the monomorphism which takes each $a_i$ to $a_i^r$ and each $s_i$ to itself. The group $\Sigma G_{r,p}$ is defined to be the associated multiple HNN extension with stable letters $u_1$ and $v_1$:

$$\Sigma G_{r,p} = \langle G_{r,p}, u_1, v_1 \mid u_1gu_1^{-1} = \phi(g), v_1gv_1^{-1} = \phi(g) \ (g \in G_{r,p}) \rangle.$$

**The spaces** $X_{r,p}^3$. These spaces will have fundamental group $\Sigma G_{r,p}$. Recall that $X_{r,p}$ is a 2-dimensional $K(G_{r,p}, 1)$ space. There is a cellular map $\Phi: X_{r,p} \to X_{r,p}$ which induces the map $\phi$ on the fundamental group. It maps the 1-cells labeled $s_i$ homeomorphically to themselves, maps the 1-cells labeled $a_i$ to themselves by degree $r$ maps, and maps each 2-cell in the obvious manner; the image of each triangular 2-cell has combinatorial area $r^2$, and the image of the remaining 2-cells (which have an $s_i$ edge in their boundaries) have combinatorial area $r$. The 3-complex $X_{r,p}^3$ with fundamental group $\Sigma G_{r,p}$ is obtained by taking two copies of the mapping torus of the map $\Phi$ and identifying them along a copy of $X_{r,p}$. From this perspective it is easy to see that $X_{r,p}^3$ is aspherical; each mapping torus is aspherical since $X_{r,p}$ is an aspherical 2-complex, and since $\Phi$ induces the monomorphism $\phi$ in $\pi_1$. We give more details of the cell structure of $X_{r,p}^3$ below.

Start with the 2-complex $X_{r,p}$ and form two copies of $X_{r,p} \times [0,1]$. Each copy is given the product cell structure, in which each $k$-cell of $X_{r,p}$ gives rise to a $(k+1)$-cell in $X_{r,p} \times (0,1)$. The “bottom” side $X_{r,p} \times \{0\}$ keeps its original cell structure and the “top” $X_{r,p} \times \{1\}$ is subdivided by pulling back under $\Phi$ the cell structure of $\Phi(X_{r,p})$. That is, each triangular 2-cell in a vertex space of $X_{r,p}$ is subdivided into $r^2$ triangles, and each edge space 2-cell (bearing the boundary label $s_jc_{\sigma(j)}s_j^{-1}a_j^r$) is subdivided into $r$ copies of the same cell.

The vertical 1-cells of the two copies of $X_{r,p} \times [0,1]$ are labeled $u_1$ and $v_1$ respectively, oriented from $X_{r,p} \times \{1\}$ to $X_{r,p} \times \{0\}$. Finally to form $X_{r,p}^3$ one attaches the bottom of each piece to $X_{r,p}$ by the identity, and the top by the map $\Phi$. Figures 5 and 6 illustrate the two types of 3-cell occurring in $X_{r,p}^3$. 
**Snowflake balls.** We define 3-dimensional balls $B^3_r$ in $\tilde{X}^3_{r,P}$ in a similar fashion to the snowflake disks constructed in section 5. An essential difference, however, is that now $r$ is an integer, and the observations of Remark 4.1 apply. That is, snowflake disks of diameter $r^i$ are unique, and the corresponding snowflake words have no "remainder" terms.

As in the proof of Theorem A we let $c$ be the diagonal element of a vertex group $V_m$ in $G_{r,P} \subset \Sigma G_{r,P}$ where $m \geq 2$. We let $w_i^+$ and $w_i^-$ denote respectively the (unique) positive and negative snowflake words representing $c^r_i$. (Note that the indexing here differs from that in section 5, where these words would be called $w_i^{\pm r}$.) Let $B_j^2$ be the snowflake disk bounded by $w_j = w_{i+1}^+(w_i^-)^{-1}$, with "diameter" $c^r_j$. Note that $B_j^2$ is the same as the snowflake disk $\Delta_{r,\star}$ of section 5.

For each positive integer $j$, we shall use a *stack of thickened van Kampen disks* to define an embedded 3-ball $B_j^3$ in the universal cover of $X^3_{r,P}$. Note that the universal cover of $X^3_{r,P}$ contains infinitely many embedded copies of the universal cover of $X_{r,P}$; one for each coset of $G_{r,P}$ in $\Sigma G_{r,P}$. We call two such copies adjacent if the cosets have representatives which differ by right multiplication by $u_j^1$ or $v_j^1$.

The map $\Phi: X_{r,P} \to X_{r,P}$ lifts to a map of universal covers which we also denote by $\Phi$. Consider the image $\Phi(B_j^2)$ of the embedded snowflake disk $B_j^2$. This image is again embedded, but its boundary word is $\phi(w_j)$. If we apply the curve shortening procedure once to the subword $\phi(w_j^+)$ we obtain $w_{i+1}^+$, which is the positive snowflake word for $c^{r+1}_j$. Similarly, if we apply curve shortening once to the subword $\phi(w_j^-)$ we obtain the negative snowflake word for $c^{r+1}_j$. Thus $\Phi(B_j^2)$ is a sub-diagram of $B_{i+1}^3$. 

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**Figure 5.** A triangular 3-cell (with $r = 2$)

**Figure 6.** A rectangular 3-cell
The top half of the ball $B^3_j$ is defined to be the union of the mapping cylinders of $\Phi$ with domain $B^2_i$ and codomain $B^2_{i+1}$ where $i$ ranges from 1 to $j$; the copies of $B^2_i$ are identified. This embeds in the universal cover of $X^3_{r,P}$ as follows. The disk $B^2_1$ embeds in some copy of the universal cover of $X_{r,P}$, $B^2_3$ embeds in the adjacent copy obtained by right multiplying by $u_1^{-1}$, and the mapping cylinder of $\Phi$: $B^2_1 \to B^2_3$ embeds in the universal cover of $X_{r,P}$ to interpolate between the images of $B^2_1$ and $B^2_2$. Note that this embedding is possible since the universal covering of $X^3_{r,P}$ can be described as an infinite union of mapping cylinders of $\Phi$: $\tilde{X}_{r,P} \to \tilde{X}_{r,P}$ which is encoded by the Bass-Serre tree $T$ corresponding to the multiple HNN description of $\Sigma G_{r,P}$.

We continue to add mapping cylinders of $\Phi$: $B^2_i \to B^2_{i+1}$ for $i = 2, \ldots, j$, as indicated in the top half of the schematic diagram in figure 7. The image of the union

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{schematic.png}
\caption{A schematic of the embedded ball $B^3_j$}
\end{figure}

of the first few embedded layers is shown in figure 8. In a similar fashion, we can embed a second copy of the union of mapping cylinders of $\Phi$: $B^2_i \to B^2_{i+1}$. However, this time we start from the copy of $B^2_j$ in the image of the previous union, and add the mapping cylinders in descending order (so $i = j, \ldots, 1$) and require that new copies of the universal cover of $X_{r,P}$ differ by right multiplication by $v_1^{-1}$. The image of this family is indicated in the lower half of the schematic diagram of figure 7, and the total union is the embedded ball $B^3_j$. It is easy to see that the union embeds, since each mapping cylinder embeds, and distinct mapping cylinders correspond to distinct layers in the 3-complex $\tilde{X}^3_{r,P}$. These layers are distinct, since they map to distinct edges of the Bass-Serre tree $T$ above. Finally, there is a 2-dimensional “fringe” at the equator $B^2_{j+1}$ level. We remove this fringe by simply replacing the two embeddings of $\Phi$: $B^2_j \to B^2_{j+1}$ by embeddings of $\Phi$: $B^2_j \to \Phi(B^2_j)$.

**Lemma 6.2.** Given $r$ and $P$ there is a positive constant $F$ such that $|\partial B^3_j| \leq \text{Area}(\partial B^3_j) \leq F|\partial B^3_j|$ for every $j$. 
Proof. The ball $B^3_j$ is a union of $2j$ mapping cylinders. See figure 7 for a schematic representation. Its boundary area is twice the area of the upper hemisphere. This latter area is estimated as follows.

For each $1 \leq i \leq j$, there are $|\partial B^2_i|$ vertical (conjugation by $u_1$) 2-cells, which interpolate between $\partial B^2_i$ and $\Phi(\partial B^2_j)$. This proves the first inequality, $|\partial B^2_j| \leq \text{Area}(\partial B^3_j)$.

For each $1 \leq i \leq j$ there are horizontal 2-cells which interpolate between $\Phi(\partial B^2_{i-1})$ and $\partial B^2_i$. In the case $i = 1$ there is no loop $\Phi(\partial B^2_0)$, and the horizontal 2-cells just fill the van Kampen diagram $B^2_1$. For any $i$, the horizontal 2-cell contribution to the area is bounded above by $|\partial B^2_i|$. To see this, note that the horizontal interpolation is a union of pieces of the form $s_j a_i \cdots a_{i_m} s_j^{-1} a_j^{-1}$ where $\{a_1, \ldots, a_m\}$ generates a vertex group $V_m$, and the stable letter $s_j$ conjugates the diagonal element of this vertex group to some generator $a_j$ of $G_{r,p}$. The area of this piece is $m$, and its contribution to $|\partial B^2_i|$ is $m + 2$.

Counting vertical and horizontal 2-cells for both hemispheres we obtain

$$\text{Area}(\partial B^3_j) \leq 4 \sum_{i=1}^{j} |\partial B^2_i| .$$

Proposition 4.5 implies that $|w^+_i| \leq C_0^{-1/\alpha} r^{i/\alpha}$ and so

$$4 \sum_{i=1}^{j} |\partial B^2_i| = 8 \sum_{i=1}^{j} |w^+_i| \leq 8 C_0^{-1/\alpha} \sum_{i=1}^{j} (r^{1/\alpha})^i .$$
The last term is a geometric series, and so is bounded above by \( F'(r^{1/\alpha})^j \) for a positive constant \( F' \) (independent of \( j \)). Proposition 4.5 also gives \( C_1^{-1/\alpha} r^{j/\alpha} \leq |w_k^+| \) and so

\[
\text{Area}(\partial B_j^3) \leq F'r^{j/\alpha} \leq \frac{F'}{2} C_1^{1/\alpha} |\partial B_j^2|.
\]

Now the desired (second) inequality holds by taking \( F = (F'/2)C_1^{1/\alpha}. \)

**The inductive supension procedure.** Having discussed \( \Sigma G_{r,p} \) we define further suspensions \( \Sigma^k G_{r,p} \) having \((k+2)\)-dimensional Eilenberg-MacLane spaces \( X_{r,p}^{k+2} \), and \((k+2)\)-dimensional snowflake balls \( B_j^{k+2} \subset \tilde{X}_{r,p}^{k+2} \). We assume that the group \( \Sigma^{k-1} G_{r,p} \), the space \( X_{r,p}^{k+1} \), and snowflake balls \( B_j^{k+1} \subset \tilde{X}_{r,p}^{k+1} \) have already been constructed.

First we define the groups \( \Sigma^k G_{r,p} \). Let \( \phi: \Sigma^{k-1} G_{r,p} \to \Sigma^{k-1} G_{r,p} \) be the monomorphism which sends \( a_i \) to \( a_i^r \) and which leaves fixed the stable letters \( s_i, u_i, \) and \( v_i \). We define \( \Sigma^k G_{r,p} \) to be the multiple ascending HNN extension with two stable letters \( u_k \) and \( v_k \), each acting by:

\[
\Sigma^k G_{r,p} = \langle \Sigma^{k-1} G_{r,p}, u_k, v_k | u_k g u_k^{-1} = \phi (g), v_k g v_k^{-1} = \phi (g) \ (g \in \Sigma^{k-1} G_{r,p}) \rangle.
\]

Next we define the spaces \( X_{r,p}^{k+2} \). The homomorphism \( \phi \) is induced by a cellular map \( \Phi^{k+1}: X_{r,p}^{k+1} \to X_{r,p}^{k+1} \). We define \( X_{r,p}^{k+2} \) to be the double mapping torus with monodromy \( \Phi^{k+1} \). That is, take two copies of \( X_{r,p}^{k+1} \times [0, 1] \), identify the “bottom” sides \( X_{r,p}^{k+1} \times \{0\} \) to \( X_{r,p}^{k+1} \) by the identity, and attach the “top” sides \( X_{r,p}^{k+1} \times \{1\} \) to \( X_{r,p}^{k+1} \) by the map \( \Phi^{k+1} \). The vertical 1-cells of the copies of \( X_{r,p}^{k+1} \times [0, 1] \) are labeled \( u_k \) and \( v_k \) respectively, oriented from \( X_{r,p}^{k+1} \times \{1\} \) to \( X_{r,p}^{k+1} \times \{0\} \). The resulting space \( X_{r,p}^{k+2} \) is given a cell structure exactly as in the definition of \( X_{r,p}^{k+2} \). As before, \( X_{r,p}^{k+2} \) is aspherical, has dimension \( k+2 \), and has fundamental group \( \Sigma^k G_{r,p} \).

Now we define the higher-dimensional snowflake balls. The map \( \Phi^{k+1} \) lifts to a map \( \tilde{X}_{r,p}^{k+1} \to \tilde{X}_{r,p}^{k+1} \) which we will also call \( \Phi^{k+1} \). We define \((k+2)\)-dimensional balls \( B_j^{k+2} \) of diameter \( r^j \) for each \( j \) as unions of mapping cylinders (called layers) of the map \( \Phi^{k+1} \) restricted to the \((k+1)\)-dimensional balls \( B_i^{k+1} \). These mapping cylinders are assembled as shown in figure 7, with \( B_i^{k+1} \) in place of \( B_2^2 \). More specifically, we assume inductively that \( \Phi^{k+1} \) maps \( B_i^{k+1} \) into a subcomplex of \( B_{i+1}^{k+1} \) for each \( i \). Then the upper hemisphere of \( B_j^{k+2} \) is the union of the mapping cylinders of \( \Phi^{k+1} \): \( B_i^{k+1} \to B_{i+1}^{k+1} \) where \( i \) ranges from 1 to \( j-1 \), and the mapping cylinder of \( \Phi^{k+1} \): \( B_j^{k+1} \to \Phi^{k+1}(B_j^{k+1}) \). The lower hemisphere is defined similarly, and the two are identified along \( \Phi^{k+1}(B_j^{k+1}) \). Note that the subspaces \( B_i^{k+1} - \Phi^{k+1}(B_{i-1}^{k+1}) \) of the domains of these mapping cylinders lie in the boundary of \( B_j^{k+2} \).
Recall that $\Phi^{k+2}$ maps $B^{k+2}_{i+1}$ to a subcomplex of $B^{k+1}_{i+1}$. There is an induced map $\Phi^{k+2}$ from the mapping cylinder of $\Phi^{k+1}: B^{k+1}_i \to B^{k+1}_{i+1}$ to the mapping cylinder of $\Phi^{k+1}: B^{k+1}_{i+1} \to B^{k+1}_{i+2}$; use $\Phi^{k+1} \times \text{id}$ on $B^{k+1}_i \times I$ and $\Phi^{k+1}$ on $B^{k+1}_{i+1}$. Then $\Phi^{k+2}$ maps layer $i$ of $B^{k+2}_j$ to layer $i+1$ of $B^{k+2}_{j+1}$ for any $i < j$ (in either hemisphere). These maps defined on the layers of $B^{k+2}_i$ join together to define the map $\Phi^{k+2}: B^{k+2}_j \to B^{k+2}_{j+1}$.

The balls $B^{k+2}_j$ embed into $X^{k+2}_{r,P}$ exactly as the balls $B^3_j$ embed into $X^3_{r,P}$. That is, we consider $X^{k+2}_r$ as a union of copies of the mapping cylinder of $\Phi^{k+1}: X^{k+1}_{r,P} \to X^{k+1}_{r,P}$ with the mapping parameter corresponding to right multiplication by $u_i^{-1}$ or $v_i^{-1}$. Then the embedding $B^{k+2}_j \to X^{k+2}_{r,P}$ is assembled from the embeddings $B^{k+1}_{i+1} \to X^{k+1}_{r,P}$ (for $i < j$) as shown in figure 7, with the upper hemisphere extending in the $u_k$ direction and the lower hemisphere in the $v_k$ direction. Under this embedding, the map $\Phi^{k+2}: B^{k+2}_j \to B^{k+2}_{j+1}$ described above is simply the restriction of $\Phi^{k+2}: X^{k+2}_{r,P} \to X^{k+2}_{r,P}$

For any $k$, we define the shell of a snowflake ball $B^k_j$ to be the subspace $B^k_j - \Phi^k(B^k_{j-1})$, or simply $B^k_j$ in the case $j = 1$.

**Lemma 6.3.** $\text{Vol}^k(\text{shell}(B^k_j)) \leq \text{Vol}^{k-1}(\partial B^k_j)$.

**Proof.** It suffices to show that every $k$-cell of the shell has a $(k-1)$-dimensional face contained in $\partial B^k_j$. Recall that $B^k_j$ is a union of layers, so consider the intersection of the shell with layer $i$ (in either hemisphere). This layer is a mapping cylinder $\mathcal{M}(\Phi^{k-1}: B^{k-1}_i \to B^{k-1}_{i+1})$ and its preimage in $B^{k-1}_{j-1}$ under $\Phi^{k}$ is layer $i-1$ of this smaller ball (or is empty in the case $i = 1$). Hence the intersection of the shell with layer $i$ is

$$\mathcal{M}(\Phi^{k-1}: B^{k-1}_i \to B^{k-1}_{i+1}) - \Phi^{k}(\mathcal{M}(\Phi^{k-1}: B^{k-1}_{i-1} \to B^{k-1}_i))$$

$$= \mathcal{M}(\Phi^{k-1}: B^{k-1}_i \to B^{k-1}_{i+1}) - \mathcal{M}(\Phi^{k}: \Phi^{k-1}(B^{k-1}_{i-1}) \to \Phi^{k-1}(B^{k-1}_i))$$

$$= \mathcal{M}(\Phi^{k-1}: (B^{k-1}_i - \Phi^{k-1}(B^{k-1}_{i-1})) \to (B^{k-1}_i - \Phi^{k-1}(B^{k-1}_i)))$$

if $i > 1$, and is $\mathcal{M}(\Phi^{k-1}: B^{k-1}_i \to B^{k-1}_{i+1})$ in the case $i = 1$. Either way, this part of shell($B^k_j$) is the mapping cylinder of the restriction of $\Phi^{k-1}$ to shell($B^{k-1}_i$). Hence each $k$-cell has a $(k-1)$-dimensional face in shell($B^{k-1}_i$), which is contained in $\partial B^{k}_i$. \hfill $\square$

The next result is a higher-dimensional analogue of Lemma 6.2.

**Lemma 6.4.** Given $r$, $P$, and $k \geq 3$ there is a positive constant $F_k$ such that $\text{Vol}^{k-2}(\partial B^k_i) \leq \text{Vol}^{k-1}(\partial B^k_j) \leq F_k \text{Vol}^{k-2}(\partial B^k_j)$ for every $j$.

**Proof.** We prove, for $k \geq 3$, the following two statements: there exist positive constants $E_k, F_k$ such that

$$1) \quad (2C_1^{-1/\alpha}) (r^{1/\alpha})^j \leq \text{Vol}^{k-2}(\partial B^{k-1}_j) \leq E_k (r^{1/\alpha})^j,$$

and
(2) \( \text{Vol}^{k-2}(\partial B_j^{k-1}) \leq \text{Vol}^{k-1}(\partial B_j^k) \leq F_k \text{Vol}^{k-2}(\partial B_j^{k-1}) \)
for all \( j \) (with \( C_1 \) given by Proposition 4.5). Statement (1) is a higher-dimensional analogue of Proposition 4.5 and (2) is the main statement of the lemma. The two statements are proved together by induction on \( k \).

If \( k = 3 \) then (1) follows from Proposition 4.5, with \( E_3 = 2C_0^{-1/\alpha} \). Statement (1) is given by Lemma 6.2 (with \( F_3 = F \)).

For \( k > 3 \) we prove (1) as follows. The induction hypothesis implies that
\[
\text{Vol}^{k-2}(\partial B_j^{k-1}) \leq F_{k-1} \text{Vol}^{k-3}(\partial B_j^{k-2})
\]
by (2) and \( \text{Vol}^{k-3}(\partial B_j^{k-2}) \leq E_{k-1}(r^{1/\alpha})^j \) by (1). Hence \( \text{Vol}^{k-2}(\partial B_j^{k-1}) \leq E_k(r^{1/\alpha})^j \)
with \( E_k = F_{k-1}E_{k-1} \). We also have (by induction) \( \text{Vol}^{k-2}(\partial B_j^{k-1}) \geq \text{Vol}^{k-3}(\partial B_j^{k-2}) \geq (2C_1^{-1/\alpha})(r^{1/\alpha})^j \) by (2) and (1). This establishes (1).

To prove (2) we count vertical and horizontal \((k-1)\)-cells of \( \partial B_j^k \) as in the proof of Lemma 6.2. In each hemisphere of \( B_j^k \), layer \( i \) is a copy of the mapping cylinder of \( \Phi^{k-1} : B_i^{k-1} \rightarrow B_i^{k-1} \). This layer meets \( \partial B_j^k \) in horizontal cells which are the \((k-1)\)-cells of shell(\( B_i^{k-1} \)), and vertical cells, each of which is the product of a \((k-2)\)-cell in \( \partial B_i^{k-1} \) with \( I \). This latter observation implies the first inequality of (2) (taking \( i = j \)) and also that the number of vertical cells in layer \( i \) is at most \( \text{Vol}^{k-2}(\partial B_i^{k-1}) \).

The number of horizontal cells is at most \( \text{Vol}^{k-2}(\partial B_i^{k-1}) \) by Lemma 6.3. Adding the contributions from all layers in both hemispheres, we obtain
\[
\text{Vol}^{k-1}(\partial B_j^k) \leq 4\sum_{i=1}^j \text{Vol}^{k-2}(\partial B_i^{k-1}).
\]
Statement (1) implies \( 4\sum_{i=1}^j \text{Vol}^{k-2}(\partial B_i^{k-1}) \leq 4E_k \sum_{i=1}^j (r^{1/\alpha})^j \) and the latter sum is a geometric series. Hence \( \text{Vol}^{k-1}(\partial B_j^k) \leq F_k'(r^{1/\alpha})^j \) for some constant \( F_k' \). Now (1) implies that \( \text{Vol}^{k-1}(\partial B_j^k) \leq \frac{(F_k'/2)}{C_1^{1/\alpha}} \text{Vol}^{k-2}(\partial B_j^{k-1}) \), establishing (2) with \( F_k = (F_k'/2)C_1^{1/\alpha} \).

\[\square\]

7. Proof of Theorem B

We will establish upper and lower bounds for the order \( k \) Dehn functions \( \delta^{(k)}(x) \) of the groups \( \Sigma^{k-1}G_{r,P} \) and these will be equal. As usual \( \lambda \) denotes the Perron-Frobenius eigenvalue of \( P \) and \( \alpha = \log_\lambda(r) \). In the case of \( \Sigma^{k-1}\mathbb{Z}^2 \) we define \( \alpha = 1 \).

The lower bound. As in the proof of Theorem A, we show that the embedded snowflake balls \( B_i^{k+1} \subset \tilde{X}_{r,P}^{k+1} \) have the correct proportions and are numerous enough to determine \( \delta^{(k)}(x) \) from below.

First we show that for every \( k \geq 1 \) there is a constant \( G_k \) such that
\[
\text{Vol}^{k+1}(B_i^{k+1}) \geq G_k \text{Vol}^k(\partial B_i^{k+1})^{2^\alpha}\tag{7.1}
\]
for all $i$. The case $k = 1$ was proved in (5.1) with $G_1 = (C_0)^{24^{-\alpha}}$. For $k > 1$ we proceed by induction. Note that $\text{Vol}^{k+1}(B^{k+1}_i) \geq \text{Vol}^k(B^k_i)$ since the latter is the volume of the mapping cylinder of $\Phi^k: B^k_i \to \Phi^k(B^k_i)$ inside $B^{k+1}_i$. We also have $\text{Vol}^k(B^k_i) \geq G_{k-1}\text{Vol}^{k-1}(\partial B^k_i)^{2\alpha}$ by the induction hypothesis. Lemma 6.4 implies that $G_{k-1}\text{Vol}^{k-1}(\partial B^k_i)^{2\alpha} \geq G_{k-1}F_{k+1}^{-2\alpha}\text{Vol}^k(\partial B^{k+1}_i)^{2\alpha}$. Equation (7.1) now follows by taking $G_k = G_{k-1}F_{k+1}^{-2\alpha}$.

Next we show that for each $k \geq 2$ the sequence $(\text{Vol}^k(\partial B^{k+1}_i))_i$ is exponentially bounded and tends to infinity. Consider first the case $k = 2$. Then we have

$$\frac{\text{Vol}^2(\partial B^3_{i+1})}{\text{Vol}^2(\partial B^3_i)} \leq \frac{F|\partial \Delta_{r+1}|}{|\partial \Delta_r|^\alpha} \leq \frac{F|\partial \Delta_{r+1}|^\alpha}{C_0} \leq \frac{Fr^{i+1}C_1}{C_0}$$

where the first inequality holds by Lemma 6.2, the second since $\alpha \geq 1$, and the third by Proposition 4.5. Thus, the sequence is exponentially bounded. For $k > 2$ we have

$$\frac{\text{Vol}^k(\partial B^{k+1}_{i+1})}{\text{Vol}^k(\partial B^{k+1}_i)} \leq \frac{F_{k+1}\text{Vol}^{k-1}(\partial B^k_{i+1})}{\text{Vol}^{k-1}(\partial B^k_i)}$$

by Lemma 6.4 and so $(\text{Vol}^k(\partial B^{k+1}_i))_i$ is exponentially bounded, by induction on $k$. It tends to infinity because

$$\text{Vol}^k(\partial B^{k+1}_i) \geq \text{Vol}^2(\partial B^3_i) \geq |\partial \Delta_r| \geq 2C_1^{-1/\alpha}(r^{1/\alpha})^i$$

by Lemma 6.4, Lemma 6.2, and Proposition 4.5. Now, using Remarks 2.1 and 2.5, we conclude from (7.1) that $\delta^k(x) \geq x^{2\alpha}$.

**The upper bound.** To establish the upper bound we must work with Dehn functions $\delta^M_G(x)$ modeled on arbitrary manifolds $M$ with boundary, as defined in section 2. Recall that the order of $\delta^M_G(x)$ is the dimension of $\partial M$, and $\delta^M_G(x)$ agrees with the usual order $k$ Dehn function when $M$ is the $(k+1)$-dimensional ball.

A function $F: \mathbb{N} \to \mathbb{N}$ is superadditive if $F(a + b) \geq F(a) + F(b)$ for all $a, b$.

**Theorem 7.2.** Let $G$ be a group of type $F_n$ and geometric dimension at most $n$, and fix a finite aspherical $n$-complex $X$ with fundamental group $G$. Suppose that every order $n-1$ Dehn function $\delta^M_G(x)$ (defined with respect to $X$) satisfies

$$\delta^M_G(x) \leq F(x)$$

where $F: \mathbb{N} \to \mathbb{N}$ is non-decreasing. Let $H$ be a multiple ascending HNN extension of $G$. Then $H$ is of type $F_{n+1}$, has geometric dimension at most $n + 1$, and

$$\delta^M_H(x) \leq F(x)$$

for every order $n$ Dehn function $\delta^M_H(x)$.

In the hypotheses we are including Dehn functions $\delta^M_G(x)$ where $M$ has more than one connected component (otherwise we should add that $F$ is superadditive). The order $n$ Dehn functions in the conclusion are defined with respect to a fixed complex.
$Y$ constructed in the proof of the theorem. We work with specific functions rather than equivalence classes, for otherwise the Dehn functions $\delta_H^M(x)$ could conceivably fail to have a uniform bound in the same class.

**Proof.** First we define the finite $(n + 1)$-dimensional complex $Y$ with fundamental group $H$ in the usual way. Suppose the multiple ascending extension has $k$ stable letters. Form $k$ copies of $X \times [-1, 1]$, give each the product cell structure, and attach each copy of $X \times \{ -1 \}$ to $X$ by the identity map. Then attach each copy of $X \times \{ 1 \}$ to $X$ by the appropriate monodromy map, and call the resulting space $Y$. Let $Z \subset Y$ be the union of the spaces $X \times \{ 0 \}$. There are natural projections along the fibers $p_0: Z \to X$ and $p_1: Z \to X$ which factor through $Z \times \{ -1 \}$ and $Z \times \{ 1 \}$ respectively. Let $\tilde{Y}$ be the universal cover of $Y$ and let $\tilde{X}$ and $\tilde{Z}$ be the preimages of $X$ and $Z$ in $\tilde{Y}$. The projections $p_i$ lift to projections $\tilde{p}_i: \tilde{Z} \to \tilde{X}$ along fibers. Note that each component of $\tilde{X}$ and $\tilde{Z}$ is a copy of the universal cover of $X$, and in fact $p_0: \tilde{Z} \to \tilde{X}$ is a homeomorphism.

Each open $k$-cell $\sigma^k$ in $\tilde{Z} \times (-1, 1) \subset \tilde{Y}$ has the form $\sigma^{k-1} \times (-1, 1)$ where $\sigma^{k-1}$ is a $(k - 1)$-cell in $\tilde{X}$, and the restriction of $p_0$ to $\sigma^k \cap \tilde{Z}$ is simply projection onto the first factor. Since $\tilde{Z}$ is not a subcomplex of $\tilde{Y}$, we measure volume in $\tilde{Z}$ by passing to $\tilde{X}$ via $p_0$. The description of $p_0$ just given leads to the following observation: if $f: M^k \to \tilde{Y}$ is a singular combinatorial map and $N = f^{-1}(\tilde{Z})$ and $M_0 = f^{-1}(\tilde{X})$ then

$$\text{Vol}_k(f) = \text{Vol}^{k-1}(p_0 \circ f|_N) + \text{Vol}^k(f|_{M_0})$$

(7.3)

where the left hand side is volume in $\tilde{Y}$ and the right hand side is volume in $\tilde{X}$.

Now suppose that $M$ is a compact orientable $(n + 1)$-manifold with boundary and let $g: M \to \tilde{Y}$ be a least-volume map with boundary $f = g|_{\partial M}$. We can arrange that $N = g^{-1}(\tilde{Z})$ is a properly embedded codimension one submanifold with a product neighborhood $N \times [-1, 1] \subset M$ such that $g^{-1}(\tilde{Z} \times (-1, 1)) = N \times (-1, 1)$. The product structure on $N \times [-1, 1]$ may be chosen so that $g|_{N \times (-1,1)}$ is the map $g|_{N \times \text{id}}$. Note that $N$ may have several connected components.

We claim that $\text{Vol}^n(p_0 \circ g|_N)$ is smallest among all $N$-fillings of $p_0 \circ f|_{\partial N}: \partial N \to \tilde{X}$. Assuming this for the moment, the theorem is proved as follows. We have $\text{Vol}^{n+1}(g) = \text{Vol}^n(p_0 \circ g|_N)$ by (7.3) because $\tilde{X}$ has dimension $n$. Then $\text{Vol}^n(p_0 \circ g|_N) = \text{FVol}^N(p_0 \circ f|_{\partial N})$ by the claim, and the latter is at most $\delta_G^N(\text{Vol}^{n-1}(p_0 \circ f|_{\partial N}))$ by the definition of $\delta_G^N$. Equation (7.3) implies that $\delta_G^N(\text{Vol}^{n-1}(p_0 \circ f|_{\partial N})) \leq \delta_G^N(\text{Vol}^n(f))$. Then we have the desired bound

$$\text{FVol}^M(f) = \text{Vol}^{n+1}(g) \leq \delta_G^N(\text{Vol}^n(f)) \leq F(\text{Vol}^n(f))$$

by the main hypothesis and we conclude that $\delta_H^M(\text{Vol}^n(f)) \leq F(\text{Vol}^n(f))$. Since $\text{Vol}^n(f)$ was arbitrary and $F$ is non-decreasing, we have $\delta_H^M(x) \leq F(x)$ for all $x$. 
Now we return to the claim that $\text{Vol}^n(p_0 \circ g|_N) = \text{FVol}^N(p_0 \circ f|_{\partial N})$. We show that if $p_0 \circ g|_N$ is not a least-volume filling of $p_0 \circ f|_{\partial N}$ then $g$ can be modified rel $\partial M$ to a map of smaller volume, contradicting the choice of $g$.

Let $M_0 = g^{-1}(\tilde{X})$, and note that the frontier of $M_0$ in $M$ is $N \times \{-1\} \cup N \times \{1\}$. These two subsets of $\partial M_0$ will be denoted $M_0^-$ and $M_0^+$ respectively.

Suppose $\text{Vol}^n(h) < \text{Vol}^n(p_0 \circ g|_N)$ for some map $h: N \to \tilde{X}$ with $h|_{\partial N} = p_0 \circ f|_{\partial N}$. Form a new copy of $M$ in which $N \times (-1,1)$ is replaced by $N \times (-2,2)$. Define a new map $g': M \to \tilde{Y}$ by letting $g'$ be $g$ on $M_0$, $(p_0^{-1} \circ h) \times \text{id}$ on $N \times (-1,1)$, and by extending to the remaining regions as follows. Note that $(p_0^{-1} \circ h) \times \text{id}$ extends continuously to $N \times [-1,1]$ as $h$ on $N \times \{-1\}$ and as $p_1 \circ p_0^{-1} \circ h$ on $N \times \{1\}$. Since each component of $\tilde{X}$ is contractible the maps $p_1 \circ p_0^{-1} \circ h$ and $g|_{M_0}$ are homotopic rel $\partial N$. We let $g'|_{N \times [1,2]}: N \times [1,2] \to \tilde{X}$ be such a homotopy. Similarly $g'|_{N \times [-2,-1]}$ is defined to be a homotopy in $\tilde{X}$ from $g|_{M_0}$ to $h$, fixing $\partial N$ pointwise. This defines the map $g': M \to \tilde{Y}$.

Now collapse each fiber of $\partial N \times [1,2]$ and $\partial N \times [-2,-1]$ to a point, to obtain a new copy of $M$ with a map $g'': M \to \tilde{Y}$ which agrees with $g$ on $\partial M$. Note that all of $M - (N \times (-1,1))$ maps by $g''$ into $\tilde{X}$ and $g''|_{N \times (-1,1)} = (p_0^{-1} \circ h) \times \text{id}$. Hence by (7.3) we have $\text{Vol}^{n+1}(g'') = \text{Vol}^n(h) < \text{Vol}^n(p_0 \circ g|_N) = \text{Vol}^{n+1}(g)$, a contradiction. \hfill $\square$

**Proposition 7.4.** If $G$ is finitely presented, $\delta_G(x) \leq F(x)$ with $F(x)$ superadditive, and $M$ is a compact orientable 2-manifold with boundary, then $\delta^M_G(x) \leq F(x)$.

In particular if $\delta_G(x)$ is superadditive then $\delta^M_G(x) \leq \delta_G(x)$ for every 2-manifold $M$.

**Proof.** If $M$ is connected with one boundary component then let $g: M \to D^2$ be a quotient map which collapses the complement of a collar neighborhood of $\partial M$ to a point. Then $\text{Area}(g \circ q) = \text{Area}(g)$ for any map $g: D^2 \to \tilde{X}$, and we have $\delta^M_G(x) \leq \delta_G(x) \leq F(x)$.

If $N$ is closed then $\delta^M_G|_{\partial M \cup N}(x) = \delta^M_G(x)$ since $N$ may be assigned zero area by mapping it to a point. So without loss of generality assume that $M$ has no closed components. For each component $M'$ of $M$ there is a quotient map to a connected, simply connected space $Z'$ which is a union of disks (one for each boundary component of $M'$) and arcs joining them. Taking a union of such spaces and maps, we have a quotient map $M \to Z$. Every map $D^2 \sqcup \cdots \sqcup D^2 \to \tilde{X}$ extends to a map $Z \to \tilde{X}$ which yields (by composition) a map $M \to \tilde{X}$ with the same area. Hence $\delta^M_G(x) \leq \delta^2_G \sqcup \cdots \sqcup \delta^2_G(x)$. Now superadditivity of $F$ implies $\delta^2_G \sqcup \cdots \sqcup \delta^2_G(x) \leq F(x)$. \hfill $\square$

**Corollary 7.5.** Let $G$ be a finitely presented group of geometric dimension 2 with $\delta_G(x)$ equivalent to a superadditive function. Let $H$ be obtained from $G$ by performing $n$ iterated multiple ascending HNN extensions. Then $\delta^{(n+1)}_H(x) \leq \delta_G(x)$.

The upper bound of Theorem B follows immediately, by Theorem A.
Proof. Let $F_0(x)$ be superadditive where $F_0(x) \simeq \delta_G(x)$. Then $\delta_G(x) \leqslant F(x) = CF_0(Cx) + Cx$ for some $C$ and $F(x)$ is superadditive. The result now follows directly from Proposition 7.4 and Theorem 7.2.

The case $n = 1$ of the Corollary was proved by Wang and Pride [14], using a more direct method.

8. PRODUCTS WITH $\mathbb{Z}$

In this section we determine higher order Dehn functions of $G \times \mathbb{Z}$ for certain groups $G$. In these cases the geometry of $G \times \mathbb{Z}$ is accurately represented by embedded balls which are products of optimal balls in $G$ with intervals, with suitably chosen lengths.

To establish an upper bound for Dehn functions of $G \times \mathbb{Z}$ we need the following result which is similar to Theorem 7.2. The proof is based on [1, Theorem 6.1].

**Theorem 8.1.** Let $G$ be a group of type $F_n$ and geometric dimension at most $n$, and fix a finite aspherical $n$-complex $X$ with fundamental group $G$. Suppose that every order $n - 1$ Dehn function $\delta^M_G(x)$ satisfies

$$\delta^M_G(x) \leqslant Cx^s$$

for some $C > 0$ and $s > 1$. Then

$$\delta^M_{G \times \mathbb{Z}}(x) \leqslant C^{1/s}x^{2 - 1/s}$$

for every order $n$ Dehn function $\delta^M_{G \times \mathbb{Z}}(x)$.

**Proof.** First note that we are in the situation of Theorem 7.2, which is valid, but no longer provides the best possible upper bound. Define $Y$, $Z$, $p_0$, and $p_1$ as in the proof of Theorem 7.2. Note that now the projections along fibers $p_0$, $p_1$: $\tilde{Z} \to \tilde{X}$ are both homeomorphisms, and $\text{Vol}^k(p_0 \circ f) = \text{Vol}^k(p_1 \circ f)$ for any $f$: $N^k \to \tilde{Z}$.

Given a compact orientable $(n+1)$-manifold $M$ with boundary, consider a map $f$: $\partial M \to \tilde{Y}$. Arrange that $L = f^{-1}(\tilde{Z})$ is a codimension one submanifold with a product neighborhood $L \times [-1,1] \subset \partial M$ such that $f^{-1}(\tilde{Z} \times (-1,1)) = L \times (-1,1)$. As before, the product structure on $L \times [-1,1]$ can be chosen so that $f|_{L \times (-1,1)}$ is the map $f|_L \times \text{id}$.

We will prove that $\delta^M_{G \times \mathbb{Z}}(x) \leqslant C^{1/s}x^{2 - 1/s}$ by induction on the number of connected components of $L$. If $L = \emptyset$ then $f(\partial M) \subset \tilde{X}$. Every component of $\tilde{X}$ is contractible so $f$ extends to a map $g$: $M \to \tilde{X}$ with $g|_{\partial M} = f$. Since $\tilde{X}$ has dimension $n$, $\text{Vol}^{n+1}(g) = 0$.

Now assume that $L \neq \emptyset$. Let $\tilde{Z}_0$ be a connected component of $\tilde{Z}$ such that $L_0 = f^{-1}(\tilde{Z}_0)$ is a non-empty union of components of $L$, and $f(L)$ lies entirely in one component of $\tilde{Y} - p_1(\tilde{Z}_0)$. (Think of $L_0$ as an innermost union of components of $L$.) Let $N_1 \subset \partial M - (L_0 \times (-1,1))$ be the union of components having boundary $L_0 \times \{1\}$.
That is, \( N_1 \) and its complement \( N_{-1} \) in \( \partial M - (L_0 \times (-1, 1)) \) map to opposite sides of \( \tilde{Z}_0 \times (-1, 1) \) in \( \tilde{Y} \), and in fact \( f(N_1) \subset p_1(\tilde{Z}_0) \subset \tilde{X} \), by the choice of \( \tilde{Z}_0 \).

Our method now is to fill \( L_0 \) with a least-volume copy of \( N_1 \) and then fill either side of \( \partial M \) efficiently by \( M \) (using the induction hypothesis) and \( N_1 \times I \). These fillings fit together to yield a filling of \( f \) by \( M \) having the required volume.

Let \( v = \text{Vol}^n(f) \) and \( u = \text{Vol}^{n-1}(p_0 \circ f|_{L_0}) \) (which is equal to \( \text{Vol}^n(f|_{L_0 \times (-1,1)}) \) by (7.3)). Let \( h: N_1 \to \tilde{X} \) be a least-volume \( N_1 \)-filling of \( p_0 \circ f|_{L_0} \). Thus, \( h|_{\partial N_1} = p_0 \circ f|_{L_0} \) and \( \text{Vol}^n(h) \leq Cu^s \). Define a new map \( f' : \partial M \to \tilde{Y} \) by first collapsing the fibers of \( L_0 \times [-1, 1] \) to points, and then sending \( N_{-1} \) by \( f \) and \( N_1 \) by \( h \). Since \( h \) is least-volume and \( L_0 \times [-1, 1] \) was collapsed we have \( \text{Vol}^n(f') \leq v - u \). Also \( (f')^{-1}(\tilde{Z}) = L - L_0 \), so by the induction hypothesis there is a map \( g_{-1} : M \to \tilde{Y} \) with \( g_{-1}|_{\partial M} = f' \) such that

\[
\text{Vol}^{n+1}(g_{-1}) \leq C^{1/s}(v - u)^{2-1/s}.
\]

Next let \( g_1 : N_1 \times [-1, 1] \to \tilde{X} \) be a homotopy which begins with \( h \) on \( N_1 \times \{-1\} \) and pushes across \( \tilde{Z}_0 \times (-1, 1) \) and then deforms within \( p_1(\tilde{Z}_0) \) to \( f|_{N_1} \), with the boundary fixed pointwise. This latter homotopy exists since \( p_1(\tilde{Z}_0) \) is contractible. Note that \( \text{Vol}^{n+1}(g_1) = \text{Vol}^n(h) \) by (7.3) since \( p_1(\tilde{Z}_0) \) has dimension \( n \).

Now join \( N_1 \subset \partial M \) to \( (N_1 \times \{-1\}) \subset N_1 \times [-1, 1] \) to get a new copy of \( M \) and a map \( g : M \to \tilde{Y} \) extending \( g_{-1} \) and \( g_1 \). Then \( g|_{\partial M} = f \) and

\[
\text{Vol}^{n+1}(g) \leq C^{1/s}(v - u)^{2-1/s} + v_h
\]

where \( v_h = \text{Vol}^n(h) \). Now \( s > 1 \) and \( v \geq u \) imply

\[
\text{Vol}^{n+1}(g) \leq C^{1/s}(v - u)v^{1-1/s} + v_h = C^{1/s}v^{2-1/s} \left( 1 - \frac{u}{v} + \frac{v_h^{(1/s) - 1}}{C^{1/s}v} \right). \tag{8.2}
\]

Recall that \( v_h = \text{Vol}^n(h) \leq \text{Vol}^n(f|_{N_1}) \leq v \) because \( h \) is least-volume. Hence

\[
1 - \frac{u}{v} + \frac{v_h^{(1/s) - 1}}{C^{1/s}v} \leq 1 - \frac{u}{v} + \frac{v_h^{(1/s) - 1}}{C^{1/s}v} = 1 - \frac{u}{v} + \frac{v_h^{1/s}}{C^{1/s}v}. \tag{8.3}
\]

The main hypothesis implies that \( v_h \leq Cu^s \), or \( v_h^{1/s} \leq C^{1/s}u \), again because \( h \) is least-volume. Thus

\[
1 - \frac{u}{v} + \frac{v_h^{1/s}}{C^{1/s}v} \leq 1 - \frac{u}{v} + \frac{u}{v} = 1. \tag{8.4}
\]

By equations (8.2), (8.3), and (8.4) we have \( \text{Vol}^{n+1}(g) \leq C^{1/s}v^{2-1/s} \) where \( v = \text{Vol}^n(g|_{\partial M}) \), which completes the proof. \( \square \)
Definition 8.5. Let $G$ be a group of type $F_{k+1}$ and geometric dimension at most $k + 1$. The order $k$ Dehn function $\delta_G^{(k)}(x)$ has embedded representatives if there is a finite aspherical $(k+1)$-complex $X$, a sequence of embedded $(k+1)$-dimensional balls $B_i \subset \widetilde{X}$, and a function $F(x) \simeq \delta_G^{(k)}(x)$, such that the sequence $(n_i) = (\text{Vol}^k(\partial B_i))$ tends to infinity and is exponentially bounded, and $\text{Vol}^{k+1}(B_i) \geq F(n_i)$ for each $i$.

The lower bounds established in this paper for various Dehn functions are all obtained by constructing embedded representatives and applying Remarks 2.1 and 2.5. In particular the order $k$ Dehn functions of $\Sigma^{k-1}G_{r,P}$ and $\Sigma^{k-1}\mathbb{Z}^2$ have embedded representatives.

Proposition 8.6. Let $G$ be a group of type $F_{k+1}$ and geometric dimension at most $k + 1$. Suppose the order $k$ Dehn function $\delta^{(k)}(x)$ of $G$ is equivalent to $x^s$ and has embedded representatives. Then $G \times \mathbb{Z}$ has order $k + 1$ Dehn function $\delta^{(k+1)}(x) \geq x^{2-1/s}$, with embedded representatives.

Proof. We establish the lower bound $\delta^{(k+1)}(x) \geq x^{2-1/s}$ for $G \times \mathbb{Z}$ as follows. Since $\delta^{(k)}_G(x)$ has embedded representatives, let $X$, $F(x)$, $B_i$, and $(n_i)$ be as in Definition 8.5; without loss of generality suppose that $F(x) = Cx^s$ for some $C > 0$. Define $m_i = 3\text{Vol}^{k+1}(B_i)$. The space $\widetilde{Y} = X \times S^1$ has fundamental group $G \times \mathbb{Z}$ and universal cover $\widetilde{Y} = \widetilde{X} \times \mathbb{R}$. Consider the $(k+2)$-dimensional balls

$$C_i = B_i \times [0,m_i/3n_i] \subset \widetilde{Y}.$$ 

The boundary of $C_i$ is $\partial B_i \times [0,m_i/3n_i] \cup B_i \times [0,m_i/3n_i]$ which implies that

$$\text{Vol}^{k+1}(\partial C_i) = m_i.$$ 

We also have $\text{Vol}^{k+2}(C_i) = \text{Vol}^{k+1}(B_i)m_i/3n_i = (m_i)^2/9n_i$ for each $i$. Since $m_i = 3\text{Vol}^{k+1}(B_i) \geq 3C(n_i)^s$ we have $(3C)^{-1/s}(m_i)^{1/s} \geq n_i$. Then

$$\text{Vol}^{k+2}(C_i) = \frac{(m_i)^2}{9n_i} \geq \left( \frac{C^{1/s}}{3^{2-1/s}} \right) (m_i)^{2-1/s}.$$ 

Note that $\widetilde{Y}$ is aspherical and has dimension $k + 2$, and so $C_i$ is a least-volume ball (cf. Remark 2.5). Therefore $\delta^{(k+1)}(m_i) \geq (C^{1/s}/3^{2-1/s})(m_i)^{2-1/s}$ for each $i$. Now it remains to check that the sequence $(m_i)$ has the required properties. It tends to infinity since $m_i \geq 3C(n_i)^s$. Also each ball $B_i \subset \widetilde{X}$ is least-volume, so there is a constant $D$ such that $m_i \leq D(n_i)^s$ for all $i$. Then $m_i/m_{i+1} \leq (D/C)(n_{i+1}/n_i)^s$, which is bounded. Now Remark 2.1 implies that $\delta^{(k+1)}(x) \geq x^{2-1/s}$. \hfill \qedsymbol

Corollary 8.7. Given $P$ with Perron-Frobenius eigenvalue $\lambda$, $r$ as in Theorem B, and $k, \ell \in \mathbb{N}$, the group $\Sigma^{k-1}G_{r,P} \times \mathbb{Z}^\ell$ has order $k + \ell$ Dehn function $\delta^{(k+\ell)}(x) \simeq x^4$.

\footnote{Here we are using the upper bound for $\delta_G^{(k)}(x)$.}
where \( s = \frac{2(\ell + 1)\alpha - \ell}{2\alpha - (\ell - 1)} \) and \( \alpha = \log_3(r) \). The group \( \Sigma^{k-1}\mathbb{Z}^2 \times \mathbb{Z}_\ell \) has order \( k + \ell \) Dehn function \( \delta^{(k + \ell)}(x) \simeq x^s \) where \( s = \frac{\ell + 2}{\ell + 1} \).

Holding \( k \) and \( \ell \) fixed and varying \( r \) and \( P \), the resulting exponents \( s \) form a dense subset of the interval \( \left[ \frac{\ell + 2}{\ell + 1}, \frac{\ell + 1}{\ell - 1} \right] \), including all rational numbers within this range. From this observation one deduces Corollary B.

**Proof.** Fix \( r, P \), and \( k \), let \( s(\ell) = \frac{2(\ell + 1)\alpha - \ell}{2\alpha - (\ell - 1)} \), and let \( G_\ell \) be the group \( \Sigma^{k-1}G_{r,P} \times \mathbb{Z}_\ell \).

(Or let \( s(\ell) = \frac{\ell + 2}{\ell + 1} \) and \( G_\ell = \Sigma^{k-1}\mathbb{Z}^2 \times \mathbb{Z}_\ell \).) We verify by induction on \( \ell \) the following statements for \( G_\ell \):

1. \( \delta^M(x) \leq Cx^{s(\ell)} \) for all \( (k + \ell + 1) \)-manifolds \( M \) and some constant \( C > 0 \),
2. \( \delta^{(k + \ell)}(x) \simeq x^{s(\ell)} \), and
3. \( \delta^{(k + \ell)}(x) \) has embedded representatives.

The first two statements together imply \( \delta^{(k + \ell)}(x) \simeq x^{s(\ell)} \).

If \( \ell = 0 \) then (1) follows from Theorem 7.2 and Proposition 7.4. Statement (2) holds by Theorem B, and we have already observed that (3) holds for these groups.

For \( \ell > 0 \) note first that \( s(\ell) = 2 - 1/s(\ell - 1) \). Then statement (1) holds by Theorem 8.1 and property (1) of \( G_{\ell-1} \). Proposition 8.6 implies (2) and (3) by properties (1)–(3) of \( G_{\ell-1} \).

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