# ON FUNDAMENTAL GROUPS OF POSITIVELY CURVED MANIFOLDS

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## 0. Introduction

In 1965, S.S.Chern posed the following question [7, p.167] (sometimes called Chern's conjecture [12, p.671]; see also [6]): Let $M$ be a compact Riemannian manifold of positive sectional curvature. Is it true that every abelian subgroup of $\pi_1(M)$ is cyclic? Since $\pi_1(M)$ is finite, this is equivalent to saying that the cohomology ring $H^*(\pi_1, \mathbb{Z})$ is periodic (cf. [2]). In this note we will point out that there exist infinitely many counterexamples by observing that the normal homogeneous Aloff-Wallach space $N_{1,1}$ (cf. [10]) and the Eschenburg space $M_{1,1}$ (cf. [3])[1] both admit free, isometric SO(3) actions. Curiously enough $N_{1,1}$ was precisely the one missed in the classification of positively curved normal homogeneous spaces (cf. [1]). So, the motivation for posing the question possibly came from looking at metric space forms (cf. [11]) or more generally (?) $\frac{1}{4}$-pinched manifolds (cf. [4]) where the fundamental groups all have periodic cohomology, and from manifolds of negative curvature where the statement is true (Preissman's Theorem).

## 1. Free, isometric SO(3) actions

Following Wilking [10] we represent the normal homogeneous Aloff-Wallach space $N_{1,1}$ as the quotient $(\mathrm{SU}(3) \times \mathrm{SO}(3))/\,\mathrm{U}^*(2)$. Here $\mathrm{U}^*(2)$ is the image under the embedding $(i, \pi) : \mathrm{U}(2) \hookrightarrow \mathrm{SU}(3) \times \mathrm{SO}(3)$ given by the natural inclusion

$$i(A) = \begin{pmatrix} A & 0 \\ 0 & \det(A)^{-1} \end{pmatrix} \quad \text{for } A \in \mathrm{U}(2)$$

and the projection $\pi : \mathrm{U}(2) \to \mathrm{U}(2)/\mathrm{S}^1 \cong \mathrm{SO}(3)$, where $\mathrm{S}^1 \subset \mathrm{U}(2)$ is the center of U(2). The metric being normal homogeneous, the entire group $\mathrm{SU}(3) \times \mathrm{SO}(3)$ acts isometrically on $N_{1,1}$ on the left. In particular, the subgroup $\{\mathrm{id}\} \times \mathrm{SO}(3)$ acts isometrically on the left.

**Proposition 1.1.** *The group* $\{\mathrm{id}\} \times \mathrm{SO}(3)$ *acts freely on* $N_{1,1}$.

*Proof*: The action is free if and only if $(\{\mathrm{id}\} \times \mathrm{SO}(3)) \cap \mathrm{Ad}(g)(\mathrm{U}^*(2))$ is trivial for all $g$ in $\mathrm{SU}(3) \times \mathrm{SO}(3)$. This is equivalent to saying that $\mathrm{Ad}(g)(\{\mathrm{id}\} \times$

[1]In [3], the space $M_{1,1}$ is denoted as $M'_{1,1}$.

SO(3)) $\cap$ U$^*$(2) is trivial for all $g$. But Ad($g$)({id} $\times$ SO(3)) = {id} $\times$ SO(3) and $(i, \pi)$(U(2)) $\cap$ {id} $\times$ SO(3) is clearly trivial. □

The Eschenburg space $M_{1,1}$ is constructed as follows: Start with the group U(3) and perturb the bi-invariant metric to a normal homogeneous metric that is left invariant and Ad(U(2) $\times$ U(1))-invariant. Consider the subgroups $Z' = \{\mathrm{diag}(z, z, \bar{z}) | z \in \mathrm{S}^1\}$ and $U_{p,q} = \{\mathrm{diag}(z^p, z^q, 1) | z \in \mathrm{S}^1\}$ where $\gcd(p, q) = 1$ and $\mathrm{diag}(a_1, a_2, \ldots, a_n)$ denotes the matrix with diagonal entries $a_1, a_2, \ldots, a_n$. It is shown in [3] that if $p \cdot q > 0$ then the double coset manifold $M_{p,q} = U_{p,q} \backslash \mathrm{U}(3)/Z'$ (also called a biquotient) has positive curvature for the submersed metric. Since the group U(2) $\times$ $Z'$ acts freely and isometrically on U(3), it follows that there is a Riemannian fibration (see [3] for details)

$$(\mathrm{U}(2) \times Z')/(U_{p,q} \times Z') \to M_{p,q} \to \mathbf{C}P^2$$

When $p = q = 1$, U(2) $\times$ $Z'$ induces an isometric but non-effective action on $M_{1,1}$ (since $U_{1,1}$ is the center of U(2)) with kernel $U_{1,1} \times Z'$. The resulting isometric action by SO(3) = $(\mathrm{U}(2) \times Z')/(U_{1,1} \times Z')$ is clearly free and we get

$$\mathrm{SO}(3) \to M_{1,1} \to \mathbf{C}P^2$$

**Proposition 1.2.** *The Eschenburg space $M_{1,1}$ admits a free, isometric* SO(3) *action.* □

In fact, it was consideration of this fibration that led to the original observation by the author.

## 2. Remarks

**1.** Up to conjugacy the finite subgroups of SO(3) are

$$\mathbb{Z}_n,\ \ n \geq 1 \qquad D_m,\ \ m \geq 2 \qquad A_4 \qquad S_4 \qquad A_5$$

where $D_m$ is the dihedral group of order $2m$, $S_n$ denotes the permutation group on $n$ letters, and $A_n \subset S_n$ is the subgroup of even permutations. Since $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \approx D_2 \hookrightarrow D_{2m}$ for all $m \geq 1$, we get two infinite families of counterexamples: one covered by $N_{1,1}$ and one covered by $M_{1,1}$. This answers Chern's question in the negative.

**2.** A finite group is said to satisfy the $m$-condition if any subgroup of order $m$ is cyclic. It is shown in [5] that a finite group acts freely on a topological sphere if and only if it satisfies all $2p$- and $p^2$-conditions where $p$ is any prime that divides the order of the group. Note that satisfying all $p^2$-conditions is equivalent to the condition in Chern's problem. The above examples show that neither the $2p$- nor the $2^2$-conditions need hold for fundamental groups of positively curved manifolds. However, it is not known whether the $p^2$-condition remains true for odd primes $p$. We may formulate the following:

**Question.** Let M be a compact Riemannian manifold of positive sectional curvature. Is it true that every abelian subgroup of $\pi_1(M)$ of odd order is cyclic?

**3.** In the context of the question, some partial answers are known when the dimension of the manifold is fixed; see for instance [8].

**4.** The remaining proper, closed subgroups of SO(3) are SO(2) $\approx S^1$ and O(2). The quotients by SO(2) are $N_{1,1}/S^1 = F$ and $M_{1,1}/S^1 = F'$ where $F$ is the space of flags over $\mathbf{C}P^2$ and $F'$ is the "twisted" Eschenburg flag (cf. [3]). The quotients by O(2) then give positively curved manifolds with fundamental group $\mathbb{Z}_2$. They are isometric $\mathbb{Z}_2$ quotients of $F$ and $F'$ respectively. It follows from Synge's theorem that they are nonorientable. It can be shown without too much difficulty that the quaternionic flag Sp(3)/(Sp(1) $\times$ Sp(1) $\times$ Sp(1)) and the Cayley flag $\mathrm{F}_4$/Spin(8) also admit isometric $\mathbb{Z}_2$ quotients. In summary all known simply connected, even dimensional manifolds with positive curvature admit isometric $\mathbb{Z}_2$ quotients if they do so topologically, since the remaining known examples are the compact, rank one, symmetric spaces (cf. [9]).

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