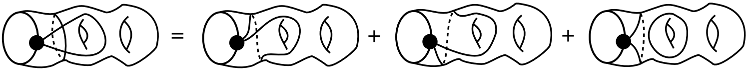
Bracelet bases are theta bases

Travis Mandel
(based joint work with Fan Qin)



Overview: skein algebras

- A skein algebra of a marked surface Σ is an algebra $\mathtt{Sk}(\Sigma)$ of arcs and loops in Σ.

Overview: skein algebras

- A skein algebra of a marked surface Σ is an algebra $\mathsf{Sk}(\Sigma)$ of arcs and loops in Σ.
- $\mathsf{Sk}(\Sigma)$ is part of the ring of functions on Fock-Goncharov's moduli of decorated twisted SL_2-local systems on Σ.

Overview: skein algebras

- A skein algebra of a marked surface Σ is an algebra $\mathsf{Sk}(\Sigma)$ of arcs and loops in Σ.
- $\mathsf{Sk}(\Sigma)$ is part of the ring of functions on Fock-Goncharov's moduli of decorated twisted SL_2-local systems on Σ.
- $\mathsf{Sk}(\Sigma)$ has a cluster algebra structure [Fock-Goncharov, Fomin-Shapiro-Thurston, Musiker-Williams].

Overview: skein algebras

- A skein algebra of a marked surface Σ is an algebra $\mathsf{Sk}(\Sigma)$ of arcs and loops in Σ.
- $\mathsf{Sk}(\Sigma)$ is part of the ring of functions on Fock-Goncharov's moduli of decorated twisted SL_2-local systems on Σ.
- $\mathsf{Sk}(\Sigma)$ has a cluster algebra structure [Fock-Goncharov, Fomin-Shapiro-Thurston, Musiker-Williams].
- $\mathsf{Sk}(\Sigma)$ has a canonical basis called the "bracelet basis" [Musiker-Schiffler-Williams, Fock-Goncharov].

Overview: skein algebras

- A skein algebra of a marked surface Σ is an algebra $\mathsf{Sk}(\Sigma)$ of arcs and loops in Σ.
- $\mathsf{Sk}(\Sigma)$ is part of the ring of functions on Fock-Goncharov's moduli of decorated twisted SL_2-local systems on Σ.
- $\mathsf{Sk}(\Sigma)$ has a cluster algebra structure [Fock-Goncharov, Fomin-Shapiro-Thurston, Musiker-Williams].
- $\mathsf{Sk}(\Sigma)$ has a canonical basis called the "bracelet basis" [Musiker-Schiffler-Williams, Fock-Goncharov].
- Cluster algebras have canonical "theta bases" [Gross-Hacking-Keel-Kontsevich].

Overview: skein algebras

- A skein algebra of a marked surface Σ is an algebra $\mathsf{Sk}(\Sigma)$ of arcs and loops in Σ.
- $\mathsf{Sk}(\Sigma)$ is part of the ring of functions on Fock-Goncharov's moduli of decorated twisted SL_2-local systems on Σ.
- $\mathsf{Sk}(\Sigma)$ has a cluster algebra structure [Fock-Goncharov, Fomin-Shapiro-Thurston, Musiker-Williams].
- $\mathsf{Sk}(\Sigma)$ has a canonical basis called the “bracelet basis” [Musiker-Schiffler-Williams, Fock-Goncharov].
- Cluster algebras have canonical “theta bases” [Gross-Hacking-Keel-Kontsevich].
- **Theorem** [M-Qin]: The bracelet basis equals the theta basis.

Overview: quantum skein algebras

- Quantum skein algebras have quantum cluster structures [G. Muller] with quantum bracelet bases [D. Thurston].

Overview: quantum skein algebras

- Quantum skein algebras have quantum cluster structures [G. Muller] with quantum bracelet bases [D. Thurston].
- Quantum cluster algebras have quantum theta bases [Davison-M].

Overview: quantum skein algebras

- Quantum skein algebras have quantum cluster structures [G. Muller] with quantum bracelet bases [D. Thurston].
- Quantum cluster algebras have quantum theta bases [Davison-M].
  - I'll assume some familiarity with these since Ben just talked about this.

Overview: quantum skein algebras

- Quantum skein algebras have quantum cluster structures [G. Muller] with quantum bracelet bases [D. Thurston].
- Quantum cluster algebras have quantum theta bases [Davison-M].
  - I'll assume some familiarity with these since Ben just talked about this.
- **Theorem** [M-Qin]: Quantum bracelet bases are quantum theta bases.

Overview: cluster Poisson algebra

- Fock-Goncharov: functions on the moduli of framed PGL_2-local systems form a cluster Poisson algebra, equipped with canonical coordinates.

Overview: cluster Poisson algebra

- Fock-Goncharov: functions on the moduli of framed PGL_2-local systems form a cluster Poisson algebra, equipped with canonical coordinates.
- Allegretti-Kim use Bonahon-Wong's quantum trace map to define quantum canonical coordinates for the corresponding quantum cluster Poisson algebra.

Overview: cluster Poisson algebra

- Fock-Goncharov: functions on the moduli of framed PGL_2-local systems form a cluster Poisson algebra, equipped with canonical coordinates.
- Allegretti-Kim use Bonahon-Wong's quantum trace map to define quantum canonical coordinates for the corresponding quantum cluster Poisson algebra.
- **Theorem** [M-Qin]: These (quantum) canonical coordinates are (quantum) theta bases.

Skein algebras

- Let $\Sigma = (\mathbf{S}, \mathbf{M})$ be a marked surface, i.e.:
  - a closed surface **S** with boundary ∂**S**, and
  - a finite collection of marked points **M** such that every component of ∂**S** is marked. Marked points in $\mathbf{S} \setminus \partial\mathbf{S}$ are called punctures.

Skein algebras

- Let $\Sigma = (\mathbf{S}, \mathbf{M})$ be a marked surface, i.e.:
  - a closed surface $\mathbf{S}$ with boundary $\partial\mathbf{S}$, and
  - a finite collection of marked points $\mathbf{M}$ such that every component of $\partial\mathbf{S}$ is marked. Marked points in $\mathbf{S} \setminus \partial\mathbf{S}$ are called punctures.
- $\mathtt{Sk}(\Sigma)$: spanned by isotopy classes of immersions $i : C \to \mathbf{S}$ such that
  - C is a closed one-manifold (i.e., a disjoint union of circles and closed intervals)
  - $i(\partial C) \subset \mathbf{M}$

  modulo certain relations (next slide).

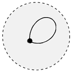
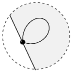
Skein algebras

- Let $\Sigma = (\mathbf{S}, \mathbf{M})$ be a marked surface, i.e.:
  - a closed surface $\mathbf{S}$ with boundary $\partial\mathbf{S}$, and
  - a finite collection of marked points $\mathbf{M}$ such that every component of $\partial\mathbf{S}$ is marked. Marked points in $\mathbf{S} \setminus \partial\mathbf{S}$ are called punctures.
- $\mathrm{Sk}(\Sigma)$: spanned by isotopy classes of immersions $i : C \to \mathbf{S}$ such that
  - C is a closed one-manifold (i.e., a disjoint union of circles and closed intervals)
  - $i(\partial C) \subset \mathbf{M}$

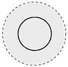
  modulo certain relations (next slide).
- The product of two elements of $\mathrm{Sk}(\Sigma)$ is the union of the corresponding immersions of curves.

The skein relations

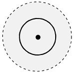
- Contractible arcs are equivalent to 0:

 $= 0$  $= 0$

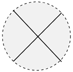
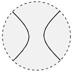
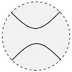
- Contractible loops are equivalent to -2:

 $= -2$

- A loop around a puncture (called a **peripheral loop**) is equivalent to 2;

 $= 2$

- The skein relation:

 $=$  $+$ 

Cluster structure of the skein algebra

- Theorem [Fock-Goncharov, Fomin-Shapiro-Thurston, Musiker-Williams]: This skein algebra $\mathrm{Sk}(\Sigma)$ has a cluster structure such that:
  - (tagged) triangulations corresponding to clusters;
  - (tagged) arcs correspond to cluster variables;

Cluster structure of the skein algebra

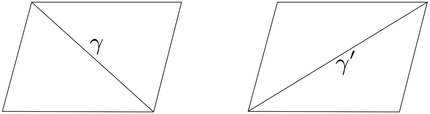
- Theorem [Fock-Goncharov, Fomin-Shapiro-Thurston, Musiker-Williams]: This skein algebra $\mathrm{Sk}(\Sigma)$ has a cluster structure such that:
  - (tagged) triangulations corresponding to clusters;
  - (tagged) arcs correspond to cluster variables;
  - Given a triangulation of Σ, the exchange pairing between arcs i and j is a signed count of the triangles containing i and j.

Cluster structure of the skein algebra

- Theorem [Fock-Goncharov, Fomin-Shapiro-Thurston, Musiker-Williams]: This skein algebra $\mathrm{Sk}(\Sigma)$ has a cluster structure such that:
  - (tagged) triangulations corresponding to clusters;
  - (tagged) arcs correspond to cluster variables;
  - Given a triangulation of Σ, the exchange pairing between arcs i and j is a signed count of the triangles containing i and j.
  - Boundary arcs correspond to frozen variables.

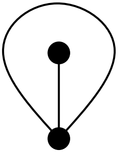
Cluster structure of the skein algebra

- Theorem [Fock-Goncharov, Fomin-Shapiro-Thurston, Musiker-Williams]: This skein algebra $\mathsf{Sk}(\Sigma)$ has a cluster structure such that:
  - (tagged) triangulations corresponding to clusters;
  - (tagged) arcs correspond to cluster variables;
  - Given a triangulation of Σ, the exchange pairing between arcs i and j is a signed count of the triangles containing i and j.
  - Boundary arcs correspond to frozen variables.
- Mutation corresponds to flipping the diagonal of a quadrilateral:

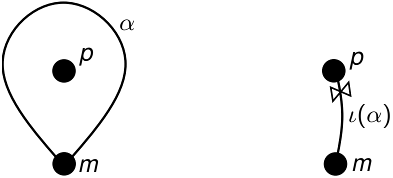


Tagged arcs

- An arc inside a self-folded triangle cannot be flipped:

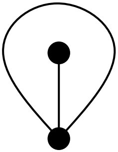


- [Fomin-Shapiro-Thurston] deals with this by introducing "tagged arcs" whose ends are tagged either plain or notched, subject to certain compatibility conditions:

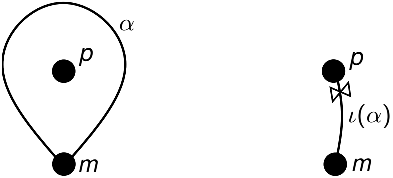


Tagged arcs

- An arc inside a self-folded triangle cannot be flipped:



- [Fomin-Shapiro-Thurston] deals with this by introducing “tagged arcs” whose ends are tagged either plain or notched, subject to certain compatibility conditions:



- Enlarge $\mathsf{Sk}(\Sigma)$ to include tagged arcs.

Quantum skein algebra

In **unpunctured** cases, Muller describes a quantization $\mathrm{Sk}_t(\Sigma)$ of $\mathrm{Sk}(\Sigma)$:

- $\mathrm{Sk}_t(\Sigma)$ is an algebra over $\Bbbk[t^{\pm 1}]$ rather than over $\Bbbk$. Denote $q = t^2$.

Quantum skein algebra

In **unpunctured** cases, Muller describes a quantization $\mathrm{Sk}_t(\Sigma)$ of $\mathrm{Sk}(\Sigma)$:

- $\mathrm{Sk}_t(\Sigma)$ is an algebra over $\Bbbk[t^{\pm 1}]$ rather than over $\Bbbk$. Denote $q = t^2$.
- The curves are now links: when strands cross, we identify which strand is on top.

Quantum skein algebra

In **unpunctured** cases, Muller describes a quantization $\mathrm{Sk}_t(\Sigma)$ of $\mathrm{Sk}(\Sigma)$:

- $\mathrm{Sk}_t(\Sigma)$ is an algebra over $\mathbb{k}[t^{\pm 1}]$ rather than over $\mathbb{k}$. Denote $q = t^2$.
- The curves are now links: when strands cross, we identify which strand is on top.
- The product $*$ is the superposition product — $L_1 * L_2$ is obtained by placing L_1 on top of L_2 (i.e., strands of L_1 always cross over strands of L_2).

Quantum skein algebra

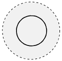
In **unpuncturedcases**, Muller describes a quantization $\mathrm{Sk}_t(\Sigma)$ of $\mathrm{Sk}(\Sigma)$:

- $\mathrm{Sk}_t(\Sigma)$ is an algebra over $\Bbbk[t^{\pm 1}]$ rather than over $\Bbbk$. Denote $q = t^2$.
- The curves are now links: when strands cross, we identify which strand is on top.
- The product $*$ is the superposition product — $L_1 * L_2$ is obtained by placing L_1 on top of L_2 (i.e., strands of L_1 always cross over strands of L_2). Multiplying arcs which share endpoints results in additional powers of t.

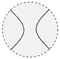
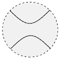
Quantum skein algebra

In **unpunctured** cases, Muller describes a quantization $\mathsf{Sk}_t(\Sigma)$ of $\mathsf{Sk}(\Sigma)$:

- $\mathsf{Sk}_t(\Sigma)$ is an algebra over $\Bbbk[t^{\pm 1}]$ rather than over $\Bbbk$. Denote $q = t^2$.
- The curves are now links: when strands cross, we identify which strand is on top.
- The product $*$ is the superposition product — $L_1 * L_2$ is obtained by placing L_1 on top of L_2 (i.e., strands of L_1 always cross over strands of L_2). Multiplying arcs which share endpoints results in additional powers of t.
- One makes the following modifications to the skein relations:
  - Contractible loops are equivalent to $-(q^2 + q^{-2})$;

 $= -(q^2 + q^{-2})$

  - The Kaufmann skein relation:

 $= q$  $+q^{-1}$ 

The resulting algebra $\mathsf{Sk}_t(\Sigma)$ is a quantum cluster algebra.

Bangles and bracelets

- Musiker-Schiffler-Williams construct “bangle” and “bracelet” bases for $\mathsf{Sk}(\Sigma)$.

Bangles and bracelets

- Musiker-Schiffler-Williams construct “bangle” and “bracelet” bases for $\mathsf{Sk}(\Sigma)$.
- Both represented by unions of pairwise-disjoint and non-isotopic $\mathbb{Z}_{\geq 1}$-weighted (tagged) arcs and non-peripheral loops:

Bangles and bracelets

- Musiker-Schiffler-Williams construct "bangle" and "bracelet" bases for $\mathrm{Sk}(\Sigma)$.
- Both represented by unions of pairwise-disjoint and non-isotopic $\mathbb{Z}_{\geq 1}$-weighted (tagged) arcs and non-peripheral loops:
  - **Bangles**: weight-k arc or loop $\leftrightsquigarrow$ k-th power of the arc or loop in $\mathrm{Sk}(\Sigma)$:

$$\langle kL \rangle_{\mathrm{Bangles}} = \langle L \rangle^{k}_{\mathrm{Bangles}}.$$

Bangles and bracelets

- Musiker-Schiffler-Williams construct “bangle” and “bracelet” bases for $\mathsf{Sk}(\Sigma)$.
- Both represented by unions of pairwise-disjoint and non-isotopic $\mathbb{Z}_{\geq 1}$-weighted (tagged) arcs and non-peripheral loops:
  - **Bangles**: weight-k arc or loop $\leftrightsquigarrow$ k-th power of the arc or loop in $\mathsf{Sk}(\Sigma)$:
  $$\langle kL \rangle_{\mathrm{Bangles}} = \langle L \rangle^k_{\mathrm{Bangles}}.$$
  - **Bracelets**: same for arcs, but a weight-k loop is given as in the figure:

Figure: A weight-5 loop viewed as a bangle (left) and a bracelet (right).

Bangles and bracelets

- Musiker-Schiffler-Williams construct "bangle" and "bracelet" bases for $\mathrm{Sk}(\Sigma)$.
- Both represented by unions of pairwise-disjoint and non-isotopic $\mathbb{Z}_{\geq 1}$-weighted (tagged) arcs and non-peripheral loops:
  - **Bangles**: weight-k arc or loop $\leftrightsquigarrow$ k-th power of the arc or loop in $\mathrm{Sk}(\Sigma)$:
  $$\langle kL\rangle_{\mathrm{Bangles}} = \langle L\rangle^k_{\mathrm{Bangles}}.$$
  - **Bracelets**: same for arcs, but a weight-k loop is given as in the figure:

Figure: A weight-5 loop viewed as a bangle (left) and a bracelet (right).

- Let T_k denote the k-th Chebyshev polynomial (of the first kind), characterized by
$$T_k(\lambda + \lambda^{-1}) = \lambda^k + \lambda^{-k}.$$
Then $\langle kL\rangle_{\mathrm{Bracelets}} = T_k\langle L\rangle_{\mathrm{Bracelets}}$.

Bangles and bracelets

- Musiker-Schiffler-Williams construct “bangle” and “bracelet” bases for $\mathrm{Sk}(\Sigma)$.
- Both represented by unions of pairwise-disjoint and non-isotopic $\mathbb{Z}_{\geq 1}$-weighted (tagged) arcs and non-peripheral loops:
  - **Bangles**: weight-k arc or loop $\leftrightsquigarrow$ k-th power of the arc or loop in $\mathrm{Sk}(\Sigma)$:
  $$\langle kL \rangle_{\mathrm{Bangles}} = \langle L \rangle^k_{\mathrm{Bangles}}.$$
  - **Bracelets**: same for arcs, but a weight-k loop is given as in the figure:

Figure: A weight-5 loop viewed as a bangle (left) and a bracelet (right).

- Let T_k denote the k-th Chebyshev polynomial (of the first kind), characterized by
$$T_k(\lambda + \lambda^{-1}) = \lambda^k + \lambda^{-k}.$$
Then $\langle kL \rangle_{\mathrm{Bracelets}} = T_k \langle L \rangle_{\mathrm{Bracelets}}$.
- Bracelets agree with Fock-Goncharov canonical coordinates:
Weight-k loop $\rightsquigarrow$ Trace of SL_2-holonomy around the loop k times.

Quantum bracelet bases

- D. Thurston constructs quantum bracelet bases for unpunctured surfaces:

Quantum bracelet bases

- D. Thurston constructs quantum bracelet bases for unpunctured surfaces:
- Again given by disjoint unions of weighted arcs and loops;

Quantum bracelet bases

- D. Thurston constructs quantum bracelet bases for unpunctured surfaces:
- Again given by disjoint unions of weighted arcs and loops;
- $\langle kL \rangle_{\text{Bracelets}} := T_k \langle L \rangle_{\text{Bracelets}}$.

Positivity definitions

- Recall that $f \in \mathsf{Sk}(\Sigma)$ has a Laurent expansion in the cluster variables (i.e., arcs) of each cluster (triangulation).

Positivity definitions

- Recall that $f \in \mathrm{Sk}(\Sigma)$ has a Laurent expansion in the cluster variables (i.e., arcs) of each cluster (triangulation).
- Nonzero f is **universally positive** if the coefficients of these Laurent expansions are always non-negative integers.

Positivity definitions

- Recall that $f \in \mathsf{Sk}(\Sigma)$ has a Laurent expansion in the cluster variables (i.e., arcs) of each cluster (triangulation).
- Nonzero f is **universally positive** if the coefficients of these Laurent expansions are always non-negative integers.
- Universally positive f is **atomic** if it cannot be decomposed as a sum of two other universally positive elements.

Positivity definitions

- Recall that $f \in \mathsf{Sk}(\Sigma)$ has a Laurent expansion in the cluster variables (i.e., arcs) of each cluster (triangulation).
- Nonzero f is **universally positive** if the coefficients of these Laurent expansions are always non-negative integers.
- Universally positive f is **atomic** if it cannot be decomposed as a sum of two other universally positive elements.
- A basis is **strongly positive** if any product of basis elements is a $\mathbb{Z}_{\geq 0}$-linear combination of basis elements.

Positivity definitions

- Recall that $f \in \mathsf{Sk}(\Sigma)$ has a Laurent expansion in the cluster variables (i.e., arcs) of each cluster (triangulation).
- Nonzero f is **universally positive** if the coefficients of these Laurent expansions are always non-negative integers.
- Universally positive f is **atomic** if it cannot be decomposed as a sum of two other universally positive elements.
- A basis is **strongly positive** if any product of basis elements is a $\mathbb{Z}_{\geq 0}$-linear combination of basis elements.
- Analogous definitions apply in the quantum setting using $\mathbb{Z}_{\geq 0}[t^{\pm 1}]$ in place of $\mathbb{Z}_{\geq 0}$.

Some past results and conjectures on positivity

- Atomic $\implies$ Strongly positive $\implies$ Universally positive

Some past results and conjectures on positivity

- Atomic $\implies$ Strongly positive $\implies$ Universally positive
- (Quantum) theta functions are atomic for the scattering atlas [Davison-M].

Some past results and conjectures on positivity

- Atomic $\implies$ Strongly positive $\implies$ Universally positive
- (Quantum) theta functions are atomic for the scattering atlas [Davison-M].
- Yurikusa: For cluster algebras from surfaces, the cluster complex is dense (except for once-punctured closed Σ) so the cluster atlas and scattering atlas agree.

Some past results and conjectures on positivity

- Atomic $\implies$ Strongly positive $\implies$ Universally positive
- (Quantum) theta functions are atomic for the scattering atlas [Davison-M].
- Yurikusa: For cluster algebras from surfaces, the cluster complex is dense (except for once-punctured closed Σ) so the cluster atlas and scattering atlas agree.
- D. Thurston: Classical bracelets are strongly positive (also cf. Fock-Goncharov). Conjectured strong positivity for quantum bracelets.

Some past results and conjectures on positivity

- Atomic $\implies$ Strongly positive $\implies$ Universally positive
- (Quantum) theta functions are atomic for the scattering atlas [Davison-M].
- Yurikusa: For cluster algebras from surfaces, the cluster complex is dense (except for once-punctured closed Σ) so the cluster atlas and scattering atlas agree.
- D. Thurston: Classical bracelets are strongly positive (also cf. Fock-Goncharov). Conjectured strong positivity for quantum bracelets.
- Fock-Goncharov: Conjectured their canonical coordinates were part of an atomic basis.

Some past results and conjectures on positivity

- Atomic $\implies$ Strongly positive $\implies$ Universally positive
- (Quantum) theta functions are atomic for the scattering atlas [Davison-M].
- Yurikusa: For cluster algebras from surfaces, the cluster complex is dense (except for once-punctured closed Σ) so the cluster atlas and scattering atlas agree.
- D. Thurston: Classical bracelets are strongly positive (also cf. Fock-Goncharov). Conjectured strong positivity for quantum bracelets.
- Fock-Goncharov: Conjectured their canonical coordinates were part of an atomic basis.
- Note: these positivity properties are known for (quantum) theta bases, so these conjectures would follow immediately from proving that the (quantum) bracelet and theta bases agree.

Bracelets = Thetas

Theorem (Qin-M)

The (quantum) bracelet bases agree with the (quantum) theta bases.

Bracelets = Thetas

Theorem (Qin-M)

*The (quantum) bracelet bases agree with the (quantum) theta bases.**

*Minor exception for the once-punctured torus: A bracelet equals 4^k times a theta function where k is the sum of the weights of the notched arcs.

Cutting/gluing

- **Gluing Lemma**: If a bracelet β in Σ equals a theta function, then if we identify two boundary arcs of Σ to get Σ', β will still be a theta function.

Cutting/gluing

- **Gluing Lemma**: If a bracelet β in Σ equals a theta function, then if we identify two boundary arcs of Σ to get Σ', β will still be a theta function.
- Proof for unpunctured Σ', classical setting:
  - Classical setting: $\beta \in \mathsf{Sk}(\Sigma')$ is universally positive [Thurston], hence a *positive* linear combination of theta functions.

Cutting/gluing

- **Gluing Lemma**: If a bracelet β in Σ equals a theta function, then if we identify two boundary arcs of Σ to get Σ', β will still be a theta function.
- Proof for unpunctured Σ', classical setting:
  - Classical setting: $\beta \in \mathsf{Sk}(\Sigma')$ is universally positive [Thurston], hence a *positive* linear combination of theta functions.
  - β has the same Laurent expansion for Σ and Σ'.

Cutting/gluing

- **Gluing Lemma**: If a bracelet β in Σ equals a theta function, then if we identify two boundary arcs of Σ to get Σ', β will still be a theta function.
- Proof for unpunctured Σ', classical setting:
  - Classical setting: $\beta \in \mathsf{Sk}(\Sigma')$ is universally positive [Thurston], hence a *positive* linear combination of theta functions.
  - β has the same Laurent expansion for Σ and Σ'.
  - Gluing boundary arcs adds a wall which can only result in more broken lines, hence possibly more positive terms or higher coefficients in each theta function.

Cutting/gluing

- **Gluing Lemma**: If a bracelet β in Σ equals a theta function, then if we identify two boundary arcs of Σ to get Σ', β will still be a theta function.
- Proof for unpunctured Σ', classical setting:
  - Classical setting: $\beta \in \mathrm{Sk}(\Sigma')$ is universally positive [Thurston], hence a *positive* linear combination of theta functions.
  - β has the same Laurent expansion for Σ and Σ'.
  - Gluing boundary arcs adds a wall which can only result in more broken lines, hence possibly more positive terms or higher coefficients in each theta function.
  - Since β has no new terms, positivity implies it must still be only a single theta function, and the theta function gained no new terms.

Cutting/gluing

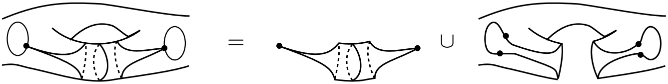
- **Gluing Lemma**: If a bracelet β in Σ equals a theta function, then if we identify two boundary arcs of Σ to get Σ', β will still be a theta function.
- Proof for unpunctured Σ', classical setting:
  - Classical setting: $\beta \in \mathsf{Sk}(\Sigma')$ is universally positive [Thurston], hence a *positive* linear combination of theta functions.
  - β has the same Laurent expansion for Σ and Σ'.
  - Gluing boundary arcs adds a wall which can only result in more broken lines, hence possibly more positive terms or higher coefficients in each theta function.
  - Since β has no new terms, positivity implies it must still be only a single theta function, and the theta function gained no new terms.
- Quantum setting:
  - Gluing resulted in no new broken lines for the classical theta function, hence no new broken lines for the quantum theta function.

Cutting/gluing

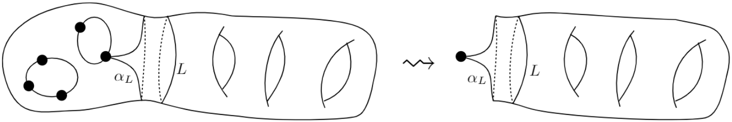
- **Gluing Lemma**: If a bracelet β in Σ equals a theta function, then if we identify two boundary arcs of Σ to get Σ', β will still be a theta function.
- Proof for unpunctured Σ', classical setting:
  - Classical setting: $\beta \in \mathrm{Sk}(\Sigma')$ is universally positive [Thurston], hence a *positive* linear combination of theta functions.
  - β has the same Laurent expansion for Σ and Σ'.
  - Gluing boundary arcs adds a wall which can only result in more broken lines, hence possibly more positive terms or higher coefficients in each theta function.
  - Since β has no new terms, positivity implies it must still be only a single theta function, and the theta function gained no new terms.
- Quantum setting:
  - Gluing resulted in no new broken lines for the classical theta function, hence no new broken lines for the quantum theta function.
  - The quantum Laurent expansion for β is also unchanged. $\square$

Reducing to annuli and once-marked cases

- By the Gluing Lemma, to show that weighted loops are theta functions, it suffices to consider:
  - the twice-marked annulus



  - and the once-marked surfaces with boundary



Annulus case

The case of the twice-marked annulus corresponds to the Kronecker-quiver (plus frozen vertices) and is understood explicitly.

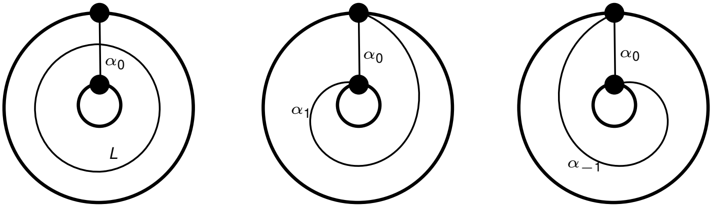


Figure: $\alpha_0 \cdot L = q\alpha_{-1} + q^{-1}\alpha_1$.

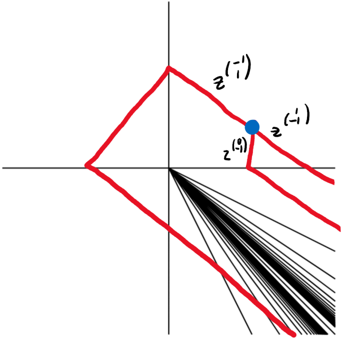
Annulus case continued

Previous slide: $\alpha_0 \cdot L = q\alpha_{-1} + q^{-1}\alpha_1$.
Now check that the same holds for theta functions:

Annulus case continued

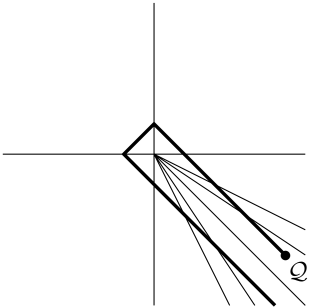
Previous slide: $\alpha_0 \cdot L = q\alpha_{-1} + q^{-1}\alpha_1$.

Now check that the same holds for theta functions:



$$\underbrace{z^{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}}_{=\vartheta_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}=\alpha_0}\vartheta_{\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)} = q\,\underbrace{z^{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}}_{=\vartheta_{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}=\alpha_{-1}} + q^{-1}\underbrace{\left(z^{\left(\begin{smallmatrix}2\\-1\end{smallmatrix}\right)} + z^{\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)}\right)}_{=\vartheta_{\left(\begin{smallmatrix}2\\-1\end{smallmatrix}\right)}=\alpha_1}$$

Weighted loops for the annulus case



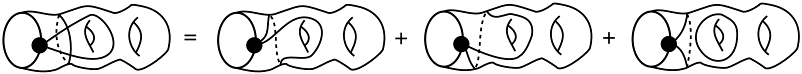
Up to any finite order, for $\mathcal{Q}$ sufficiently close to $p = (1, -1)$, compute

$$\vartheta_{kp,\mathcal{Q}} = z^{kp} + z^{-kp}.$$

The Chebyshev relation follows.

Surfaces with a single boundary marking

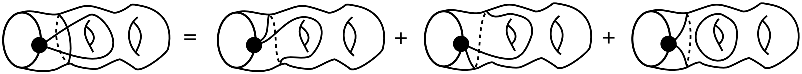
- The following shows that the boundary loop L times an arc γ equals a sum of three theta functions:



Combining several properties of theta functions and bracelets (positivity, pointed-ness, bar-invariance, etc.) we use this to deduce L is a theta function.

Surfaces with a single boundary marking

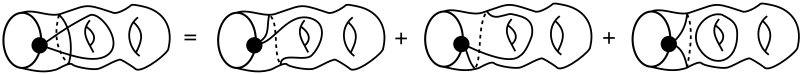
- The following shows that the boundary loop L times an arc γ equals a sum of three theta functions:



  Combining several properties of theta functions and bracelets (positivity, pointed-ness, bar-invariance, etc.) we use this to deduce L is a theta function.
- With quite a bit more work we show that each $\langle kL \rangle_{\text{Bracelets}}$ is a theta function as well.

Surfaces with a single boundary marking

- The following shows that the boundary loop L times an arc γ equals a sum of three theta functions:

  Combining several properties of theta functions and bracelets (positivity, pointed-ness, bar-invariance, etc.) we use this to deduce L is a theta function.
- With quite a bit more work we show that each $\langle kL\rangle_{\text{Bracelets}}$ is a theta function as well.
  - Lemma: $L^k = \sum_{b\in\mathbb{Z}_{\geq 0}} c_b \vartheta_{bg(L)}$.
  - Proof idea: The mapping class group Γ acts trivially on L^k and equivariantly on theta functions. All other g-vectors have infinite orbits under Γ (ignoring the frozen variable).

Disjoint unions of bracelets

- Finally, we must show that if $\beta_1, \ldots, \beta_s$ are disjoint bracelets corresponding to theta functions $\vartheta_{g_1}, \ldots, \vartheta_{g_s}$, then

$$\vartheta_{g_1} \cdots \vartheta_{g_s} = \vartheta_{g_1+\ldots+g_s}.$$

Disjoint unions of bracelets

- Finally, we must show that if $\beta_1, \ldots, \beta_s$ are disjoint bracelets corresponding to theta functions $\vartheta_{g_1}, \ldots, \vartheta_{g_s}$, then

$$\vartheta_{g_1} \cdots \vartheta_{g_s} = \vartheta_{g_1+\ldots+g_s}.$$

- Useful tool:
  - By Yurikusa, Dehn twists by L shear the g-vectors of arcs by a multiple of $g(L)$.

Disjoint unions of bracelets

- Finally, we must show that if $\beta_1, \ldots, \beta_s$ are disjoint bracelets corresponding to theta functions $\vartheta_{g_1}, \ldots, \vartheta_{g_s}$, then

$$\vartheta_{g_1} \cdots \vartheta_{g_s} = \vartheta_{g_1+\ldots+g_s}.$$

- Useful tool:
  - By Yurikusa, Dehn twists by L shear the g-vectors of arcs by a multiple of $g(L)$.
  - Use this to find a chamber of the cluster complex arbitrarily close to Span$\langle g_1, \ldots, g_s \rangle$.

Disjoint unions of bracelets

- Finally, we must show that if $\beta_1, \ldots, \beta_s$ are disjoint bracelets corresponding to theta functions $\vartheta_{g_1}, \ldots, \vartheta_{g_s}$, then

$$\vartheta_{g_1} \cdots \vartheta_{g_s} = \vartheta_{g_1+\ldots+g_s}.$$

- Useful tool:
  - By Yurikusa, Dehn twists by L shear the g-vectors of arcs by a multiple of $g(L)$.
  - Use this to find a chamber of the cluster complex arbitrarily close to $\mathrm{Span}\langle g_1, \ldots, g_s\rangle$.
  - Up to finite order, we can find such a chamber where each ϑ_{g_i} is either z^{g_i} or $z^{g_i} + z^{-g_i}$.

Punctured surfaces

- Use gluing and the previous arguments to prove bracelets without notches are theta functions.
- In most cases, notched arcs are cluster variables, so extending to include these is not hard.

Punctured surfaces

- Use gluing and the previous arguments to prove bracelets without notches are theta functions.
- In most cases, notched arcs are cluster variables, so extending to include these is not hard.
- Once-punctured surfaces: notched arcs are not cluster variables.
    - Fomin-Thurston: interpret (notched) arcs in terms of lambda lengths between (conjugate) horocycles.

Punctured surfaces

- Use gluing and the previous arguments to prove bracelets without notches are theta functions.
- In most cases, notched arcs are cluster variables, so extending to include these is not hard.
- Once-punctured surfaces: notched arcs are not cluster variables.
  - Fomin-Thurston: interpret (notched) arcs in terms of lambda lengths between (conjugate) horocycles.
  - We relate the once-punctured surfaces to covering spaces and use a generalized version of quiver folding.

Punctured surfaces

- Use gluing and the previous arguments to prove bracelets without notches are theta functions.
- In most cases, notched arcs are cluster variables, so extending to include these is not hard.
- Once-punctured surfaces: notched arcs are not cluster variables.
  - Fomin-Thurston: interpret (notched) arcs in terms of lambda lengths between (conjugate) horocycles.
  - We relate the once-punctured surfaces to covering spaces and use a generalized version of quiver folding.
  - Then using the digon relations we find that bracelets agree with theta functions except in genus 1.

Punctured surfaces

- Use gluing and the previous arguments to prove bracelets without notches are theta functions.
- In most cases, notched arcs are cluster variables, so extending to include these is not hard.
- Once-punctured surfaces: notched arcs are not cluster variables.
  - Fomin-Thurston: interpret (notched) arcs in terms of lambda lengths between (conjugate) horocycles.
  - We relate the once-punctured surfaces to covering spaces and use a generalized version of quiver folding.
  - Then using the digon relations we find that bracelets agree with theta functions except in genus 1.
  - For the once-puncture torus, comparison to Y. Zhou's description of theta functions shows that a notched arc is actually 4 times a theta function.