GROSS-HACKING-KEEL I

TRAVIS MANDEL

This talk was prepared for the the MIT-RTG Mirror Symmetry Workshop in Big Bear Lake, CA, May 2013.

1. Log Calabi-Yau Surfaces with Maximal Boundary

Let (Y, D) denote a smooth projective surface Y over an algebraically closed field $\Bbbk$ with characteristic 0, and a singular nodal anti-canonical divisor $D = D_1 + \ldots + D_n$ (D_i the irreducible components, cyclically ordered).

Denote $U := Y \setminus D$.

I may call (Y, D) a Looijenga pair, and U a Looijenga interior.

Examples 1.1.
- $Y \cong \mathbb{P}^2$, D a nodal cubic.
- Y a complete nonsingular toric surface, D the toric boundary (i.e, the complement of the big torus orbit $\mathbb{G}_m^2$).
- Blowups: Given a Looijenga pair $(\overline{Y}, \overline{D})$, we can take

  "toric blowups," where Y is the blowup of $\overline{Y}$ at a nodal point of $\overline{D}$ and D is the inverse image of $\overline{D}$). Note that toric blowups do not change U.

  "non-toric blowups," where $\widetilde{Y}$ is the blowup of $\overline{Y}$ at a non-nodal point of $\overline{D}$, and $\widetilde{D}$ is the proper transform of $\overline{D}$.

Proposition 1.2. *Every Looijenga pair can be obtained from a toric variety by a sequence of non-toric blowups and toric blowdowns.*

Goal: Construct a mirror family for these spaces which is a smoothing of $\mathbb{V}^n := \mathbb{A}^2_{x_1,x_2} \cup \mathbb{A}^2_{x_2,x_3} \cup \ldots \cup \mathbb{A}^2_{x_n,x_1} \subset \mathbb{A}^n$. If D supports a (D-)ample divisor, we can actually get a mirror family containing the original U as a fiber.

2. Tropicalizing U

We now define an integral linear manifold (transition maps in $SL_2(\mathbb{Z})$) U^{trop}. This will generalize $N_{\mathbb{R}}$ for (Y, D) and $M_{\mathbb{R}}$ for the mirror.

Let $(\widetilde{Y}, \widetilde{D}) \to (\overline{Y}, \overline{D})$ be a toric model of a toric blowup. Let N be the cocharacter lattice corresponding to $\overline{Y}$, and $\widetilde{\Sigma} \subset N_{\mathbb{R}} := N \otimes \mathbb{R}$ the corresponding fan.

Ray $\widetilde{\rho}_i$ corresponds to $\widetilde{D}_i$. 2-cells $\widetilde{\sigma_{i,i+1}}$ correspond to $D_i \cap D_{i+1}$.

As topological spaces, $U^{\text{trop}} = N_{\mathbb{R}}$, but the affine structure is different.

Let $\widetilde{v}_i$ be primitive generator of $\widetilde{\rho}_i$. Define $\widetilde{U}_i := \widetilde{\sigma}_{i-1,i} \cup \widetilde{\sigma}_{i,i+1}$

Date: May 30, 2013.

Charts:

$$\psi_i : \widetilde{U}_i \to \mathbb{R}^2$$
$$\widetilde{v}_{i-1} \mapsto (1,0) \qquad \widetilde{v}_i \mapsto (0,1) \qquad \widetilde{v}_{i+1} \mapsto (-1, -\widetilde{D}_i^2)$$

ψ_i linear on $\widetilde{\sigma}_{i-1,i}$, $\widetilde{\sigma}_{i,i+1}$.

Thus, the affine structure of $N_{\mathbb{R}}$ is modified so that

$$\widetilde{v}_{i-1} + D_i^2 \widetilde{v}_i + \widetilde{v}_{i+1} = 0 \tag{1}$$

in $\widetilde{U}_i$.

Define $U^{\text{trop}}(\mathbb{Z})$ to be N viewed as a subset of U^{trop} (alternatively, $\bigcup_i \psi_i^{-1}(\mathbb{Z}^2)$).

Remarks 2.1.
- U^{trop} depends only on the intersection matrix $(D_i \cdot D_j)_{ij}$. So the choice of toric model is unimportant.
- Toric blowups correspond to refinements of the fan, but do not change the affine structure. Thus, U^{trop} really depends only on U.

  Define Σ to be the fan in U^{trop} with a ray ρ_i for each $D_i \in D$ (rather than in $\widetilde{D}$), and similarly for $\sigma_{i,i+1}$'s.
- If (Y,D) is toric, $U^{\text{trop}} = N_{\mathbb{R}}$. This is because for any class $[C] \subset A_1(Y)$, $\sum_i [C] \cdot [D_i] v_i = 0$, so taking $[C] = [D_i]$ yields Equation [1].
- Points of $U^{\text{trop}}(\mathbb{Z})$ correspond to non-negative multiples of boundary divisors on (Y,D) and its toric blowups, with divisors on different toric blowups identified if they correspond to the same valuation.
- There is a canonical lattice Λ in TU^{trop} which has non-trivial monodromy about the singular point 0. We can identify cones σ in U^{trop} with cones in $T_p U^{\text{trop}}$, $p \in \sigma$. This identification takes integer points of U^{trop} to points in Λ.

Alternate Construction: Fix a toric model. Define a scattering diagram $\mathfrak{D}_{in}$ in $N_{\mathbb{R}}$ with lines (ρ_i, f_i), $\rho_i := \mathbb{R} v_i$, $f_i := \prod_{j=1}^{b_i} (1 + c_{ij} t_i z^{-v_i})$.

To define the integral linear structure, it suffices to identify the straight lines. These are the "broken lines" which bend towards the origin as much as is allowed.

3. The Mumford Degeneration

Let P^{gp} be a finite-rank free Abelian group, $P^{gp}_{\mathbb{R}} := P^{gp} \otimes \mathbb{R}$, and $P \subset P^{gp}$ a sub-monoid.

We say a function $\varphi : U^{\text{trop}} \to P^{gp}_{\mathbb{R}}$ is integral Σ-piecewise-linear if it is piecewise linear with bends only along rays of Σ, and $\varphi(U^{\text{trop}}(\mathbb{Z})) \subset P^{gp}$.

Let n_i be a primitive element of the dual to $T_{v_i} U^{\text{trop}}$ with $n_i(v_i) = 0$ and $n_i(\sigma_{i,i+1}) > 0$ (so $n_i = v_i \wedge \cdot$). Define the bending parameter p_i of φ along ρ_i to be the element of P^{gp} such that $\varphi|_{\sigma_{i,i+1}} = \varphi_{i,i-1} + n_i p_i$. We say φ is convex if all bending parameters are in P.

Let φ be a multi-valued convex integral Σ-piecewise-linear function on U^{trop}. Multi-valued means φ is in the sheaf $\mathcal{PL}/\mathcal{L}$, where $\mathcal{L}$ is the sheaf of linear functions and $\mathcal{PL}$ is the sheaf of piecewise-linear functions. φ is uniquely determined by its bending parameters.

Examples 3.1.
- $P^{gp} = A_1(Y)$, $P = NE(Y)$, $\widetilde{\varphi}$ has bending parameter $[D_i]$ along ρ_i.
- Let $\eta : A_1(Y) \to P^{gp}$ be a homomorphism such that $\eta[NE(Y)] \subset P$. Then take $\varphi := \eta \circ \widetilde{\varphi}$.

- For example, we could have $P^{gp} = \mathbb{Z}$, $P = \mathbb{Z}_{\geq 0}$, $\eta([C]) := W \cdot [C]$, where W is the class of an ample divisor on Y.

Let $\Gamma_{\mathbb{R}} \subset U^{\text{trop}} \times P^{gp}_{\mathbb{R}}$ be the graph of φ, and $\Gamma := \Gamma_{\mathbb{R}} \cap [U^{\text{trop}}(\mathbb{Z}) \times P^{gp}]$. If U^{trop} is non-singular (i.e, (Y, D) toric), then $\Gamma + P$ is a monoid, and the desired mirror family is $\mathcal{X} := \operatorname{Spec} \Bbbk[\Gamma + P] \to \operatorname{Spec} \Bbbk[P]$, with the morphism coming from the inclusion of $P \mapsto (0, P)$. [REFERENCE MARK GROSS' BOOK].

The fiber over 0 is just $\mathbb{V}^n$, and the general fiber in this toric case is isomorphic to $U \cong \mathbb{G}_m^2$.

Unfortunately, the singularity at 0 complicates this in general. Instead, do a local version of this construction as follows:

Identify $U_i \times P^{gp}_{\mathbb{R}}$ with a cone in $T_{v_i} U^{\text{trop}}$. Let σ be any cone in Σ containing ρ_i. Using this identification, define a cone

$$\Gamma_{\rho_i, \sigma, \mathbb{R}} := \{x - y \in T_{v_i} U^{\text{trop}} \times P^{gp}_{\mathbb{R}} | x \in \varphi(\rho_i), y \in \varphi(\sigma)\}.$$

Let $\Gamma_{\rho_i, \sigma}$ denote the integer points. Define $R_{\rho_i, \sigma} := \operatorname{Spec} \Bbbk[\Gamma_{\rho_i, \sigma}]$, $U_{\rho_i, \sigma} = \operatorname{Spec} \Bbbk[R_{\rho_i, \sigma}]$.

Now, we want to glue each U_{ρ_i, ρ_i} to $U_{\rho_{i+1}, \rho_{i+1}}$ by identifying $U_{\rho_i, \sigma_{i,i+1}}$ with $U_{\rho_{i+1}, \sigma_{i,i+1}}$.

Fixing a representative of φ on $U_i \cup U_{i+1}$ and using parallel transport in $\sigma_{i,i+1}$ to identify tangent spaces, the naive approach is to note that $\Gamma_{\rho_i, \sigma_{i,i+1}} = \Gamma_{\rho_{i+1}, \sigma_{i,i+1}}$, and use this obvious identification.

Problem: This gives a smoothing of $\mathbb{V}^n_0 := \mathbb{V}^n \setminus \{0\}$ which does not extend across the origin. The issue is essentially that the non-trivial monodromy of Λ around $0 \in U^{\text{trop}}$ prevents functions from patching.

Solution: Use a scattering diagram to modify the gluing.

4. Scattering Diagrams

Using the alternative description of U^{trop} above, a scattering diagram in U^{trop} is just one as in [GPS], with all lines and rays passing through 0.

So we have a collection of pairs (ρ, f_ρ) where ρ is a ray emanating from 0, and $f_\rho \in \widehat{\Bbbk[P_{\varphi_\rho}]}$ (P_{φ_ρ} is the cone "siting above the graph of φ along ρ," and the hat means the completion with respect to, say, the maximal ideal $\mathfrak{m}$ of P) satisfies $f_\rho \equiv 1$ modulo the maximal ideal $\mathfrak{m}$ of P. Furthermore, for any $k \in \mathbb{Z}$, there are only finitely many scattering rays with $f_\rho \not\equiv 1$ modulo $\mathfrak{m}^k$.

NOTE: [GHK] is slightly more general, allowing $\mathfrak{m}$ to be replaced with a radical ideal J containing the bending parameters, and $\mathfrak{m}^k$ to be an ideal I such that $\sqrt{I} = J$.

Now for the modified gluing, to glue $U_{\rho_i, \sigma_{i,i+1}}$ to $U_{\rho_{i+1}, \sigma_{i,i+1}}$ we use the path ordered product along a path γ in $\sigma_{i,i+1}$ from v_i to v_{i+1} which only crosses rays in the counterclockwise direction. That is, whenever γ crosses a scattering ray ρ, apply

$$z^v \mapsto z^v f_\rho^{n_\rho(v)}$$

.

Note: If ρ_i is a scattering ray, maybe use notation $U^{\pm}_{\rho_i, \sigma}$ to indicate whether or not the scattering automorphism for crossing ρ_i counterclockwise has been applied.

Problem: There are often infinitely many scattering rays between ρ_i and ρ_{i+1}. So work modulo $\mathfrak{m}^k$. Then there are only finitely many rays, by the definition of a scattering diagram. Maybe use subscript k.

Now, gluing all of the $U^{\pm}_{\rho_i, \sigma, k}$'s as above, we construct an infinitesimal deformation $\mathcal{X}^k_0$ of $\mathbb{V}^n_0$. Taking the inverse limit over all k yields a formal thickening $\widetilde{\mathcal{X}}_0$.

If the scattering diagram is "consistent," then $\mathcal{X}^k := \Gamma(\mathcal{X}_0^k, \mathcal{O}_{\mathcal{X}_0^k})$ gives an infinitesimal smoothing of $\mathbb{V}^n$ over $\operatorname{Spec} \Bbbk[P/\mathfrak{m}^k]$. Similarly, if we also have $\dim \Gamma(\widetilde{\mathcal{X}}_0, \mathcal{O}_{\mathcal{X}_0}) = \dim \widetilde{\mathcal{X}}_0$ (turns out to be true for a "consistent" scattering diagram whenever D supports an ample divisor), define $\mathcal{X} := \operatorname{Spec} \Gamma(\widetilde{\mathcal{X}}_0, \mathcal{O}_{\widetilde{\mathcal{X}}_0})$. This gives an affine smoothing of $\mathbb{V}^n$.

Consistency is designed to gaurantee that we have global sections (modulo the issue of U not being affine). We examine this now.

5. Broken Lines and Theta Functions

We now define a canonical set of global functions on the mirror family, called theta functions. First we must understand "broken lines," as these record the monomial terms in local expressions of the theta functions.

[*Draw picture].

Attached to each straight segment L is a monomial $c_L z^{q_L}$ with $c_L \in \Bbbk$ and $q_L \in \Gamma(L, \mathcal{P}_{\gamma^{-1}(L)})$ (i.e, q_L corresponds to an integer point in the graph of φ over some point in L, and at other points is given by parallel transport). The projection of q_L to Λ gives the (reverse) direction of flow for L. Has "ends" (q, Q) ($q \in U^{\text{trop}}(\mathbb{Z})_0$ and $Q \in U_0^{\text{trop}}$) if the initial (unbounded) monomial is z^q) and the end point is Q.

Breaking: Broken lines can "break" when they cross a ray (ρ, f_ρ) of the scattering diagram. To get from $c_L z^{q_L}$ to the next monomial $c_{L'} z^{q_{L'}}$, multiply by some monomial term in the power series expansion of $f_\rho^{n_\rho(r(q_L))}$ (r denotes the projection to Λ, and n_ρ chosen to make the exponent positive). I.e, the new monomial is one term in the expansion obtained by applying the path-ordered product to $c_L z^{q_L}$ for a path crossing only ρ in the same direction as the broken line.

Defining the Theta Functions: For each $q \in U^{\text{trop}}(\mathbb{Z})$, we have a canonical theta function ϑ_q on the mirror. Define $\vartheta_0 = 1$. For $q \neq 0$, define $\vartheta_q|_{U_{\rho,\rho}}$ (maybe with subscript k and superscript $\pm$) by picking a point Q "arbitrarily close to ρ" and defining

$$\vartheta_q|_{U_{\rho,\rho}} := \sum_{\gamma | \text{ends}(\gamma) = (q,Q)} c_{\gamma,Q} z^{q_{\gamma,Q}}$$

where $c_{\gamma,Q} z^{q_{\gamma,Q}}$ denotes the monomial attached to the final straight segment.

For this to be well-defined, it must commute with the gluings. When this happens, we say the scattering diagram is "consistent."

Theorem 5.1. $\Gamma(\mathcal{X}_k, \mathcal{O}_{\mathcal{X}_k}) = \bigoplus_{q \in U^{\text{trop}}(\mathbb{Z})} \Bbbk[P] \cdot \vartheta_q$ *(similarly for $\mathcal{X}$ if there are enough global sections).*

References

[GHK] M. Gross, P. Hacking, S. Keel, *Mirror Symmetry for Log Calabi-Yau Surfaces I*, arXiv:1106.4977v1 [math.AG].
[GPS] M. Gross, R. Pandharipande, B. Siebert, *The tropical vertex*, arXiv:0902.0779v1 [math.AG].