INTRO TO GIT

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This talk (for the UT Austin, Student geometry seminar, Nov 26, 2013) is based on Mukai's book, *An Introduction to Invariants and Moduli*, particularly Chapters 5 and 6,

1. Affine Quotients

Let $X = \operatorname{Spec}(R)$ (or $m\operatorname{Spec}(R)$), with R an integral domain over $\Bbbk$. Let G be an (linear) algebraic group (i.e., multiplication and inversion are polynomial morphisms) acting[1] on X. How do we understand $X /\!\!/ G$?.

Consider the map $\phi : X \to \mathbb{A}^n$, $x \mapsto (f_1(x), \ldots, f_n(x))$ for some collection of G-invariant functions $f_1, \ldots, f_n \in R^G$. Note that G-invariance implies that point in the same orbit are identified. So we hope to define a quotient $X /\!\!/ G := \operatorname{Spec} R^G$.

Question 1.1. When is R^G nice (finitely generated)?

Definition 1.2. A group G is **reductive** if it contains no nontrivial connected unipotent normal subgroups.

Examples 1.3. General linear groups, Semisimple algebraic groups, Algebraic tori.

Theorem 1.4 (Hilbert)**.** *If R is a finitely generated $\Bbbk$-algebra and G is reductive, then R^G is finitely generated.*

From now on, assume G is reductive.

Question 1.5. Is $\phi : X \to X /\!\!/ G$ (the quotient map, induced by $R^G \hookrightarrow R$) injective on the set of orbits?

Answer: No. In fact,

Theorem 1.6. *Let O_1 and O_2 be two orbits. Then $\phi(O_1) = \phi(O_2)$ if and only if $\overline{O_1} \cap \overline{O_2} \neq \emptyset$.*

Example 1.7. Consider $X = \mathbb{A}^{n+1}$, $G = \mathbb{C}^*$, with $g \cdot (x_1, \ldots, x_{n+1}) = (gx_1, \ldots, gx_{n+1})$. Removing the origin, we would expect the quotient to be $\mathbb{P}^n$. But in fact, $R^G = \Bbbk$, so $X /\!\!/ G$ is a point. Note that 0 is contained in the closure of every orbit. We'll come back to this issue.

Some properties:

Proposition 1.8. *ϕ is surjective, giving a bijection between "closure equivalence classes" of orbits in X and points in $X /\!\!/ G$. Furthermore, it takes closed sets to closed sets.*

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[1] I'm not defining exactly what is meant by a group action here. The idea is to dualize the usual notion of group action to get a coaction of the funcitons.

2. Stability

Question 2.1. Over what locus does the quotient give something nice?

Definition 2.2. Suppose X is a vector space and G is linearly reductive. Let $x \in X$. Then

- x is **unstable** if 0 is in the closure of its orbit (more generally, replace 0 with the fixed points of the action).
- x is **semistable** otherwise.
- x is **stable** if its orbit is closed and its stabilizer is finite (note that this makes sense for more general X).

Let X^s denote the stable points of X, X^{ss} the semistable points.

Proposition 2.3. *Stable implies semistable (assuming G is nontrivial).*

Proof. Suppose x is unstable. If $x = 0$, its stabilizer is G, so it is not stable. Otherwise, since $0 \in \overline{O(x)} \setminus O(x)$, the orbit is not closed, and x is still not stable. □

Proposition 2.4. *X^s and $\phi(X^s) \subset X /\!\!/ G$ are open sets.*

Theorem 2.5. *$\phi|_{X^s} : X^s \to X^s /\!\!/ G$ gives a bijection between the orbits of G in X^s and points in $X^s /\!\!/ G =: X^s/G$.*

In general, ϕ is the **categorical quotient**, meaning that it is the universal morphism which is equivariant with respect to the G-action (other such morphisms all factor through it).

$\phi|_{X^s} \to X^s/G$ is a **geometric quotient** (meaning the fibers are orbits, the topology is the quotient topology, and the map can locally be described in the same way as ϕ).

Geometric quotients are categorical quotients, but not vice-versa.

Throwing in some or all of the semistable points gives us nice compactifications.

Examples 2.6.

- Let $\mathbb{V}_{n,d}$ be the vector space of homogeneous degree d polynomials in $n+1$ variables. Mukai describes a moduli space of semistable hypersurfaces of degree d in $\mathbb{P}^n$ as $\mathbb{V}_{n,d} /\!\!/ GL(n+1)$ (this is really a projective quotient).
- Stable and semistable vector bundles on curves are used when constructing the moduli space of bundles topologically equivalent to some vector bundle E, up to isomorphism. The **slope** of a vector bundle E is $\mu(E) := \deg(E)/\text{rank}(E)$. E is stable if $\mu(E) > \mu(F)$ for all sub-bundles F of E ($\geq$ for semistable).

3. Projective Quotients

Recall Example 1.7. The only regular functions in $\Bbbk[x_1, \dots, x_n]$ invariant under the $\mathbb{G}_m$-action are the constant functions. But if we consider $\Bbbk(x_1, \dots, x_n)$, then the invariant rational funcitons are those of the form f/g, where f and g are homogeneous polynomials of the same degree. So allowing rational functions should help us get a better quotient.

Construction: Let $G = \mathbb{G}_m$ act on a space $X = m\,\text{Spec}(R)$ as before. R admits a grading[2] $R = \bigoplus_{k=0}^{\infty} R_k$, such that for $f \in R_k$, $f(g \cdot x) = g^k f(x)$ (so we have graded R by the weights of the G-action—this is possible by Proposition 4.7 in Mukai). Then we can define $X /\!\!/_{\text{Proj}}\, G := \text{Proj}(R)$.

[2]It is possible that the grading is over all of $\mathbb{Z}$, in which case me might ignore the negative integers to take the quotient. When the grading is over $\mathbb{N}$, we say the action is of *ray type*.

Example 3.1. $\mathbb{P}^n = \mathbb{A}^{n+1} /\!\!/_{\mathrm{Proj}} (\mathbb{G}_m)$, so this is much better than before.

Note that $R_0 = R^G$, so $X /\!\!/_{\mathrm{Proj}} G$ is projective over $X /\!\!/ G$.

Let $R_+ = R \setminus R_0$ be the irrelevant ideal. Let $F \subset X$ be the corresponding closed set, which is equal to the fixed point set of the action. For any $a \in R_+$, we have a principal open set $D(a) := \operatorname{Spec} R_a$ (the complement of $a = 0$). We get a commutative diagram:

$$\begin{array}{cccc} \operatorname{Spec} R_a = & D(a) & \hookrightarrow & X \setminus F \\ & \downarrow & & \downarrow \\ \operatorname{Spec} R_{a,0} = & D(a) /\!\!/ G & \hookrightarrow & \operatorname{Proj}(R) \end{array}$$

So locally (over affine open subsets of the quotient), the projective quotient looks like an affine quotient.

More generally, suppose G is an arbitrary group with a character $\chi : G \to \mathbb{G}_m$ (for $G = \mathbb{G}_m$, we used $\chi = \mathrm{id}$). We can grade R by $f \in R_{\chi^k}$ if $f(g \cdot x) = \chi(g)^k f(x)$ (f is then called a semiinvariant of weight χ^k).

We can then take Proj of R with this grading to get the quotient $X /\!\!/_\chi G$.

Example 3.2. If $G = \mathrm{GL}(n)$, then the only characters are of the form $\det^k$, $k \in \mathbb{Z}$. The choice of $k \neq 0$ used to define χ does not matter, except for the sign.

Definition 3.3. Let $G_\chi := \ker \chi$. Let $F_\chi = \operatorname{Spec} R_+$ be the fixed point set of G_χ.

$x \in X$ is *unstable with respect to* χ if $\overline{G_\chi \cdot x} \cap F \neq \emptyset$, and *semistable with respect to* χ otherwise. If $\chi \cdot x$ is closed and the stabilizer $\{g \in G | g \cdot x = x\}$ is finite, we say x is *stable with respect to* χ Denote the semistable (stable) points by X_χ^{ss} (resp., X_χ^s).

Equivalently, $x \in X$ is *semistable* with respect to χ if there is some $f \in R_{\chi^k}$, $k > 0$, such that $f(x) \neq 0$. Otherwise, x is *unstable.*

The semistable points form an open subset $X^{ss} \subset X$. The map $\Phi : X^{ss} \to X /\!\!/_\chi G$ is the *projective quotient map* with respect to χ.

Example 3.4. If we use the trivial character $\chi = 1$, then $X /\!\!/_\chi G = X /\!\!/ G$. Note that the the stability definitions agree with those from before.

Proposition 3.5. *If X_χ^s is nonempty, then every invariant rational function can be expressed as a (product of?) ratios of χ-semiinvariants. In particular, $k(X /\!\!/_\chi G) = k(X)^G$.*

Proposition 3.6. *If $X_\chi^{ss} = X_\chi^s$, then the fibers of $\Phi|_{X^s} : X^s \to X^s /\!\!/_\chi G$ are closed orbits. So $\Phi|_{X^s}$ is a geometric quotient.*

3.1. **Linerization.** It is possible that distinct orbits fail to be distinguished even when there are no characters of G. Mukai gives an example with $G = SL(n)$. So the above is still not general enough.

Idea: Find a G-equivariant line bundle $\mathcal{L}$ on $X = \operatorname{Spec} R$ which is generated by global sections. Take the tensor algebra of the ring of global sections, and then take the graded subring of invariants. Proj of this is the quotient we want.

Remark 3.7. Mukai approaches this by talking about co-actions on invertible GR-modules. I've just translated this to the geometric side for simplicity.

Example 3.8. In the cases where we had a character χ, we use the equivariant line bundle corresponding to the GR-module $R \otimes \Bbbk[G]$, with the coaction $R \to R \otimes \Bbbk[G]$, $r \mapsto r \otimes \chi^{-1}$.

How to construct $\mathcal{L}$?: Let $\{f_i\}_{i=1,\dots,n}$ be a generating set for the invariant rational functions, and let D be the sum of the divisors of poles of these functions. Take $\mathcal{L}$ to be the line bundle corresponding to $\mathcal{O}(D)$ (using X locally factorial to say $\mathcal{O}_X(D)$ is a Cartier divisor).

3.2. **Wall Crossing.** Scaling by a positive integer does not have any effect. However, other changes might significantly affect the quotient.

Example 3.9. Suppose $X = \mathbb{A}^n$ and $t \in G = \mathbb{G}^m$ acts by, say,

$$t \cdot (x_1, \dots, x_p, y_1, \dots, y_q) \mapsto (tx_1, \dots, tx_p, t^{-1}y_1, \dots, t^{-1}y_q).$$

Let R_+ be the positive graded part of R. Similarly for R_0 and R_-. Let $X_+ := \operatorname{Proj} R_+$, etc. Then X_+ and X_- are birationally equivalent, but this birational maps do not extend to morphisms in either direction. This map is called a *flop*. The case $p = q = 2$ is the classical Atiyah flop.

The phenomenon where χ crossing some wall changes your space by a birational map, like the flops above when you cross $\chi = 0$, is called *wall crossing*.

Example 3.10. Toric varieties can be constructed as quotients for certain choices of χ (in the "Nef polyhedron"). Other choices of χ might get you toric subvarieties, or even the empty set. Cox's construction is a bit different from this approach, and doesn't involve these choices.