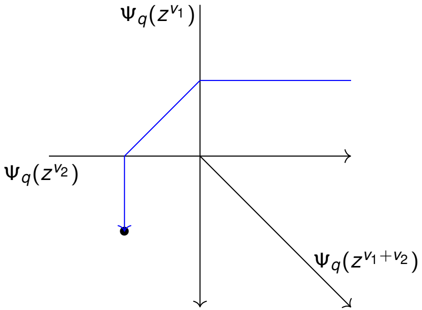
Mirror symmetry and canonical bases for quantum cluster algebras

Travis Mandel



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- The A-model for one space $X \cong$ The B-model for a "mirror" space Y.
- A-model captures structure from symplectic geometry.
- B-model captures structure from algebraic geometry.

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- 1996: Strominger-Yau-Zaslow (SYZ) Conjecture — More direct geometric relationship between mirrors.
- 2002: Gross-Siebert Program — Interpret SYZ using tropical and log geometry to give an explicit systematic construction of mirrors.

SYZ Conjecture

Mirrors should have dual torus fibrations:

$$\begin{array}{ccc} X & & Y \\ & {\scriptstyle \mu_X}\searrow \quad \swarrow{\scriptstyle \mu_Y} & \\ & B & \end{array}$$

- $\dim_{\mathbb{R}}(B) = \frac{1}{2}\dim_{\mathbb{R}}(X) = \frac{1}{2}\dim_{\mathbb{R}}(Y)$,
- generic fibers of μ_X are tori, and
- generic fibers of μ_Y are the dual tori.

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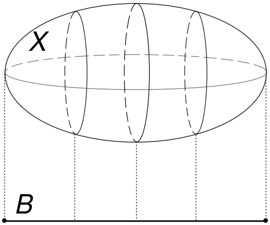
Example:

- Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.
- Let $X = (\mathbb{C}^*)^n$ (the “algebraic torus”).
- Then $Y = \mathrm{Hom}((\mathbb{C}^*)^n, \mathbb{C}^*) \cong (\mathbb{C}^*)^n$.

$$\mu : (\mathbb{C}^*)^n \to \mathbb{R}^n, \qquad \mu(a_1, \ldots, a_n) := (\log|a_1|, \ldots, \log|a_n|)$$

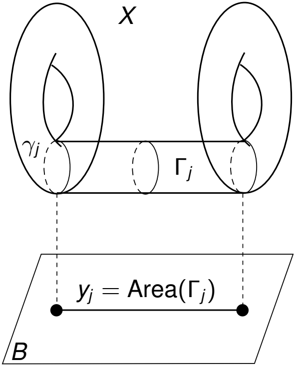
SYZ map

- Example: $\mu : S^2 \to [0, 1]$.



Local coordinate on B

Let $\gamma_1, \ldots, \gamma_n$ be a basis for $\pi_1(S_1^n) = \pi_1(\mu_X^{-1}(Q))$.



$\{y_j | j = 1, \ldots, n\}$ form local coordinates on B.

Local coordinates for Y

- The y_j's form local coordinates on B.
- Let $x_j := dy_j$. This determines lattices $T^*_{\mathbb{Z}}B \subset T^*B$ and $T_{\mathbb{Z}}B \subset TB$.
- Locally,

$$X = T^*B/T^*_{\mathbb{Z}}B \qquad \text{and} \qquad Y = TB/T_{\mathbb{Z}}B.$$

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$$X = T^*B/T^*_{\mathbb{Z}}B \qquad \text{and} \qquad Y = TB/T_{\mathbb{Z}}B.$$

- $w_j := x_j + iy_j$ gives local holomorphic coordinates for Y.
- $z_j := \exp(2\pi i w_j)$ gives local algebraic coordinates.

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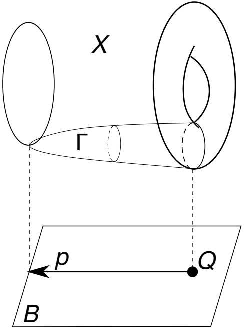

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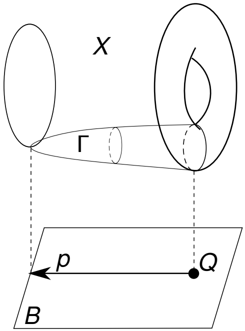

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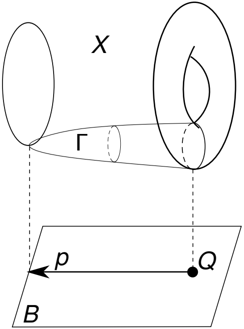

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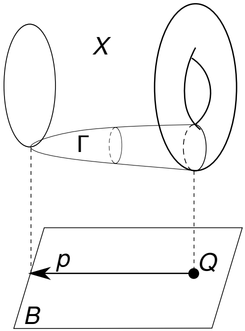

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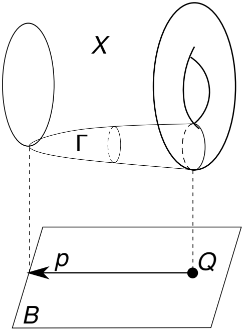

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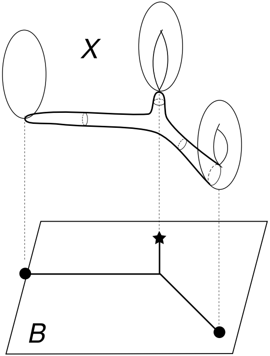
- Varying Q makes y_Γ a local function on B. Let $x_\Gamma := dy_\Gamma$.
- Let $z_\Gamma := \exp(2\pi i(x_\Gamma + iy_\Gamma))$.
- Local expression for ϑ_p near torus over Q given by:

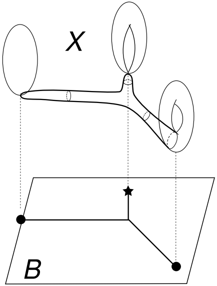
$$\vartheta_{p,Q} := \sum_{\Gamma \in D_{p,Q}} z_\Gamma.$$

Holomorphic disks

Typically, some fibers are singular (e.g., pinched tori). This results in more holomorphic disks.



The Gross-Siebert program



The graph in *B* is called a **tropical disk**.

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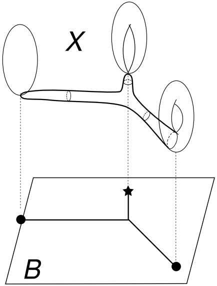


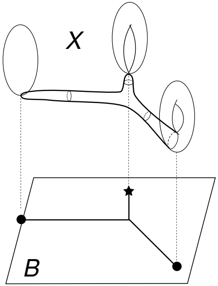
The graph in B is called a **tropical disk**.

The Gross-Siebert Program:

- Use the tropical picture to construct mirrors Y with canonical theta function bases for their rings of global functions.
- Use log geometry to relate these bases to curve counts in X.

Wall-crossing

Holomorphic disks over B result in walls in B where our local coordinate system changes:



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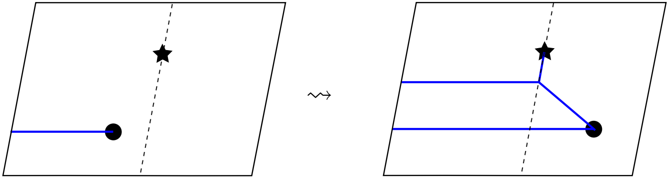
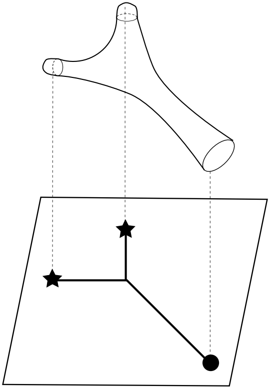


$$\text{E.g.,} \quad (\mathbb{C}^*)^2 \dashrightarrow (\mathbb{C}^*)^2, \quad x \mapsto x(1+y).$$

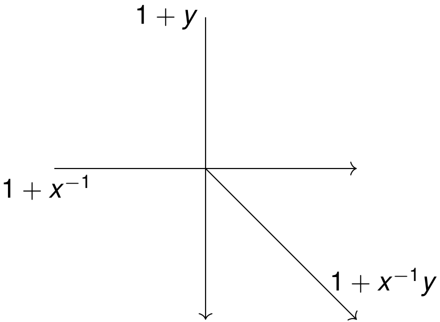
Scattering

The initial walls can interact to form new walls:



Scattering diagrams

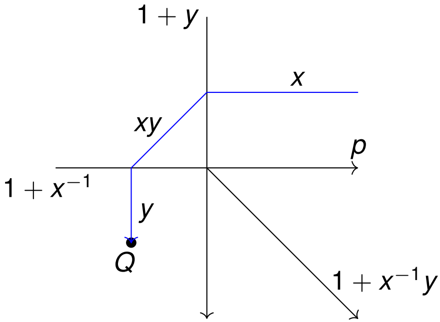
The data of these walls is encoded in a "scattering diagram."



Walls labelled with functions indicating the corresponding transition functions.

Broken lines

Broken lines with ends (p, Q) — tropical version of the holomorphic disks whose behavior at ∞ is determined by p, and whose boundary is on the fiber over Q.



Theta functions

- Theta function for each $p \in T_{\mathbb{Z}}B$, given locally by:

$$\vartheta_{p,Q} := \sum_{\mathrm{Ends}(\gamma)=(p,Q)} a_{\gamma} z^{m_{\gamma}},$$

where $a_{\gamma} z^{m_{\gamma}}$ is the monomial attached to the last straight segment of γ.

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- Gross-Hacking-Keel-Kontsevich (JAMS, 2018): Used this to define canonical bases for cluster algebras.

Cluster algebras

Cluster algebras — certain combinatorially constructed commutative rings.

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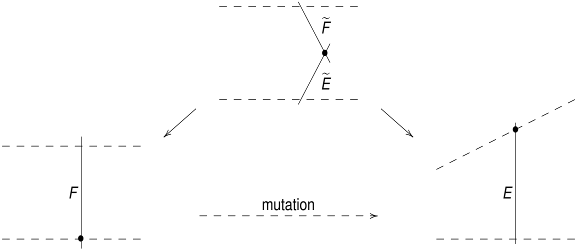
Applications:

- Representation theory and quantum groups;
- (Higher) Teichmüller theory and Poisson geometry;
- Discrete integrable systems;
- DT–theory and quiver representations;
- Mirror symmetry;
- ...

Cluster varieties

- Fock-Goncharov (Ann. Sci. Éc. Norm. Supér. 2009):
  Cluster varieties constructed by gluing together algebraic tori, called **clusters**, via certain birational maps called **mutations**.



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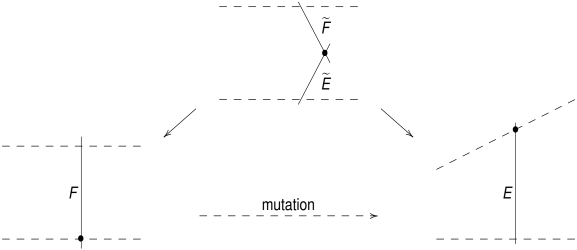

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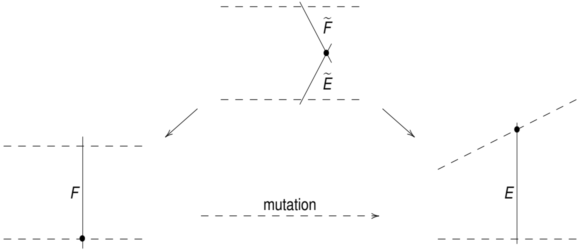

- Upper cluster algebra — ring of global regular functions on the cluster variety.
- Cluster algebra — subring generated by the "cluster monomials," i.e., elements which are monomials in some cluster.

Examples

Many important spaces have cluster structures, including:

- Semisimple Lie groups;
- Grassmannians, other partial flag varieties, and Schubert varieties;
- Higher Teichmüller spaces;
- All log Calabi-Yau surfaces;
- ...

Example: $SL_2(\mathbb{C})$

Consider $SL_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, ad - bc = 1 \right\}$.

- Two clusters $\mathbb{C}^* \times \mathbb{C}^2$:
  - One where $a \neq 0$: coordinate ring $\mathbb{C}[a^{\pm 1}, b, c]$
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- **Canonical basis** for coordinate ring $\mathbb{C}[a, b, c, d]/\langle ad - bc = 1\rangle$:

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  - **Universally positive**: $\mathbb{Z}_{\geq 0}$–coefficients in each cluster. E.g., $d = a^{-1} + a^{-1}bc$.
  - **Strongly positive**: structure constants in $\mathbb{Z}_{\geq 0}$. E.g. both equal 1 for $a \cdot d = 1 + bc$.
  - **Atomic**: basis elements cannot be decomposed as sums of other universally positive elements.

The Fock-Goncharov Conjecture

Conjecture (Fock-Goncharov, 2003)

The atomic elements of a (classical or quantum) cluster algebra form a canonical basis for the algebra. This basis includes all the cluster monomials.

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- Universal positivity for the cluster monomials was conjectured in the original cluster algebras paper (Fomin-Zelevinsky, 2002).
  - Universal positivity for the classical cluster monomials proved by Lee-Schiffler (Annals 2015).
  - Universal positivity for the quantum cluster monomials proved by Davison (Annals 2018).

Why care about positive bases?

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- Positivity is useful for proving other properties.
- Positivity suggests the coefficients have some deeper meaning:
  - Mirror symmetry: counts of holomorphic disks;
  - Categorification: dimensions of vector spaces.

Classical theta bases

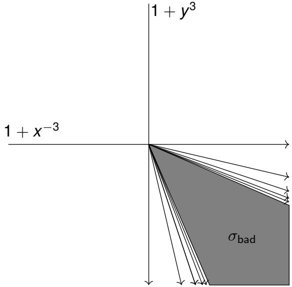
Using scattering diagrams and broken lines:

Theorem (Gross-Hacking-Keel-Kontsevich, JAMS 2018)

A modified version of the Fock-Goncharov conjecture is true for classical cluster algebras. I.e., there are strongly positive, universally positive bases $\{\vartheta_p\}_p$ *which include the cluster monomials.*

The cluster complex and atomicity

Clusters $\leftrightsquigarrow$ chambers in the “nice” part of the scattering diagram.



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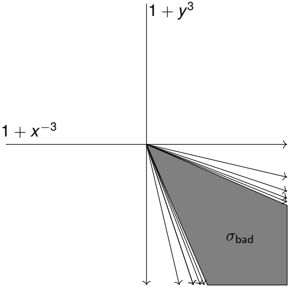

Theorem (M, Compositio 2017)

$\{\vartheta_p\}_p$ is atomic with respect to the scattering atlas — i.e., the local coordinate systems associated to the scattering diagram (more charts than just clusters).

Geometric interpretation of theta bases

Theorem (M, Keel-Yu, Gross-Siebert, 2019)

The theta functions really are determined by counts of holomorphic curves as predicted by mirror symmetry.

- Keel-Yu: Structure constants are positive because they literally are counts of curves (at least for affine cluster varieties).

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$$n = 2: \qquad \mathbb{C}[q^{\pm 1/2}][x^{\pm 1}, y^{\pm 1}]/\langle xy = qyx \rangle.$$

    Recovers classical version under $q^{1/2} \mapsto 1$.

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Motivation for quantum cluster algebras

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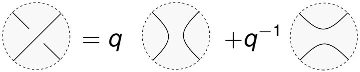
  - Skein algebras [G. Muller, Quantum Topol. 2016], quantum higher Teichmüller spaces;



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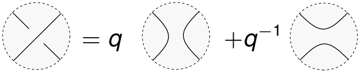

  - Quantum integrable systems
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- Motivation from physics.
- Reveals more structure:
  - Poincaré polynomials instead of Euler characteristics;
  - *B*-field contributions in mirror symmetry.

Quantum theta bases

Using a quantum version of scattering diagrams and broken lines, we prove the quantum Fock-Goncharov conjecture:

Theorem (M-Davison, Oct. 2019)

Quantum cluster algebras admit a strongly positive, universally positive canonical basis $\{\vartheta_p\}_p$, *consisting precisely of the elements which are atomic with respect to the scattering atlas, including all cluster monomials.*

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  - Points towards approach via categorification.

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- Connections to representation theory and Teichmüller theory.
- Relation to other bases (greedy bases, bracelet bases, dual canonical bases,...).

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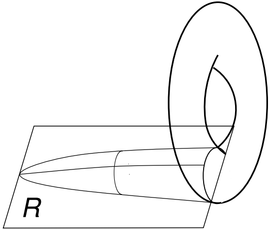

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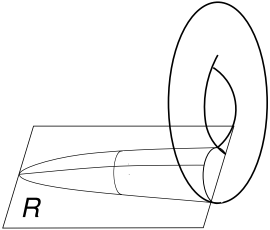

- For quantum theta functions, the powers of q should come from the areas of the disks with respect to a "B-field."