THETA FUNCTIONS AND LOG GROMOV-WITTEN INVARIANTS

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Contents

1. Talk 1: Tropical correspondence

The Gross-Siebert program approaches mirror symmetry by tropicalizing the A- and B-models. This talk relates to tropicalizing the A-model. The A-model is concerned with Gromov-Witten theory (counting holomorphic curves). I'll explain how certain Gromov-Witten type invariants can be computed tropically.

The first papers on this are [Mik05] (all genus in dimension 2) and [NS06] (genus 0 in all dimensions). I'll explain some cases from [MR16] (see the paper for details I skip), where we followed the [NS06] log degeneration approach.

1.1. **Tropical curves.** We'll stick to genus 0 since I don't yet know how higher genus plays into the Gross-Siebert program.

Let Γ denote a tree with its 1-valent vertices removed. Let $\Gamma^{[0]}$, $\Gamma^{[1]}$, and $\Gamma^{[1]}_{\infty}$, and $\Gamma^{[1]}_c$ denote the the sets of vertices, edges, non-compact edges, and compact edges, respectively. Equip Γ with a weight function $w:\Gamma^{[1]}\rightarrow \mathbb{Z}_{\geq 0}$. A **marking** of Γ is a map $\mu:\{1,\ldots,m\}\rightarrow \Gamma^{[0]}$ plus a bijection $\epsilon:\{1,\ldots,e_{\infty}\}\xrightarrow{\sim}\Gamma^{[1]}_{\infty}$. We denote $V_i:=\mu(i)$ and $E_j:=\epsilon(j)$. We only allow bivalent vertices if they are marked.

Definition 1.1. A **parametrized marked (genus** 0**) tropical curve** (Γ,μ,ϵ,h) is data (Γ,μ,ϵ) as above, along with a proper continuous map $h:\Gamma\rightarrow N_{\mathbb{R}}$ such that

(1) For each edge $E\in\Gamma^{[1]}$, $h|_E$ is an embedding into an affine line with rational slope.
(2) For all $V\in\Gamma^{[0]}$, $h(V)\in N_{\mathbb{Q}}$, and the **balancing condition** holds: For any edge $E\ni V$, denote by $u_{(V,E)}$ the primitive integral vector emanating from $h(V)$ into $h(E)$. Then

$$\sum_{E\ni V} w(E)u_{(V,E)}=0.$$

A (marked, genus 0) tropical curve is then a parametrized marked tropical curve up to isomorphism (morphisms are homeomorphisms of trees commuting with the markings, weights, and h's).

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The **degree** Δ of $(\Gamma, \mu, \epsilon, h)$ is the set

$$\Delta(\Gamma, \mu, \epsilon, u) = \{(w(E_j)u_{E_j}, j) \mid 1 \leq j \leq e_\infty\}. \tag{1}$$

For each $V \in \Gamma^{[0]}$, denote $m_V := \#\mu^{-1}(V)$, $\mathrm{ov}(V) := \mathrm{val}(V) + m_V - 3$ (motivation: $\mathrm{ov}(V) = \dim \mathcal{M}_{0,\mathrm{val}(V)+m_V}$).

1.1.1. *Incidence and psi-class conditions.*

Definition 1.2. Incidence conditions: Let $\mathbf{A}$ be a tuple $(A_1, \ldots, A_m; B_1, \ldots, B_{e_\infty})$ of affine subspaces of $N_\mathbb{Q}$. $(\Gamma, \mu, \epsilon, h)$ **matches the constraints A** if $h(V_i) \in A_i$ for $i = 1, \ldots, m$ and $h(E_j) \subset B_j$ for $j = 1, \ldots, e_\infty$.

Psi-class conditions: Consider another tuple $\Psi := (s_1, \ldots, s_m) \in \mathbb{Z}^m_{\geq 0}$. $(\Gamma, \mu, \epsilon, h)$ satisfies Ψ if for each marked vertex V,

$$\mathrm{ov}(V) \geq \sum_{i \in \mu^{-1}(V)} s_i. \tag{2}$$

Generically this will actuall be an equality.

Definition 1.3. Assuming $\sum s_i + \sum \mathrm{codim}(A_i) + \sum \mathrm{codim}(B_i) = e_\infty + m + n - 3$,

$$\mathrm{GW}^{\mathrm{trop}}_{0,m,e_\infty,N_\mathbb{Q},\Delta}(\mathbf{A}, \Psi) := \sum \mathrm{Mult}(\Gamma),$$

where the sum is over all $(\Gamma, \mu, \epsilon, h)$ with m marked points and degree Δ satisfying $\mathbf{A}$ and Ψ. $\mathrm{Mult}(\Gamma)$ will be defined later.

1.2. Descendant log Gromov-Witten invariants.

1.2.1. *Toric degenerations.* Fix a rational polyhedral decomposition $\mathcal{P}$ of $N_\mathbb{R}$ whose 1-skeleton contains all tropical curves contributing to $\mathrm{GW}^{\mathrm{trop}}_{0,m,e_\infty,N_\mathbb{Q},\Delta}(\mathbf{A}, \Psi)$. Assume $\mathcal{P}$ also contains each A_i and B_j.

Taking the closure of the cones over $\mathcal{P}$ in $(N \times \mathbb{Z})_\mathbb{R}$ gives a fan $\widetilde{\Sigma}$ which has a natural map to the fan for $\mathbb{A}^1$. (This is a toric degeneration and might be covered in Mandy's talk). Let $\mathcal{X} := \mathrm{TV}(\Sigma)$. Have $\mathcal{X} \to \mathbb{A}^1_t$. Let $\mathcal{X}_t$ denote the fiber over t. For $t \neq 0$, $\mathcal{X}_t = \mathrm{TV}(\Sigma)$ where $\Sigma = \widetilde{\Sigma} \cap N_\mathbb{R} \cap \{0\}$ is the asymptotic fan of $\mathcal{P}$. $\mathcal{X}_0 = \bigcup_{V \in \mathcal{P}^{[0]}} \mathcal{X}_V$, where $\mathcal{X}_V$ is the toric variety whose fan looks like the star of V in $\mathcal{P}$. If $V, V' \in E$, then $\mathcal{X}_V$ and $\mathcal{X}_{V'}$ intersect along a common divisor D_E.

1.2.2. *The moduli space.* Denote $\Delta = \{(w_i u_i, i) | i = 1, \ldots e_\infty\}$, u_i primitive. Let $\mathcal{M}_{0,m,e_\infty}(\mathcal{X}_t, \Delta)$ denote the moduli space parametrizing stable (basic, log) maps $\varphi : (C; x_1, \ldots, x_m; y_1, \ldots, y_{e_\infty}) \to \mathcal{X}_t$ such that C has genus 0 and each y_i maps to D_{u_i} with intersection multiplicity w_i (and these are all the intersections with the boundary). See [MR16, §3.2] for a more precise definition.

See [GS13] and [AC14] for full details on moduli of basic stable log maps.

1.2.3. *Incidence conditions.* For affine $A \subset N$ with rational slope, the cone over A determines a subspace $\mathrm{LC}(A) \subseteq (N \times \mathbb{Z})_\mathbb{R}$, hence a subtorus $\mathrm{LC}(A) \otimes \mathbb{C}^*$ of the big torus orbit $T_{N'} := (N \times \mathbb{Z}) \otimes \mathbb{C}^*$ of $\mathcal{X}$. Fixing generic $Q_A \in T_{N'}$ determines a subvariety

$$Z_{A,Q_A} := \overline{(\mathrm{LC}(A) \otimes \mathbb{C}^*) \cdot Q_A} \subset \mathcal{X}.$$

Equivalently, Z_{A,Q_A} is cut out by polynomials of the form $z^m = z^m(Q_A)$ with $m \in \mathrm{L}(A)^\perp$ (L meaning the linear span), where $z^m(Q_A)$ means z^m evaluated at Q^A.

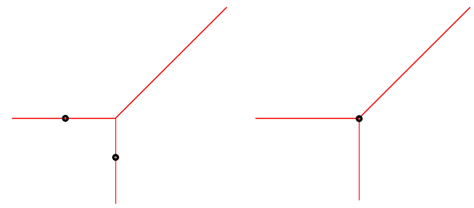

FIGURE 1.1. Unique tropical line through two given points, respectively through a single point with a ψ-class condition. Algebraically, the first of these corresponds to the fact that two points determine a line. The second shows that a a line in $\mathbb{P}^2$ is uniquely determined by specifying the image of one point along with the cross-ratio of this point with the coordinate-axis intersections.

Similarly, for a vector $u_B \in N$ parallel to B, denote

$$Z_{B,u_B,Q_B} := Z_{B,Q_B} \cap D_{u_B}.$$

We will use the same notation for the intersections with $\mathcal{X}_t$.

Fact: For $A \subset \mathrm{Supp}(\mathcal{P})$, $\mathcal{X}_V \cap Z_A \neq \emptyset$ if and only if $V \in A$.

1.2.4. *Psi-class conditions.* Let $\mathcal{L}_{\psi_i}$ be the line bundle over $\mathcal{M}_{g,m,e_\infty}(\mathcal{X}_t, \Delta)$ whose fiber over $[\varphi : (C; x_1, \dots, x_m; y_1, \dots, y_{e_\infty}) \to \mathcal{X}_t]$ is $T^*_{x_i}C$. Define

$$\psi_i := c_1(\mathcal{L}_{\psi_i}).$$

Let $\overline{\psi}_i$ denote the corresponding psi-class on $\overline{\mathcal{M}_{0,m+e_\infty}}$.

Proposition 1.4. $\psi_i = \mathrm{Forget}^* \, \overline{\psi}_i$.

Proof. Typically these differ by the locus where Forget destabilizes and contracts the irreducible component of C containing x_i, but since all intersections with the toric boundary are marked for log curves, this destabilization never happens for us. □

1.2.5. *The algebraic curve counts.*

$$\mathrm{GW}^{\log}_{g,m,e_\infty,\mathcal{X}_t^\dagger,\Delta}(\mathbf{A},\Psi) := \int_{[\mathcal{M}_{g,m,e_\infty}(\mathcal{X}_t,\Delta)]^{\mathrm{vir}}} \psi_1^{s_1} \cup \mathrm{ev}^*_{x_1}[Z_{A_1}] \cup \ldots \cup \psi_m^{s_m} \cup \mathrm{ev}^*_{x_m}[Z_{A_m}] \cup \mathrm{ev}^*_{y_1}[Z_{B_1}] \cup \ldots \cup \mathrm{ev}^*_{y_{e_\infty}}[Z_{B_{e_\infty}}].$$

This is equivalent to the count of torically transverse maps in $\mathcal{M}_{g,m,e_\infty}(\mathcal{X}_t, \Delta)$ with x_i (resp. y_j) mapping to a generic translate of A_i (resp. B_j) whose image under Forget is in the locus cut out by some generic ψ-class conditions.

Proposition 1.5. $\mathrm{GW}^{\log}_{g,m,e_\infty,\mathcal{X}_t^\dagger,\Delta}(\mathbf{A},\Psi)$ *is independent of t, so we can just compute it for $t = 0$. Also, the virtual and actual fundamental classes agree, so (after an application of Bertini) the log count is an actual naive enumerative count.*

Theorem 1.6.

$$\mathrm{GW}^{\mathrm{trop}}_{g,m,e_\infty,N_\mathbb{Q},\Delta}(\mathbf{A},\Psi) = \mathrm{GW}^{\log}_{g,m,e_\infty,\mathcal{X}_t^\dagger,\Delta}(\mathbf{A},\Psi).$$

1.3. **Proof sketch.**

1.3.1. *Log $\rightsquigarrow$ tropical.* Tropicalization $[\varphi : (C; x_1, \ldots, x_m; y_1, \ldots, y_{e_\infty}) \to \mathcal{X}_0] \rightsquigarrow (\Gamma, \mu, \epsilon, h)$ as follows:

- Irreducible component $C_V \subset \mathcal{X}_V \rightsquigarrow V \in N_{\mathbb{Q}}$.
- Node $C_V \cap C_{V'} \rightsquigarrow$ compact edge $E \ni V, V'$.
- $y_j \in C_V \rightsquigarrow$ unbounded edge $E_j \ni V$.
- $x_i \in C_V \rightsquigarrow \mu(i) = V$.

We see that $x_i \in Z_{A_i,Q_i} \implies V_i \in A_i$. Similarly with y_j's. So satisfying algebraic $\mathbf{A} \rightsquigarrow$ satisfying tropical $\mathbf{A}$.

C satisfying ψ's $\implies$ for each V, C_V satisfies $\bigcup_{i\in\mu^{-1}(V)} \mathrm{Forget}^* \overline{\psi_i}^{s_i}$, where here $\overline{\psi_i}$ is a class on $\overline{\mathcal{M}}_{0,\mathrm{val}(V)+m_V}$. By dimension counts, this forces $\sum_{i\in\mu^{-1}(V)} s_i \leq \dim(\overline{\mathcal{M}}_{0,\mathrm{val}(V)+m_V}) = \mathrm{ov}(V)$, and this is the tropical ψ-class condition.

1.3.2. *Tropical $\rightsquigarrow$ log.* Let $\mu^{-1}(V) = \{s_{i_1}, \ldots, s_{i_{m_V}}\}$, $\sum s_{i_j} = \mathrm{ov}(V)$. Then on $\overline{\mathcal{M}}_{0,\mathrm{ov}(V)}$,

$$\bigcup_{j=1}^{m_V} \psi_{i_j} = \binom{\mathrm{ov}(V)}{s_{i_1}, \ldots, s_{i_{m_V}}}_{i_j\in\mu^{-1}(V)} [\mathrm{pt}] = \frac{\mathrm{ov}(V)!}{\prod_{i\in\mu^{-1}(V)} s_i!}[\mathrm{pt}].$$

Denote this coefficient by $\langle V\rangle$.

Say we want to describe the log stable maps with tropicalization Γ. From the above, ψ-class conditions determine the domains of these maps up to $\prod_{V\in\Gamma^{[0]}} \langle V\rangle$ choices.

Fixing the domain curve, the possibilities for the (pre-log) map satisfying the incidence conditions are a fiber of

(3)

$$\Phi_{\mathbb{C}^*} : \prod_{V\in\Gamma^{[0]}} N\otimes\mathbb{C}^* \to \left(\prod_{E\in\Gamma_c^{[1]}} [(N/\mathbb{Z}u_E)\otimes\mathbb{C}^*]\right) \times \left(\prod_{i=1}^{m} [((N/\,\mathrm{L}(A_i))\cap N)\otimes\mathbb{C}^*]\right) \times \left(\prod_{j=1}^{e_\infty} [((N/\,\mathrm{L}(B_j))\cap N)\otimes\mathbb{C}^*]\right)$$
$$(H)_V \mapsto ((H_{\partial^+E}/H_{\partial^-E})_{E\in\Gamma_c^{[1]}}, (H_{\mu^{-1}(i)})_{i=1,\ldots,m}, (H_{\partial E_j})_{j=1,\ldots,e_\infty}).$$

Here, the V-components in the domain of $\Phi_{\mathbb{C}^*}$ corresponds to evaluation of φ at some chosen point on C_V (this evaluation and the marked domain curve C_V are in fact enough to determine $\varphi|_{C_V}$). Being in a certain fiber of the first factors of the codomain corresponds to requiring that adjacent C_V's actually meet so they can be glued at nodes. Being in certain fibers of the other factors of course corresponds to satisfying the incidence conditions.

Hence, the number of such pre-log curves is the degree of $\Phi_{\mathbb{C}^*}$, or equivalently, the index of the map without the $\otimes\mathbb{C}^*$'s:

$$\Phi_\Gamma := \prod_{V\in\Gamma^{[0]}} N \to \left(\prod_{E\in\Gamma_c^{[1]}} N/\mathbb{Z}u_{\partial^-E,E}\right) \times \left(\prod_{i=1}^{m} N/\,\mathrm{L}_N(A_i)\right) \times \left(\prod_{j=1,\ldots,e_\infty} N/\,\mathrm{L}_N(B_j)\right) \tag{4}$$
$$H \mapsto ((H_{\partial^+E} - H_{\partial^-E})_{E\in\Gamma_c^{[1]}}, (H_{\mu(i)})_{i=1,\ldots,m}, (H_{\partial E_j})_{j=1,\ldots,e_\infty}).$$

Finally, for a pre-log curve associated to Γ, there are $\prod_{E\in\Gamma_c^{[1]}} w(E)$ possible distinct log structures we can put on the curve ($w(E)$ at the node corresponding to E). We thus get that the tropical and log counts agree for

$$\mathrm{Mult}(\Gamma) := \mathrm{index}(\Phi) \prod_{E\in\Gamma_c^{[1]}} w(E) \prod_{V\in\Gamma^{[0]}} \langle V\rangle.$$

2. Talk 2: Broken lines and theta functions

The B-model side of mirror symmetry concerns sheaves and their sections, in particular sections of line bundles. Here we sketch [GHKK14]'s construction of theta functions (certain canonical bases of sections of line bundles) on cluster varieties. See also [GHK15] for the surfaces case and [GS12] for other contexts. The construction is in terms of scattering diagrams and broken lines. The construction of these objects is motivated by the A-model data and the idea that this should give B-model information.

The broad heuristic picture is that a holomorphic disk in $N \otimes \mathbb{C}^*$ corresponds to a tropical disk in $N_{\mathbb{R}}$, the final segment of which has a tangent direction and weight giving a vector $u \in N$, hence a function $z^u \in \mathbb{C}[N]$ on the mirror.

2.1. **Cluster-type varieties.** [GHK15] dealt with log Calabi-Yau surfaces, which they show can all be obtained from toric surfaces by blowing up some non-nodal points of the toric boundary (and then possibly doing some toric blowdowns). We'll look at the higher-dimensional analog where we allow blowups along "hypertori" in the boundary i.e., subvarieties $Z_{B,u,Q}$ as in my previous talk.

More precisely, we consider space $\mathcal{Y}$ which can be obtained as follows. Let $\overline{\mathcal{Y}}$ be a toric variety with cocharacter lattice N, D_e denoting the boundary divisor associated to $e \in N$. Let $\{(e_i, u_i) | i \in I\}$ be a finite collection of pairs $e_i \in N \setminus \{0\}$ contained in rays of the fan for $\overline{\mathcal{Y}}$, and $u_i \in M \setminus \{0\}$ such that the dual pairing $\langle e_i, u_i \rangle = 0$. We obtain $\mathcal{Y}$ by blowing up $\mathcal{Y}$ along the loci $\{(1 + z^{u_i})^{|e_i|}\} \cap D_{e_i}$ for each i. We note that $\{(1 + z^{u_i})^{|e_i|}\} \cap D_{e_i}$ is the same as d_i times what we called $Z_{u_i^{\perp}, e_i, \mathbf{1}}$.

Remark 2.1. We should assume that all the e_i's are contained in a convex cone in M. There are technical issues without this which can sometimes be worked around (e.g., by working with the universal torsor over $\mathcal{Y}$ instead) but which sometimes result in non-convergent theta functions.

Also, instead of $1 + z^{u_i}$ we could allow $a_i + z^{u_i} = a_i$ for some constants $a_i \in \mathbb{C}^*$, but this would complicate the results with extra coefficients. Besides, using "principal coefficients" reduces all cases to $a_i = 1$ cases (and to cases where the e_i's are part of a basis for M, taking care of the convexity issue above).

Example 2.2. For [GHKK14], [Mana], and pretty much anything on cluster varieties, there is a skew-symmetric form $\{\cdot, \cdot\}$ on N or M such that $u_i = \{e_i, \cdot\}$ or $e_i = \{u_i, \cdot\}$, or something similar, maybe up to scaling. I believe the methods of [GS11] make it possible to avoid this, but let me give some more details for the cluster setup.

In Mandy's talks she will associate scattering diagrams to a quiver Q. Here's how her quivers are viewed in the approach here. $N = \mathbb{Z}^{Q^{[0]}}$ (i.e., one dimension for each vertex) with basis $e_1, \ldots, e_n$ parametrized by the vertices of Q. Then we have a skew-symmetric form on N defined by $\{e_i, e_j\}$ =the number of arrows from i to j minus the number of arrows from j to i. The u_i's are then given by by $\{\cdot, e_i\} \in M$. The mirror construction is then given by switching the roles of N and M and of the e_i's and u_i's, and using $\{e_i, \cdot\}$ instead of $\{\cdot, e_i\}$. [Warning: I don't promise that I'm not mixing anything up or making any sign errors.]

Note: I don't just write $-\{e_i, \cdot\}$ instead of $\{\cdot, e_i\}$ because there is a generalization to "skew-symmetrizable" forms, and I believe [GS11] can be used to generalize even further to any $\mathbb{Z}$-valued bilinear form $\{\cdot, \cdot\}$ for which each $\{e_i, e_i\} = 0$.

Note: When defining theta functions from the perspective Mandy will discuss, I think one actually uses principal coefficients, meaning that N is replaced by $N_{\text{prin}} := N \oplus M$, the e_i's are as above, and $\{(n_1, m_1), (n_2, m_2)\}_{\text{prin}} := \{n_1, n_2\} + \langle n_1, m_2 \rangle - \langle n_2, m_1 \rangle$.

Why study these things? Cluster varieties make nice toy versions of a lot of what we can study with the Gross-Siebert program, but more than that, they include a lot of important spaces. The trivial cases are toric varieties. More generality, there are semi-simple Lie groups, their quotients by Borel subgroups (flag and partial flag varieties, like Grassmannians), quotients by unipotent radicals, $\overline{\mathcal{M}}_{0,n}$, various moduli spaces of local systems on punctured Riemann surfaces (cf. [FG06], these examples might include all the others), and I believe many other examples I don't know much about.

2.2. **The initial scattering diagram.** To construct theta functions on the mirror $\mathcal{Y}^\vee$ to $\mathcal{Y}$, we use the initial scattering diagram

$$\mathfrak{D}_{\text{in}} = \{(u_i^\perp, 1 + z^{e_i}, u_i) | i \in I\}$$

The u_i at the end is to denote (a version of) the "direction" of the wall. In other words, the wall-crossing function for crossing $u_i^\perp \subset N_\mathbb{R}$ is

$$z^n \mapsto z^n(1 + z^{e_i})^{-\langle n, u_i \rangle}.$$

Note that the roles of e_i and u_i have been switched, so gluing via wall-crossing now produces the mirror space $\mathcal{Y}^\vee$.

Then let $\mathfrak{D}$ be the consistent scattering diagram that can be obtained by adding only outgoing walls to $\mathfrak{D}_{\text{in}}$. (Consistent meaning that path-ordered products only depend on the initial and final points of the path).

2.3. **Broken lines.**

Definition 2.3. Let $q \in N \setminus \{0\}$, $Q \in N_\mathbb{R} \setminus \mathrm{Supp}(\mathfrak{D})$. A **broken line** γ with **ends** (q, Q) is the data of a continuous map $\gamma : (-\infty, 0] \to N_\mathbb{R} \setminus \mathrm{Joints}(\mathfrak{D})$, values $-\infty < t_0 \leq t_1 \leq \ldots \leq t_\ell = 0$, and for each $i = 0, \ldots, \ell$, an associated monomial $c_i z^{v_i} \in \Bbbk[N]$ with $c_i \in \Bbbk$, $v_i \in N$, such that:

- $\gamma(0) = Q$.
- For $i = 1 \ldots, \ell$, $\gamma'(t) = -v_i$ for all $t \in (t_{i-1}, t_i)$. Similarly, $\gamma'(t) = -v_0$ for all $t \in (-\infty, t_0)$.
- $c_0 z^{v_0} = z^q$.
- For $i = 0, \ldots, \ell - 1$, $\gamma(t_i)$ is contained in a wall $\mathfrak{d}$ of $\mathfrak{D}$. $c_{i+1} z^{v_{i+1}}$ is a monomial in the Laurent series expansion of $\theta_\mathfrak{d}(c_i z^{v_i})$ ($\theta_\mathfrak{d}$ denoting the wall-crossing automorphism associated to crossing $\mathfrak{d}$).

2.4. **Theta functions.** For $q \in N \setminus \{0\}$ and $Q \in N_\mathbb{R} \setminus \mathrm{Supp}(\mathfrak{D})$, we define

$$\vartheta_{q,Q} := \sum_{\mathrm{Ends}(\gamma) = (p,Q)} c_\gamma z^{n_\gamma},$$

where $c_\gamma z^{n_\gamma}$ denotes the final monomial attached to γ. (In general these theta functions are Laurent series). Define $\vartheta_0 = 1$. For each Q, the $\vartheta_{q,Q}$'s form an additive (topological) basis for a subalgebra A_Q of the Laurent series ring. Different choices of Q are related by path-ordered product ([CPS] shows this follows from consistency of the scattering diagram), so we have a canonical abstract algebra A with a canonical basis $\{\vartheta_q\}_{q \in N}$. One should view $\mathrm{Spec}\, A$ (or $\mathrm{Spf}\, A$) as the mirror to the space $\mathcal{Y}$ we started with.

2.5. **Multiplication rule.** The ϑ_q coefficient in $\vartheta_{q_1}\cdots\vartheta_{q_s}$ is given by the z^q coefficient of $\vartheta_{q_1,Q}\cdots\vartheta_{q_s,Q}$ for Q sufficiently close to q. In particular, for the constant (ϑ_0) coefficient c_0, one can use any $Q\in N_{\mathbb{R}}\setminus \mathrm{Supp}(\mathfrak{D})$, and then one has

$$c_0 = \sum_{\substack{\mathrm{Ends}(\gamma_i)=(q_i,Q)\\ ,}i=1,\ldots,s}\sum_{i=1}^{s} n_{\gamma_i} = 0c_1\cdots c_s.$$

Note that the condition $\sum_{i=1}^s n_{\gamma_i} = 0$ means the broken lines meet at a balanced s-valent vertex. This suggests a connection to tropical curve counts with a ψ^{s-2}-condition and a point condition at Q. We'll see this in the next talk.

3. Talk 3: Theta functions in terms of log GW invariants

This talk will combine the ideas from the previous two talks to explain how the theta functions can be defined in terms of certain descendant log GW numbers.

3.1. **Broken lines $\rightsquigarrow$ tropical curves.** When a broken line bends, it's easy to see how one can glue a ray at the break to "balence" the broken line, making it into a tropical curve. However, this approach is too simplistic (two much data is hidden in each wall). To actually turn sums of coefficients of the final monomials of broken lines into tropical GW invariants, one first has to modify the scattering diagram. Here's a sketch of the approach (due to [GPS10, §1.4, §2]):

- **Change the base** ring—for each initial wall $\mathfrak{d}_i$, adjoin a variable t_i. Replace the initial scattering functions $1+z^{e_i}$ with $1+t_iz^{e_i}$. Base change again, replacing each t_i with $\sum_{j=1}^k u_{ij}$, $u_{ij}^2=0$.
- The initial scattering functions can now be **factored** (cf. [GPS10, pg 16]). One then factors the initial scattering diagram, associating each factor of the initial scattering functions to a different wall.
- One then perturbs the new scattering diagram, translating each wall some small generic amount.
- A **recursive** procedure produces the corresponding consistent scattering diagram. Basically, when two walls meet, they form a new wall. The directions of the walls. The monomial of the new wall and the negatives of the monomials of the old walls are "balanced" (sum to 0).
- Broken lines with respect to this scattering diagram can be turned into tropical curves by gluing the broken line to segments and rays contained in scattering walls, with weighted directions being the exponents of monomials attached to the walls.
- Careful bookkeeping gives the theorem below.

Recall from the previous talk that our initial scattering diagram is $\mathfrak{D}_{\mathrm{in}} = \{(u_i^{\perp}, 1+z^{e_i}, u_i) | i\in I\}$. Let $\mathbf{e} := \{e_i | i\in I\}$. For $\mathbf{p} := (p_1,\ldots,p_s$, let $\mathfrak{W}_{\mathbf{e},\mathbf{p}}(0)$ be the set of weight vectors $\mathbf{w} := (\mathbf{w}_i)_{i\in I}$, $\mathbf{w}_i := (w_{i1},\ldots,w_{il_i})$ with $0<w_{i1}\leq\ldots\leq w_{il_i}$, such that

$$\sum_{i\in I}\sum_{j=1}^{l_i} w_{ij}e_i + \sum_{k=1}^{s} p_k = n.$$

Let $\Delta_{\mathbf{e},\mathbf{p},\mathbf{w}}$ denote the tropical degree consisting of pairs $(w_{ij}m_i,(i,j))$ for each i,j, (p_k,k) for each $k=1,\ldots,s$.

For $w \in \mathbb{Z}_{>0}$ define

$$R_{w,d} := \frac{d(-1)^{w-1}}{w^2}$$

Let $d_i := |u_i|$ (the index of u_i, i.e., u_i is $d_i > 0$ times a primitive vector). Define

$$R_{\mathbf{w}} := \prod_{i,j} R_{w_{ij},d_i}.$$

The following is a special case of [Mana, Thm 3.7 plus Thm. 2.11] (there I also deal with non-constant coefficients and quantum theta functions. I do use the skew-symmetry assumption there, but I think this can be avoided. See also Thm. 3.13 there for an analogous tropical description of the scattering diagram instead of the theta functions). See [GPS10, 2.8] and [CPS, Prop. 5.15] for related results.

Theorem 3.1 ([Mana])**.** *The constant (ϑ_0) coefficient of $\prod_{i=1}^s \vartheta_{p_i}$ is given by*

$$\sum_{\mathbf{w}\in\mathfrak{W}_{\mathbf{e},\mathbf{p}}(0)} \mathrm{GW}^{\mathrm{trop}}_{0,1,\Delta_{\mathbf{e},\mathbf{p},\mathbf{w}}}(\mathbf{A}_{\mathbf{w}},(s-2))\frac{R_{\mathbf{w}}}{|\operatorname{Aut}(\mathbf{w})|}\prod_{i,j} w_{ij}, \tag{5}$$

where $\mathbf{A}_{\mathbf{w}} := ((A_1),((B_{ij})_{ij},(B_k)_k))$ for A_1 a generic point in $\mathcal{N}_{\mathbb{Q}}$, B_{ij} a generic translate of $u_i^{\perp}$, and $B_k = \mathcal{N}_{\mathbb{Q}}$ for $k = 1,\ldots,s$. Furthermore, the constant coefficients for the cases with $s = 2$ and $s = 3$ are sufficient to uniquely determine the algebra generated by the theta functions.

Remark 3.2. We note that the constant coefficient of a function f is equal to $\int_\gamma f\Omega$, where γ is the class of the compact torus $(S^1)^n \subset (\mathbb{C}^*)^n = T_N$ (conjecturally the class of an SYZ fiber) and Ω is the holomorphic volume form with log poles along the boundary divisors. See [Manb, §6].

3.2. **Tropical curves $\rightsquigarrow$ log GW invariants.** Recall that we know from [MR16] that $\mathrm{GW}^{\mathrm{trop}} = \mathrm{GW}^{\log}$. So Theorem 3.1 can immediately be written using descendant log GW numbers. However, these are invariants of $\overline{\mathcal{Y}}$, not of $\mathcal{Y}$ (recall $\overline{\mathcal{Y}}$ was a toric variety and $\mathcal{Y}$ is obtained by blowing up $\overline{\mathcal{Y}}$ along $\{(1+z^{u_i})^{|e_i|}\}\cap D_{e_i}$ for each i). Relatedly, we have these ugly $\frac{R_{\mathbf{w}}}{|\operatorname{Aut}(\mathbf{w})|}\prod_{i,j} w_{ij}$ factors. We deal with both these issues now using the degeneration formula of [KL].

The degeneration: Assume the fan Σ for $\overline{\mathcal{Y}}$ contains rays through each e_i, and that no two e_i's share a cell (so $D_{e_i}\cap D_{e_j} = \emptyset$ for $i \neq j$. Take $\mathcal{P}$ to be a rational polyhedral decomposition refining Σ whose vertices consist of the origin and e_i for each $i \in I$ (still thinking about how exactly to do this). In the resulting space $\mathcal{X}$ (as in Talk 1) we blowup $Z_{u_i^{\perp},e_i,\mathbf{1}}$ for each $i \in I$. In other words, we take $\overline{\mathcal{Y}}\times\mathbb{A}^1$, blow up $(D_{e_i},0)$ for each i, then blow up the closure of $d_i Z_{u_i^{\perp},e_i,\mathbf{1}}\times(\mathbb{A}^1\setminus\{0\})$.

I'll probably draw pictures to explain the next part, but the point is that the curves we want to count in $\mathcal{X}_0$ are made up of components in $\overline{\mathcal{Y}}$ glued to components in the flaps. The contributions from the components in $\overline{\mathcal{Y}}$ is the $\mathrm{GW}^{\mathrm{trop}}_{0,1,\Delta_{\mathbf{e},\mathbf{p},\mathbf{w}}}(\mathbf{A}_{\mathbf{w}},(s-2))$ part of (5). $R_{\mathbf{w}}$ gives the contributions from the flaps ($R_{w,d}$ is d times the contribution of a w-fold cover of a $\mathbb{P}^1$-fiber in the flap with full ramification at a boundary point. We multiply by d because we blew up d_i times $Z_{u_i^{\perp},e_i,\mathbf{1}}$). Then $\frac{\prod_{i,j} w_{ij}}{|\operatorname{Aut}(\mathbf{w})|}$ comes from the degeneration formula (w_{ij} counts log structures at the nodes, $|\operatorname{Aut}(\mathbf{w})|$ makes up for over-labelling). Hence, from the [KL] log degeneration formula,[1] we get what [GHK15] called the Frobenius structure conjecture:

Theorem 3.3. *The constant (ϑ_0) coefficient of $\prod_{i=1}^{s} \vartheta_{p_i}$ is given by*

$$\mathrm{GW}^{\log}_{0,1,\mathcal{Y},\Delta_{\mathbf{p}}}(([\mathrm{pt}]),(s-2)),$$

and knowing these constant terms for the $s=2$ and $s=3$ cases completely determines the multiplication.

Remark 3.4. For the skew-symmetrizable cases (and other cases?) there is a quantization of the construction of the theta functions, which I showed in [Mana] can be described in a quantum count of tropical curves. In some cases, [Mik16] has shown that these quantum counts relate to certain counts of real curves (the power of q corresponds to the "logarithmic area" of the real curve). I suspect that there should be a description of the quantum theta functions in terms of some sort of real descendant log GW invariants. Helge and I are looking into this, but we still haven't quite managed to actually define the invariants.

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[1] There are some technical issues before their formula can be applied. We need the relative divisor to be smooth, e.g., just the union of the D_{e_i}'s which we assume are disjoint, so we need to check that forgetting the log structure at the other boundary divisors doesn't change our invariants. This amounts to checking that the curves we're counting are torically transverse. This is *not* a simple Bertini-type argument because the $Z_{u_i^\perp, e_i, \mathbf{1}}$'s are not generic.