

LOG GEOMETRY, TROPICAL GEOMETRY, AND MIRROR SYMMETRY FOR CLUSTER VARIETIES

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ABSTRACT. I will begin with a brief introduction to log geometry and log Gromov-Witten theory, and I will explain tropicalization from this log perspective. I will then give an overview of cluster varieties, which are defined by gluing together many copies of an algebraic torus, but which can also be understood in terms of blowups of toric varieties. I will then combine these ideas and explain how to construct a canonical basis of functions on a cluster variety in terms of the log Gromov-Witten theory of the mirror cluster variety.

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1. LOG GEOMETRY

Here we review the basics of log geometry, mostly following [Gro11, §3.2]. Roughly, a log structure on a space X is some sheaf-theoretic data which keeps track of:

- some divisors living in X , and/or
- how X fits into some larger space.

More precisely:

Definition 1.1. A **pre-log structure** on a scheme X is a sheaf of monoids \mathcal{M}_X on X together with a homomorphism

$$\alpha_X : \mathcal{M}_X \rightarrow \mathcal{O}_X,$$

where we use multiplication for the monoid structure on \mathcal{O}_X . A pre-log structure is a **log structure** if the restriction

$$\alpha_X : \alpha^{-1}(\mathcal{O}_X^\times) \rightarrow \mathcal{O}_X^\times$$

is an isomorphism. Here, \mathcal{O}_X^\times is the sheaf of invertible elements of \mathcal{O}_X .

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A **log scheme** X^\dagger is a scheme X equipped with a log structure. A **morphism of log schemes** $f^\dagger : X^\dagger \rightarrow Y^\dagger$ is a morphism of schemes $f : X \rightarrow Y$ together with a morphism of sheaves of monoids $f^b : f^{-1}\mathcal{M}_Y \rightarrow \mathcal{M}_X$ such that the following diagram commutes:

$$\begin{array}{ccc} f^{-1}\mathcal{M}_Y & \xrightarrow{f^b} & \mathcal{M}_X \\ \downarrow \alpha_Y & & \downarrow \alpha_X \\ f^{-1}\mathcal{O}_Y & \xrightarrow{f^\#} & \mathcal{O}_X \end{array}$$

Example 1.2 (The trivial log structure). Given any scheme X , the trivial log structure on X is given by taking $\mathcal{M}_X := \mathcal{O}_X^\times$ and α the inclusion of \mathcal{O}_X^\times into \mathcal{O}_X .

Example 1.3 (The divisorial log structure). Let $D \subset X$ be a closed subset of pure codimension 1. Then the **divisorial log structure induced by D** is the subsheaf $\alpha_X : \mathcal{M}_{(X,D)} \hookrightarrow \mathcal{O}_X$ of \mathcal{O}_X consisting of regular functions on X which are invertible on $X \setminus D$. I.e., if $j : X \setminus D \hookrightarrow X$ denotes the inclusion, then

$$\mathcal{M}_{(X,D)} = \mathcal{O}_X \cap (j_*\mathcal{O}_{X \setminus D}^\times).$$

Definition 1.4. The **ghost sheaf** $\overline{\mathcal{M}}_X$ of X^\dagger is defined by the short exact sequence

$$0 \rightarrow \mathcal{O}_X^\times \xrightarrow{\alpha_X^{-1}} \mathcal{M}_X \rightarrow \overline{\mathcal{M}}_X \rightarrow 0.$$

Examples 1.5 (Ghost sheaves for toric log schemes).

- (1) For example, if $X = \text{Spec } \mathbb{k}[x]$ and $D = \{0\}$, then

$$\Gamma(X, \mathcal{M}_{(X,D)}) = \mathbb{k}^\times \oplus \mathbb{N}$$

with $m \in \mathbb{N} = \mathbb{Z}_{\geq 0}$ corresponding to x^m . On the other hand, $\Gamma(X, \mathcal{O}_X^\times) = \mathbb{k}^\times$, so the quotient gives $\Gamma(X, \overline{\mathcal{M}}_X) = \mathbb{N} = \mathbb{Z}_{\geq 0}$. The log scheme of this example is denoted $(\mathbb{A}^1)^\dagger$.

- (2) Similarly, replacing \mathbb{A}^1 with \mathbb{A}^n and using the divisorial log structure with respect to the coordinate planes, we have $\Gamma(X, \overline{\mathcal{M}}_X) = \Gamma(X, \mathcal{M}_{(X,D)}) / \Gamma(X, \mathcal{O}_X^\times) \cong \mathbb{k}^\times \oplus \mathbb{N}^r / \mathbb{k}^\times = \mathbb{N}^r$.
- (3) More generally, let N be a lattice with dual lattice M , let Σ be a fan in $N_{\mathbb{R}} := N \otimes \mathbb{R}$, and let $\text{TV}(\Sigma)^\dagger$ be the corresponding toric variety $\text{TV}(\Sigma)$ equipped with the divisorial log structure induced by its toric boundary D . Then for σ a cone of Σ and x a closed point in the interior of the corresponding closed toric stratum of X , we have

$$\overline{\mathcal{M}}_{(X,D),x} = (\sigma^\vee / \sigma^\perp) \cap M.$$

Hence, $\sigma = \text{Hom}(\overline{\mathcal{M}}_{(X,D),x}, \mathbb{R}_{\geq 0})$. Patching these spaces $\text{Hom}(\overline{\mathcal{M}}_{(X,D),x}, \mathbb{R}_{\geq 0})$ together for all x (modulo certain identifications) recovers the support of the fan Σ .

This motivates the following (cf. [GS13, Appendix B]):

Definition 1.6. Given a log scheme X^\dagger , the **tropicalization of X^\dagger** is

$$X^{\text{trop}} := \bigsqcup_{x \in X} (\text{Hom}(\overline{\mathcal{M}}_{X,x}, \mathbb{R}_{\geq 0})) / \sim$$

where the disjoint union is over all scheme-theoretic points $x \in X$, and the equivalence relation \sim comes from the identifications of faces induced by the duals of the generization maps $\overline{\mathcal{M}}_{X,x} \rightarrow \overline{\mathcal{M}}_{X,x'}$ (x a specialization of x').

So, as we saw in Example 1.5, $\mathrm{TV}(\Sigma)^{\mathrm{trop}}$ is the support of the fan Σ .

Remark 1.7. Similarly, the **integral tropical points** $X^{\mathrm{trop}}(\mathbb{Z})$ are defined in the same way but replacing $\mathbb{R}_{\geq 0}$ with $\mathbb{Z}_{\geq 0}$, i.e.,

$$X^{\mathrm{trop}}(\mathbb{Z}) := \bigsqcup_{x \in X} (\mathrm{Hom}(\overline{\mathcal{M}}_{X^\dagger, x}, \mathbb{Z}_{\geq 0})) / \sim.$$

Suppose X has the divisorial log structure with respect to a normal crossings divisor D . We can blowup a stratum of D and then replace D with its proper transform, and then repeat, giving what we call **toric blowups** (\tilde{X}, \tilde{D}) of (X, D) . For each such (\tilde{X}, \tilde{D}) , components of \tilde{D} with non-negative integer multiplicity give discrete valuations on the function field of X , and these discrete valuations correspond bijectively to points of $X^{\mathrm{trop}}(\mathbb{Z})$.

As an example, $\mathrm{TV}(\Sigma)^{\mathrm{trop}}(\mathbb{Z})$ equals the intersection of N with the support of Σ . Recall that refinements of Σ correspond to blowups of strata of the toric boundary, and rays of rational slope in $N_{\mathbb{R}}$ (thus primitive elements of N) correspond to the possible toric boundary divisors.

So far we have seen how log structures arise from divisors. But there is one more important source of log structures, namely, pullback log structures. First though, we need another definition.

Definition 1.8. Let $\alpha : P_X \rightarrow \mathcal{O}_X$ be a pre-log structure on X . The **logification** of this pre-log structure is the log structure given by

$$\mathcal{M}_X := \frac{P_X \oplus \mathcal{O}_X^\times}{\{(p, \alpha(p)^{-1}) : p \in \alpha^{-1}(\mathcal{O}_X^\times)\}}$$

and

$$\alpha_X(p, f) := \alpha(p) \cdot f$$

(i.e., the fibered coproduct $P_X \oplus_{\alpha^{-1}(\mathcal{O}_X^\times)} \mathcal{O}_X^\times$ as sheaves of monoids equipped with morphisms to \mathcal{O}_X).

Definition 1.9. If $f : X \rightarrow Y$ is a morphism of schemes and Y has a log structure $\alpha_Y : \mathcal{M}_Y \rightarrow \mathcal{O}_Y$, then the composition

$$\alpha : f^{-1}\mathcal{M}_Y \xrightarrow{\alpha_Y} f^{-1}\mathcal{O}_Y \xrightarrow{f^\#} \mathcal{O}_X$$

gives a pre-log structure on X . The **pullback log structure** $f^*\mathcal{M}_Y$ on X is the log structure associated to this pre-log structure.

Example 1.10. For $(\mathbb{A}^r)^\dagger$ as in Example 1.5(2), and $i : \mathrm{Spec} \mathbb{k} \xrightarrow{0} \mathbb{A}^r$ the inclusion of the origin, the pullback log structure $i^*\mathcal{M}_{\mathbb{A}^r}$ on $X = \mathrm{Spec} \mathbb{k}$ is given by

$$i^*\mathcal{M}_{\mathbb{A}^r} = \mathbb{k}^\times \oplus \mathbb{N}^r$$

with α_X given by

$$(1) \quad \alpha_X(x, m) = \begin{cases} x & \text{if } m = 0 \\ 0 & \text{if } m \neq 0. \end{cases}$$

When $r = 0$, this is just $\mathrm{Spec} \mathbb{k}$ with the trivial log structure. When $r = 1$, the resulting log scheme is called the **standard log point**, denoted $\mathrm{Spec} \mathbb{k}^\dagger$.

More generally, we can replace \mathbb{N}^r here by any monoid Q with $Q^\times = 0$ to get another log structure on $\mathrm{Spec} \mathbb{k}$ given as in (1). Denote this by $\mathrm{Spec} \mathbb{k}_Q^\dagger$.

We now briefly mention a few technical conditions which may be mentioned later but not used explicitly. For the next set of definitions we reference [Ogu18].

Definition 1.11. Let P be a monoid. P is called **integral** if it satisfies the cancellation law, i.e., if $x + y = x' + y$ implies $x = x'$. P is called **fine** if it is integral and finitely generated. P is called **saturated** if it is integral and if, whenever¹ $p \in P^{\text{gp}}$ and $m \in \mathbb{Z}_{>0}$, $mp \in P$ implies $p \in P$. A fine saturated monoid P is called **toric** if P^{gp} is torsion free.

Given a monoid P , let \underline{P} denote the corresponding constant sheaf on X . Then a pre-log structure $\alpha : \underline{P} \rightarrow \mathcal{O}_X$ is said to be a **chart** for the corresponding log structure. The obvious example is $P \rightarrow \mathbb{k}[P]$, $p \mapsto z^p$ as a chart for a log structure on $\text{Spec } \mathbb{k}[P]$.

A log structure is called **quasi-coherent** if it can étale locally be given by charts. If the underlying monoids for these local charts are finitely generated, integral, fine, saturated, or toric, then the log structure is said to be **coherent**, integral, fine, saturated, or toric, respectively.

We now return to referencing [Gro11, §3.2].

Definition 1.12. A morphism of monoids $h : Q \rightarrow P$ is called **integral** if whenever $q_1, q_2 \in Q$, $p_1, p_2 \in P$, and $h(q_1) + p_1 = h(q_2) + p_2$, there exists some $q_3, q_4 \in Q$ and $p \in P$ such that $p_1 = p + h(q_3)$, $p_2 = p + h(q_4)$, and $q_1 + q_3 = q_2 + q_4$. One says that a morphism of fine log schemes is **integral** if the induced maps on stalks of the ghost sheaves are integral.

Definition 1.13. A morphism of log schemes $f^\dagger : X^\dagger \rightarrow Y^\dagger$ is called **strict** if $f^\# : f^* \mathcal{M}_Y \rightarrow \mathcal{M}_X$ is an isomorphism of log structures on X .

Definition 1.14. A morphism of fine log schemes $f^\dagger : X^\dagger \rightarrow Y^\dagger$ is **log smooth** if étale locally it fits into a commutative diagram

$$\begin{array}{ccc} X^\dagger & \longrightarrow & \text{Spec } \mathbb{Z}[P] \\ \downarrow f^\dagger & & \downarrow \\ Y^\dagger & \longrightarrow & \text{Spec } \mathbb{Z}[Q] \end{array}$$

such that:

- (1) The horizontal maps induce charts $\underline{P} \rightarrow \mathcal{O}_X$ and $\underline{Q} \rightarrow \mathcal{O}_Y$ for X^\dagger and Y^\dagger ,
- (2) The induced morphism $X \rightarrow Y \times_{\text{Spec } \mathbb{Z}[Q]} \text{Spec } \mathbb{Z}[P]$ is a smooth morphism of schemes, and
- (3) The right-hand vertical arrow is induced by a monoid homomorphism $Q \rightarrow P$ such that $\ker(Q^{\text{gp}} \rightarrow P^{\text{gp}})$ and the torsion part of $\text{coker}(Q^{\text{gp}} \rightarrow P^{\text{gp}})$ are finite groups of orders invertible on X .

Example 1.15. Let P be a toric monoid, and fix $\rho \in P \setminus P^\times$. If $X^\dagger = \text{Spec } \mathbb{k}[P]$ and $Y^\dagger = \text{Spec } \mathbb{k}[\mathbb{N}]$ (with the obvious log structures), then the map $\mathbb{N} \rightarrow P$, $1 \mapsto \rho$ induces a log smooth morphism $f^\dagger : X^\dagger \rightarrow Y^\dagger$. Furthermore, the central fiber with its pullback log structure X_0^\dagger is log smooth over $\text{Spec } \mathbb{k}^\dagger$.

For example, take $P = \langle (1, 0), (0, 1), (-1, a) \rangle \subset \mathbb{Z}^2$, $\rho = (0, a)$. We get that the family of log curves $\text{Spec } \mathbb{k}[x, y, t] / \langle xy = t^a \rangle^\dagger$ over $\text{Spec } \mathbb{k}[t]^\dagger$ is log smooth. Furthermore, the central fiber $\text{Spec } \mathbb{k}[x, y] / \langle xy \rangle$ with log structure induced by $P \rightarrow \mathbb{k}[x, y] / \langle xy \rangle$, $(1, 0) \mapsto x$, $(0, 1) \mapsto 0$, $(-1, a) \mapsto y$, is log smooth over $\text{Spec } \mathbb{k}^\dagger$ (with $1 \in \mathbb{N}$ for this log structure mapping to $(0, 1) \in P$).

¹Here, P^{gp} is the Groethendieck group of P , consisting of symbols $p - p'$ for $p, p' \in P$, modulo the relations $p - p' = q - q'$ whenever $p + q' = q + p'$.

2. LOG AND TROPICAL GROMOV-WITTEN THEORY

Here we review log Gromov-Witten theory as developed in [GS13] and [Che14, AC14]. We will mostly follow [GS13].

We work in the category of finite-type schemes over an algebraically closed field \mathbb{k} of characteristic 0, with log structures making them fine saturated log schemes over some fixed fine saturated log scheme S^\dagger .

Definition 2.1. A **pre-stable marked log curve** over W^\dagger is a proper log smooth integral morphism $\pi^\dagger : C^\dagger \rightarrow W^\dagger$ of fine saturated log schemes over S^\dagger with reduced connected curves as the geometric fibers, and with a tuple of sections $\mathbf{x} = (x_1, \dots, x_s)$ of π such that for U the non-critical locus of π , $\overline{\mathcal{M}}_C|_U = \pi^* \overline{\mathcal{M}}_W \oplus \bigoplus_i (x_i)_* \mathbb{N}_W$. By a theorem of F. Kato [Kat00], if 0 is a geometric point of W and $\sigma : Q \rightarrow \mathcal{O}_W$ is a local chart, then each point p of the fiber C_0 of C over 0 étale locally looks like one of the following:

- (1) a smooth unmarked point with log structure pulled back from the base (so $\overline{\mathcal{M}}_{C,p} = \overline{\mathcal{M}}_{W,0}$),
- (2) a smooth marked point $x_i(0)$ where $\overline{\mathcal{M}}_{C,x_i(0)} = \overline{\mathcal{M}}_{C,0} \oplus \mathbb{N}$, and $1 \in \mathbb{N}$ maps to a local defining equation for $x_i(0)$, or
- (3) a degeneration to a node, i.e., C locally looks like $V = \text{Spec } A[z, w]/(zw - t)$, where A the étale local ring for W at 0 with t in its maximal ideal—let $Q = \overline{\mathcal{M}}_{W,0}$, and require $\sigma(\rho_q) = t$ for a fixed nonzero $\rho_q \in Q$. The log structure is induced by

$$Q \oplus_{\mathbb{N}} \mathbb{N}^2 \rightarrow A[z, w]/(zw - t) \quad (q, (a, b)) \rightarrow \sigma(q)z^a w^b,$$

where $\mathbb{N} \rightarrow \mathbb{N}^2$ is the diagonal embedding and $\mathbb{N} \rightarrow Q$ is given by $1 \mapsto \rho_q$. In the case $W^\dagger = \text{Spec } \mathbb{k}^\dagger$, this means that C^\dagger/W^\dagger étale locally looks like the degeneration $xy = t^a$ in Example 1.15.

In particular, this implies that forgetting the log structures yields an ordinary pre-stable marked curve. A **stable marked log curve** is a pre-stable marked log curve whose underlying pre-stable marked curve is stable.²

Definition 2.2. A **stable log map** to X^\dagger with base W^\dagger is a pre-stable marked log curve $\psi^\dagger : C^\dagger \rightarrow W^\dagger$ together with a map $\varphi^\dagger : C^\dagger \rightarrow X^\dagger$ such that

$$\begin{array}{ccc} C^\dagger & \xrightarrow{\varphi^\dagger} & X^\dagger \\ \downarrow \psi^\dagger & & \downarrow \\ W^\dagger & \longrightarrow & S^\dagger \end{array}$$

commutes, and such that the underlying pre-stable map is stable.³

Suppose now that $W^\dagger = \text{Spec } \mathbb{k}_Q^\dagger$ (like in Example 1.10). Given a stable log map as above, let $\mathcal{M} := \varphi^* \mathcal{M}_X$. For x a point of C , let $P_x := \overline{\mathcal{M}}_x$.

²A pre-stable curve C is a connected nodal marked curve. Let \tilde{C} be the normalization of C and call a point $x \in \tilde{C}$ special if it corresponds to a marked point or a node in C . Then C is called stable if each genus 0 component of \tilde{C} contains at least three special points, and each genus one component contains at least one special point.

³Stable meaning that the normalization of each contracted component contains at least three special points, cf. Footnote 2.

- For each generic point η of C , we have a map

$$\varphi_\eta^\sharp : P_\eta \rightarrow Q.$$

- For x a marked point, we have $\varphi_x^\sharp : P_x \rightarrow Q \oplus \mathbb{N}$, with the projection to Q being determined by φ_η^\sharp for η the generic point whose closure contains x . So the new data from the marked point is the projection of φ_x^\sharp to the second coordinate,

$$u_x : P_x \rightarrow \mathbb{N}.$$

If X^\dagger has the divisorial log structure with respect to some D , then u_x tells us the intersection multiplicities of C with the components of D at the point x .

- For q a node of C , let η_1, η_2 be the generic points for the components of C containing q . Let $\chi_i : P_q \rightarrow P_{\eta_i}$ denote the two generization maps. Then $\varphi_q^\sharp : P_q \rightarrow Q \oplus \mathbb{N}^2$ is uniquely determined by $\varphi_{\eta_1}^\sharp, \varphi_{\eta_2}^\sharp$, and this determines a map $u_q : P_q \rightarrow \mathbb{Z}$ such that for each $m \in P_q$,

$$\varphi_{\eta_2}^\sharp(\chi_2(m)) - \varphi_{\eta_1}^\sharp(\chi_1(m)) = u_q(m) \cdot \rho_q.$$

Let Γ_C denote the dual graph to C^\dagger , i.e., Γ has a vertex V_η for every component C_η of C , a compact edge E_q for every node $q \in C$, and a half-edge E_x for every marked point $x \in C^\dagger$. The data Γ_C , plus the data \mathbf{u} of the maps u_x and u_q as above for all marked points x and nodes q is called the **type** of $\varphi^\dagger : C^\dagger \rightarrow X^\dagger$.

2.1. Tropicalization of log curves over the standard log point. Suppose $Q = \mathbb{N}$, i.e., W^\dagger is the standard log point $\text{Spec } \mathbb{k}^\dagger$. Then each φ_η^\sharp can be viewed as a point p_η in the dual cone $\text{Hom}(P_\eta, \mathbb{N})$, i.e. a point in $X^{\text{trop}}(\mathbb{Z})$. Similarly, for each marked point x , u_x can be viewed as a point p_x in the dual cone P_x^\vee of P_x . This determines a map $h : \Gamma_C \rightarrow X^{\text{trop}}$ such that $h(V_\eta) = p_\eta$ for each vertex V_η , and such that $h(E_x)$ points in the direction p_x . We give Γ_C the additional data of a weight function $w : \Gamma_C^{[1]} \rightarrow \mathbb{N}$ assigning a weight to edge: for x a marked point, $w(E_x)$ is the index⁴ of the point p_x in P_x^\vee . For q a node, $w(E_q) = |u_q| \in \mathbb{Z}_{\geq 0}$. Note that every compact edge has a positive lattice length given by the corresponding ρ_q , including any contracted self-adjacent edges.

The weighted graph Γ_C , together with the map $h : \Gamma_C \rightarrow X^{\text{trop}}$, is called the **tropicalization** φ^{trop} of $\varphi^\dagger : C^\dagger \rightarrow X^\dagger$.

I will not have time for the reverse direction of the tropical correspondence results, i.e., going from tropical curves to log curves. For this, see [NS06, MR] for the perspective I use, or [Ran17, Gro15] for a perspective using tropical intersection theory.

2.2. The basic log point. Suppose we have the data of stable map $\varphi : C \rightarrow X$, plus the additional type data \mathbf{u} . In general there are many ways to lift this to the data of a log stable map $\varphi^\dagger : C^\dagger \rightarrow X^\dagger$ with base $W^\dagger = \text{Spec } \mathbb{k}^\dagger$ the standard log point, and these different choices may have different tropicalizations. The **basic** (or **minimal** in [AC14] terminology) log structure on W comes from taking Q^\vee to be essentially the monoid of all such tropicalizations.

For each generic point η , $\varphi^{\text{trop}}(V_\eta)$ could be any point of P_η^\vee , subject to the condition that of $q = C_{\eta_1} \cap C_{\eta_2}$ is a node, then $\varphi^{\text{trop}}(C_{\eta_1}) - \varphi^{\text{trop}}(C_{\eta_2})$ is a positive integer multiple of u_q . We thus see

⁴A nonzero element u' of a monoid is called primitive if it is not equal to a positive multiple of any other element of the monoid. If $u = ku'$ for u' primitive and $k \in \mathbb{N}$, we call k the index of u .

that Q^\vee should be given by:⁵

$$Q^\vee := \left\{ ((V_\eta)_\eta, (e_q)_q) \in \bigoplus_\eta P_\eta^\vee \oplus \bigoplus_q \mathbb{N} \mid \forall q : V_{\eta_1} - V_{\eta_2} = e_q u_q \right\}.$$

Then W with the log structure corresponding to the monoid Q is the **basic log point** for the data $\varphi : C \rightarrow X$ and \mathbf{u} , and a stable log map is called **basic** if the base W^\dagger is the basic log point.

Theorem 2.3 ([GS13]). *The category of basic stable log maps to X^\dagger/S^\dagger forms an algebraic stack. If we impose a “combinatorially finite” collection of conditions β (e.g., specifying the degree and certain intersection multiplicities to ensure that only finitely many types are possible), the resulting algebraic stack $\mathcal{M}(X^\dagger/S^\dagger, \beta)$ is proper over S^\dagger . If, moreover, X^\dagger is log smooth over S^\dagger , then $\mathcal{M}(X^\dagger/S^\dagger, \beta)$ admits a virtual fundamental class allowing one to define log Gromov-Witten invariants (which are invariant in log smooth families).*

Log Gromov-Witten invariants are nice because they:

- are invariant in log smooth families,
- easily allow one to impose tangency conditions with boundary divisors,
- naturally are related to tropical curve counts (so can be understood combinatorially), and
- they seem to be enumerative (rather than just virtual counts) more often than ordinary Gromov-Witten invariants. E.g., all genus 0 log Gromov-Witten invariants of toric varieties are enumerative, essentially because the log cotangent bundle of a toric variety is trivial.

3. CLUSTER VARIETIES

Cluster algebras were first defined in [FZ02] in order to better certain canonical bases and positivity properties arising in representation theory. A geometric version called cluster varieties was then defined in [FG09], and these were interpreted from the birational geometry viewpoint in [GHK13]. Cluster varieties have become very important in geometric representation theory, with examples of cluster varieties including double Bruhat cells of reductive Lie groups [BFZ05], partial flag varieties [GLS08], and moduli of decorated local systems on punctured Riemann surfaces [FG06]. They also include all of the two-dimensional log Calabi-Yau varieties as studied in [GHK15], cf. [GHK13, §5]. Cluster structures are also closely related to the DT-theory of quiver representations [Rei10, Nag13, KS, KS14, Bri17]. And there are still many more applications which I am leaving out.

[GHKK14] used ideas from mirror symmetry (as developed in [GS11, CPS, GHK15, GHS]) to construct the conjectured totally positive canonical bases of functions on these cluster varieties. In [Manb], the author showed that these “theta bases” are uniquely determined by certain descendant log Gromov-Witten invariants. We will briefly review these constructions here.

3.1. Seeds and mutations. A **seed** is a collection of data \mathbf{S} of the form

$$(2) \quad \mathbf{S} := (N, I, E := \{e_i\}_{i \in I}, F, B),$$

⁵It is possible that some $u_q = 0$, in which case identifying the V_η subject to these constraints is not sufficient to determine an element of Q^\vee —we still need to separately specify e_q (tropically, this means specifying a lattice length for contracted edges).

where N is a lattice of finite rank, I is an index set with $|I| = \text{rank}(N)$, E is a basis for N , F is a subset of I , and B is a \mathbb{Z} -valued skew-symmetric⁶ bilinear pairing on N . If $i \in F$, we say e_i is **frozen**. Let $I_{\text{uf}} := I \setminus F$, and let N_{uf} be the span of $\{e_i\}_{i \in I_{\text{uf}}}$.

Let M denote the dual lattice $N^\vee = \text{Hom}(N, \mathbb{Z})$, and let $\{e_i^*\}_{i \in I} \subset M$ denote the dual basis to E . We use $\langle \cdot, \cdot \rangle$ to denote the dual pairing between N and M . For any lattice L , let $T_L := L \times \mathbb{C}^* = \text{Spec } \mathbb{C}[L^\vee]$. Note that the seed \mathbf{S} determines functions

$$A_i := z^{e_i^*} \in \mathbb{C}[M]$$

on $\mathcal{A}_{\mathbf{S}} := T_N$ (called cluster variables), and also

$$X_i := z^{e_i} \in \mathbb{C}[N]$$

on $\mathcal{X}_{\mathbf{S}} := T_M$. Let π_1 and π_2 denote the maps $N \rightarrow M$ given by $n \mapsto B(n, \cdot)$ and $n \mapsto B(\cdot, n)$, respectively, and let $K_i := \ker \pi_i$ (these may be distinct if B is only skew-symmetrizable instead of skew-symmetric).

For $a \in \mathbb{R}$, define $[a]_+ := \max(a, 0)$. Given a seed \mathbf{S} as above and a choice of $j \in I \setminus F$, we define the **mutation** of \mathbf{S} with respect to j to be the seed $\mu_j(\mathbf{S}) := (N, I, E' = \{e'_i\}_{i \in I}, F, B)$, where the vectors e'_i are defined by

$$(3) \quad e'_i := \mu_j(e_i) := \begin{cases} e_i + [B(e_i, e_j)]_+ e_j & \text{if } i \neq j \\ -e_j & \text{if } i = j. \end{cases}$$

We also have birational maps $\mu_j^{\mathcal{A}} : \mathcal{A}_{\mathbf{S}} \rightarrow \mathcal{A}_{\mu_j(\mathbf{S})}$ and $\mu_j^{\mathcal{X}} : \mathcal{X}_{\mathbf{S}} \rightarrow \mathcal{X}_{\mu_j(\mathbf{S})}$ given by

$$\mu_j^{\mathcal{A}}(z^m) = z^m (1 + z^{\pi_1(e_j)})^{\langle e_j, m \rangle}$$

and

$$\mu_j^{\mathcal{X}}(z^n) = z^n (1 + z^{e_j})^{\langle \pi_2(e_j), n \rangle}.$$

Remark 3.1. We could compactify each $\mathcal{A}_{\mathbf{S}}$ and $\mathcal{X}_{\mathbf{S}}$ to get nonsingular compact toric varieties with log structure corresponding to their toric boundaries. Then $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ would be the respective tropicalizations. The seed mutation $e_i \mapsto e_i + [B(e_i, e_j)]_+$ of (3) can then be viewed as the tropicalization of the map $\mu_j^{\mathcal{A}}$, cf. [GHK13, Rmk. 2.3].

3.2. Geometric picture of cluster varieties. In general, for L a lattice with dual L^\vee , $u \in L$ and $v \in L^\vee$, the map

$$\mu_{L,u,v} : T_L \rightarrow T_L, \quad z^w \mapsto z^w (1 + z^v)^{\langle u, w \rangle}$$

can be interpreted geometrically as follows: Let Σ_u be the fan in $L_{\mathbb{R}}$ with rays generated by u and $-u$. Let u' denote the primitive vector in the direction u . The map $L \rightarrow L/\mathbb{Z}u'$ induces a \mathbb{P}^1 fibration of the toric variety $\text{TV}(\Sigma_u)$ over $T_{L/\mathbb{Z}u'}$. Then $\mu_{L,u,v}$ is given by including T_L into $\text{TV}(\Sigma_u)$, blowing up the (possibly reduced) subscheme $H_v := D_{u'} \cap Z\langle (1 + z^v)^{|u|} \rangle$, contracting the proper transform \tilde{F} of the fibers F which hit H_v , and then taking the complement of the proper transforms of the boundary divisors. See Figure 3.1.

We can use the mutations $\mu_j^{\mathcal{A}}$ and to glue $\mathcal{A}_{\mathbf{S}}$ to $\mathcal{A}_{\mu_j(\mathbf{S})}$ for each $j \in I \setminus F$. Furthermore, since $\mu_j(\mathbf{S})$ is again a seed, we could mutate again, repeatedly. Gluing together all tori $\mathcal{A}_{\mathbf{S}}$ obtained in this

⁶More generally, B might only be “skew-symmetrizable” rather than skew-symmetric, meaning that there exists a skew-symmetric form ω and some positive numbers d_j such that $B(e_i, e_j) = d_j \omega(e_i, e_j)$ for each $i, j \in I$ which are not both in F . We stick to skew-symmetric cases for simplicity of exposition.

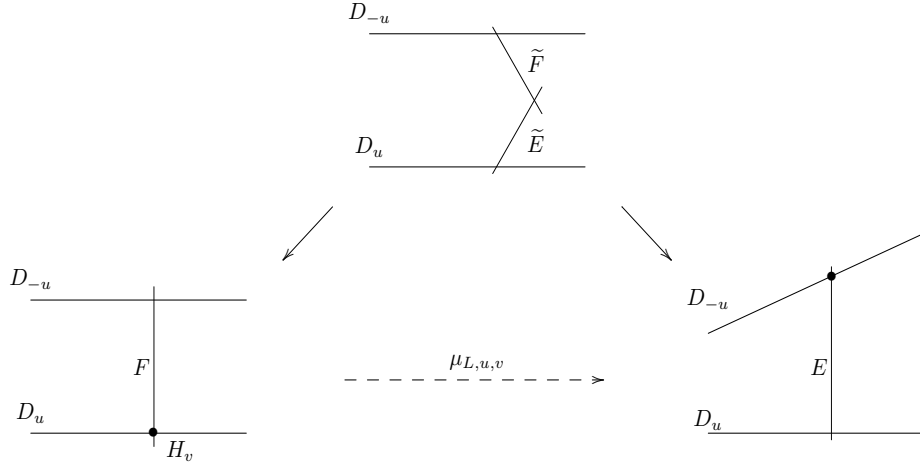


FIGURE 3.1.

way via the birational maps induced by the sequences of mutations, we obtain the cluster \mathcal{A} -variety \mathcal{A} . Then $\Gamma(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$ is called the **upper cluster algebra**, while the subalgebra generated by the cluster variables for all the different seeds is called the **cluster algebra** (the fact that this really is a subalgebra is called the *Laurent phenomenon*).

We can of course apply the same procedure to the \mathcal{X} -tori using the \mathcal{X} -mutations, yielding a cluster \mathcal{X} -variety \mathcal{X} .

Theorem 3.2 ([GHK13]). *The mutations $\mu_j^{\mathcal{X}}$ preserve the centers of the blowups corresponding to $\mu_i^{\mathcal{X}}$ for $i \in I \setminus F$. Hence, \mathcal{X} is covered up to codimension 2 by $\mathcal{X}_{\mathbf{s}} \cup \bigcup_{j \in I \setminus F} \mathcal{X}_{\mu_j(\mathbf{s})}$. I.e., up to codimension 2, \mathcal{X} is given by taking a toric compactification of $\mathcal{X}_{\mathbf{s}}$, blowing up the hypertori $D_{\pi_2(e_i)} \cap Z \langle (1+z^{e_i})^{|\pi_2(e_i)|} \rangle$ for each $i \in I$, and then forgetting the toric boundary.*

A similar statement holds for \mathcal{A} if B is non-degenerate, or if we generically deform the coefficients in the definition of $\mu_j^{\mathcal{A}}$.

3.3. The exact sequence of cluster varieties. Consider the (not necessarily exact) sequence

$$0 \rightarrow K_2 \rightarrow N \xrightarrow{\pi_2} M \xrightarrow{\lambda} K_1^* \rightarrow 0.$$

This becomes exact after tensoring with \mathbb{C}^* , and furthermore, the resulting sequence commutes with mutations. We thus obtain the sequence

$$0 \rightarrow T_{K_2} \rightarrow \mathcal{A} \xrightarrow{\pi_2} \mathcal{X} \xrightarrow{\lambda} T_{K_1^*} \rightarrow 0.$$

This sequence has a lot of nice properties:

- \mathcal{X} has the structure of a Poisson manifold via $\{z^{n_1}, z^{n_2}\} := B(n_1, n_2)z^{n_1+n_2}$. The fibers of λ form the symplectic leaves of \mathcal{X} .
- Let \mathcal{X}_t denote a fiber of λ , and let $\overline{\mathcal{X}}_t$ denote a (partial) compactification whose boundary components correspond to primitive vectors $\pi_2(e_i)$ for $i \in F$. Then K_2 can be identified with $A_1(\overline{\mathcal{X}}_t)$, cf. [Manb, Thm 2.8] (or [GHK13, Thm. 5.5] for rank 2 cases without frozen vectors).
- Similarly, the lattice $\text{coker}(\pi_2)$ can be identified with $\text{Pic}(\overline{\mathcal{X}}_t)$, cf. [Man18, Thm. 3.3] (or [GHK13, Thm. 4.1] in the absence of frozen vectors).

- $\overline{\mathcal{A}}_t$ (i.e., a partial compactification of \mathcal{A} with some coefficients so that it maps to \mathcal{X}_t instead of \mathcal{X}_e) is the universal torsor over $\overline{\mathcal{X}}_t$. Roughly, this means that

$$\Gamma(\overline{\mathcal{A}}_t, \mathcal{O}_{\overline{\mathcal{A}}_t}) = \bigoplus_{\mathcal{L} \in \text{Pic}(\overline{\mathcal{X}}_t)} \Gamma(\overline{\mathcal{X}}_t, \mathcal{O}_{\overline{\mathcal{X}}_t}(\mathcal{L}))$$

(called the Cox ring of $\overline{\mathcal{X}}_t$), cf. [Man18, §3], or [GHK13, §4] in the absence of frozen vectors.

3.4. Theta bases. We sketch a version of the [GHKK14] construction here. Let $\overline{N} := \pi_2(N)$. After possibly modifying the frozen part of our seed data, we can assume⁷ that \overline{N} is saturated in M and that the vectors $\pi_2(e_i)$ for $i \in F$ form the rays for a complete nonsingular fan Σ in \overline{N} , which as a set is equal to $\overline{\mathcal{X}}_t^{\text{trop}}$. We assume that the boundary divisors of $\overline{\mathcal{X}}_t$ are the ones corresponding to these rays. Let $\varphi : \overline{N} \rightarrow N$ denote the Σ -piecewise-linear section of π_2 taking $\pi_2(e_i)$ to e_i for each $i \in F$.

Let N^\oplus be the submonoid of N generated by $\{e_i\}_{i \in I \setminus F}$, and let $N^+ := N^\oplus \setminus \{0\}$. Let \mathfrak{m} be the maximal monomial ideal of $\mathbb{k}[N^\oplus]$.

A **wall** (\mathfrak{d}, f) in $\overline{N}_\mathbb{R}$ is the data of a function

$$f \in \mathbb{k}[[z^n]] \subset \mathbb{k}[[N^\oplus]]$$

for some $n \in N^+$, and a convex (but not necessarily strictly convex) rational polyhedral cone

$$\mathfrak{d} \subset \overline{N}_\mathbb{R}$$

such that $f \equiv 1$ modulo \mathfrak{m} and such that the linear span of \mathfrak{d} contains $\pi_2(n)$. The vector $-\pi_2(n) \in \overline{N}$ is called the **direction** of the wall. The wall is called **incoming** if \mathfrak{d} contains $\pi_2(n)$ and **outgoing** otherwise.

A scattering diagram \mathfrak{D} is a collection of walls such that, for each $k > 0$, all but finitely many walls are trivial modulo \mathfrak{m}^k . If a path γ crosses a wall $(\mathfrak{d}, f) \in \mathfrak{D}$ at time t , we have a wall-crossing automorphism

$$\theta_{\mathfrak{d}, \gamma}(z^p) = z^p f^{\langle u, \pi_2(p) \rangle}$$

where $u \in \overline{N}^\vee$ is a primitive element of \mathfrak{d}^\perp which is positive on $\gamma'(t)$. We then define the path-ordered product along γ as the composition of the wall-crossing automorphisms for all the walls crossed by γ . \mathfrak{D} is called consistent if path-ordered products only depend on the endpoints of the path.

Let

$$\mathfrak{D}_{\text{in}} := \{e_i^\perp \cap \overline{N}_\mathbb{R}, (1 + z^{e_i})^{|\pi_1(e_i)|} | i \in I \setminus F\}.$$

Up to equivalence, there exists a unique consistent scattering \mathfrak{D} whose only incoming walls are \mathfrak{D}_{in} .

Given an element $p \in \overline{N}$ and a generic $Q \in \overline{N}_\mathbb{R}$, a broken line with ends (p, Q) is a piecewise straight path $\gamma : (-\infty, 0] \rightarrow \overline{N}_\mathbb{R}$ such that each straight segment L has an attached monomial $c_L z^{n_L} \in \mathbb{C}[N]$ with $\pi_2(n_L) = -\gamma'|_L$, the first straight segment has attached monomial $z^{\varphi(p)}$, and such that if L_1, L_2 are two consecutive straight segments, then the bend occurs on a wall (\mathfrak{d}, f) of \mathfrak{D} , and $c_{L_2} z^{n_{L_2}}$ is a term of $\theta_{\mathfrak{d}, \gamma}(c_{L_1} z^{n_{L_1}})$. Let $c_\gamma z^{n_\gamma}$ denote the final monomial of γ . We define $\vartheta_{p, Q} := \sum_\gamma c_\gamma z^{n_\gamma}$, where the sum is over all broken lines with ends (p, Q) . In particular, $\vartheta_{0, Q} = 1$.

For any fixed Q , the theta functions $\{\vartheta_{p, Q}\}_{p \in \overline{N}}$ form a topological $\mathbb{k}[[K_2 \cap N^\oplus]]$ -module basis for $\mathbb{k}[[N^\oplus]]$, hence also for the subalgebra X they generate. X with this basis does not depend on Q (only

⁷Such assumptions are not necessary for the [GHKK14] construction, but they are useful when making the connections to geometry.

the embedding into $\mathbb{k}[[N^\oplus]]$ does). Furthermore, X can be identified with $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, or at least with a formal version of this.⁸ This gives the [GHKK14] theta basis for the \mathcal{X} -space.⁹

Theorem 3.3 ([Manb], the Frobenius structure conjecture for cluster varieties). *The ϑ_0 -coefficient of $\vartheta_{p_1} \cdots \vartheta_{p_s}$ is given by $\sum_{\beta \in \text{NE}(\overline{\mathcal{X}}_t)} z^\beta N_\beta(p_1, \dots, p_s)$, where $N_\beta(p_1, \dots, p_s)$ is the descendant log Gromov-Witten number which counts maps $\varphi : (C, x_1, \dots, x_s, x) \rightarrow \overline{\mathcal{X}}_t$ with $\varphi_*[C] = \beta$, with generically specified domain marked curve, with $\varphi(x)$ a generically specified point, and with $\varphi(x_i)$ mapping to D_{p_i} with intersection multiplicity $|p_i|$. Furthermore, these ϑ_0 -coefficients for $s = 2, 3$ uniquely determine the theta functions.*

Here, z^β is interpreted as a monomial in $\mathbb{k}[K_2]$ using the identification of K_2 with $A_1(\overline{\mathcal{X}}_t)$ mentioned in §3.3.

Proof sketch.

Combinatorial techniques as in [GPS10, CPS] are used in [Mana] to relate the broken line enumeration to counts of tropical curves in $\overline{N}_{\mathbb{R}}$. Then a tropical correspondence theorem (from [MR] or [Gro15], which extend [NS06] to descendant invariants) is used to relate these tropical counts to log Gromov-Witten invariants in $\text{TV}(\Sigma)$. Then we take a log smooth degeneration of $\overline{\mathcal{X}}_t$ to $\text{TV}(\Sigma)$ with some easily understood “flaps” glued on, and we use a degeneration formula as in [KLR, Li02, Ran] to relate to the desired invariants of the cluster variety. \square

Remark 3.4 (Related works). [KY] proves a version of Theorem 3.3 for affine log Calabi-Yau varieties which contain a Zariski dense algebraic torus. Their approach uses rigid analytic disks instead of log curves. Also, [GS18, GS] show that certain “punctured invariants” (a generalization of log Gromov-Witten invariants) determine mirrors with theta bases in a very general setting.

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⁸For skew-symmetrizable B , we actually get the Langland’s dual \mathcal{X} -space here, i.e., the \mathcal{X} -space obtained when we replace B with $-B^{-T}$.

⁹The [GHKK14] construction gives a basis for the upper [cluster algebra with “principle coefficients” and recovers the bases for the \mathcal{A} - and \mathcal{X} -algebras from this. In this principle coefficients setting, the \mathcal{A} - and \mathcal{X} -spaces are the same, so all the theta bases can be recovered from the construction here.

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