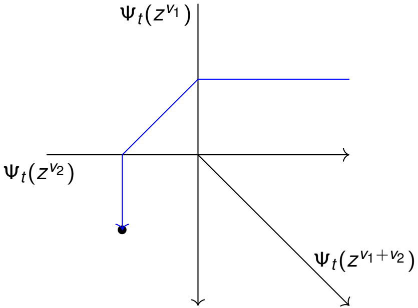
Quantum theta bases

Travis Mandel

(based joint work with Ben Davison and work in progress with Fan Qin)



Outline

- Gross-Hacking-Keel-Kontsevich (GHKK):
  Ideas from mirror symmetry (the Gross-Siebert program) $\rightsquigarrow$ canonical "**theta bases**" for classical cluster algebras.

Outline

- Gross-Hacking-Keel-Kontsevich (GHKK):
  Ideas from mirror symmetry (the Gross-Siebert program) $\rightsquigarrow$ canonical "**theta bases**" for classical cluster algebras.

- Davison-M:
  Quiver DT-theory + GHKK arguments $\rightsquigarrow$ **quantum theta bases.**

Outline

- Gross-Hacking-Keel-Kontsevich (GHKK):
  Ideas from mirror symmetry (the Gross-Siebert program) $\rightsquigarrow$ canonical "**theta bases**" for classical cluster algebras.

- Davison-M:
  Quiver DT-theory + GHKK arguments $\rightsquigarrow$ **quantum theta bases.**

- Qin-M (in progress):
  Bracelet bases for (quantum) Skein algebras coincide with the (quantum) theta bases.

Seeds

A (skew-symmetric) **seed** is the data $S = (N, I, E, F, B)$, where

- N is a finite-rank lattice;
- I is an index-set;
- $E = \{e_i | i \in I\}$ is a basis for N;
- $F \subset I$ (the frozen indices);
- $B(\cdot, \cdot)$ is a $\mathbb{Z}$-valued skew-symmetric form on N.

The dual lattice

Denote $M = \mathrm{Hom}(N, \mathbb{Z})$.

- Let $\langle \cdot, \cdot \rangle : N \oplus M \to \mathbb{Z}$ denote the dual pairing.
- Let $v_i := B(e_i, \cdot) \in M$.
- Let $M^{\oplus}$ denote the positive span of the v_i's.
- To quantize, need a $\mathbb{Z}$-valued skew-symmetric form Λ on M such that

$$\Lambda(\cdot, v_i) = e_i \qquad \text{for all } i \in I \setminus F.$$

Algebraic and quantum tori

Given a lattice L equipped with a $\mathbb{Z}$-valued skew-symmetric form ω, define the **quantum torus algebra**

$$\mathbb{C}_t^{\omega}[L] := \mathbb{C}[t^{\pm 1}][z^u | u \in L] / \langle z^u z^v = t^{\omega(u,v)} z^{u+v} \rangle.$$

In particular, associated to a seed S, we have:

$$\mathcal{X}_t^S := \mathbb{C}_t^B[N],$$

and if we fix a compatible Λ,

$$\mathcal{A}_t^S := \mathbb{C}_t^{\Lambda}[M].$$

Algebraic and quantum tori

Given a lattice L equipped with a $\mathbb{Z}$-valued skew-symmetric form ω, define the **quantum torus algebra**

$$\mathbb{C}_t^{\omega}[L] := \mathbb{C}[t^{\pm 1}][z^u | u \in L]/\langle z^u z^v = t^{\omega(u,v)} z^{u+v} \rangle.$$

In particular, associated to a seed S, we have:

$$\mathcal{X}_t^S := \mathbb{C}_t^B[N],$$

and if we fix a compatible Λ,

$$\mathcal{A}_t^S := \mathbb{C}_t^{\Lambda}[M].$$

Let $\mathring{\mathcal{X}}_t^S$ and $\mathring{\mathcal{A}}_t^S$ denote the corresponding skew-fields of fractions.

Seed mutation

Given S and $j \in I \setminus F$, define a new seed $\mu_j(S)$ by replacing each e_i with

$$e_i' := \mu_j(e_i) := \begin{cases} e_i + \max(0, B(e_i, e_j))e_j & \text{if } i \neq j \\ -e_j & \text{if } i = j. \end{cases}$$

while keeping the rest of the seed data the same.

Quantum binomial coefficients

- For $k \in \mathbb{Z}_{\geq 0}$, define

$$[k]_t := \frac{t^k - t^{-k}}{t - t^{-1}} = t^{-k+1} + t^{-k+3} + \ldots + t^{k-3} + t^{k-1} \in \mathbb{C}[t^{\pm 1}].$$

  Note $\lim_{t \to 1} [k]_t = k$.
- Define

$$[k]_t! := [k]_t[k-1]_t \cdots [2]_t[1]_t.$$

- For $r, k \in \mathbb{Z}_{\geq 0}$, $r \geq k$, define

$$\binom{r}{k}_t := \frac{[r]_t!}{[k]_t![r-k]_t!}$$

Quantum cluster $\mathcal{X}$ mutation

Recall $\mathcal{X}_t^S := \mathbb{C}_t^B[N]$. Define [Fock-Goncharov]

$$\mu_j^{\mathcal{X}} : \mathring{\mathcal{X}}_t^S \to \mathring{\mathcal{X}}_t^{\mu_j(S)}$$

by saying that for $B(n, e_j) \geq 0$,

$$\mu_j^{\mathcal{X}} : z^n \mapsto \sum_{k=0}^{B(n,e_j)} \binom{B(n,e_j)}{k}_t z^{n+ke_j}.$$

Classical limit ($t = 1$): $z^n \mapsto z^n(1 + z^{e_j})^{|B(n,e_j)|}$.

Quantum cluster $\mathcal{A}$ mutation

Recall $\mathcal{A}_t^S := \mathbb{C}_t^\Lambda[M]$. Define [Berenstein-Zelevinsky]

$$\mu_j^{\mathcal{A}} : \mathring{\mathcal{A}}_t^S \to \mathring{\mathcal{A}}_t^{\mu_j(S)}$$

by saying that for $\langle e_j, m\rangle \geq 0$, we have

$$\mu_j^{\mathcal{A}} : z^m \mapsto \sum_{k=0}^{\langle e_j,m\rangle} \binom{\langle e_j, m\rangle}{k}_t z^{m+kv_j}.$$

Classical limit: $z^m \mapsto z^m(1+z^{v_j})^{|\langle e_j,m\rangle|}$.

Quantum cluster varieties

- Given $\vec{\jmath} = (j_1, \ldots, j_k) \in (I \setminus F)^k$, let

$$\mu_{\vec{\jmath}} = \mu_{j_k} \circ \cdots \circ \mu_{j_1}.$$

  Denote $S_{\vec{\jmath}} := \mu_{\vec{\jmath}}(S)$.

- Similarly define $\mu^{\mathcal{X}}_{\vec{\jmath}}$ and $\mu^{\mathcal{A}}_{\vec{\jmath}}$.

Quantum cluster varieties

- Given $\vec{\jmath} = (j_1, \ldots, j_k) \in (I \setminus F)^k$, let

$$\mu_{\vec{\jmath}} = \mu_{j_k} \circ \cdots \circ \mu_{j_1}.$$

  Denote $S_{\vec{\jmath}} := \mu_{\vec{\jmath}}(S)$.
- Similarly define $\mu^{\mathcal{X}}_{\vec{\jmath}}$ and $\mu^{\mathcal{A}}_{\vec{\jmath}}$.
- Define

$$\mathcal{X}_t^{\mathrm{up}} := \left\{ f \in \mathcal{X}_t^S \middle| \, \mu^{\mathcal{X}}_{\vec{\jmath}}(f) \in \mathcal{X}_t^{S_{\vec{\jmath}}} \text{ for all tuples } \vec{\jmath} \text{ of indices in } I \setminus F \right\}.$$

- Define

$$\mathcal{A}_t^{\mathrm{up}} := \left\{ f \in \mathcal{A}_t^S \middle| \, \mu^{\mathcal{A}}_{\vec{\jmath}}(f) \in \mathcal{A}_t^{S_{\vec{\jmath}}} \text{ for all tuples } \vec{\jmath} \text{ of indices in } I \setminus F \right\}.$$

Positivity

Let $f \in \mathcal{A}_t^{\mathrm{up}} \setminus \{0\}$ (or $\mathcal{X}_t^{\mathrm{up}} \setminus \{0\}$).

- f is **universally positive** if each $\mu_{\vec{\jmath}}^{\mathcal{A}}(f)$ (respectively, $\mu_{\vec{\jmath}}^{\mathcal{X}}(f)$) has positive integer coefficients.
- f is **atomic** if it is universally positive, but is not a sum of two other universally positive elements.

Positivity

Let $f \in \mathcal{A}_t^{\mathrm{up}} \setminus \{0\}$ (or $\mathcal{X}_t^{\mathrm{up}} \setminus \{0\}$).

- f is **universally positive** if each $\mu_{\vec{\jmath}}^{\mathcal{A}}(f)$ (respectively, $\mu_{\vec{\jmath}}^{\mathcal{X}}(f)$) has positive integer coefficients.
- f is **atomic** if it is universally positive, but is not a sum of two other universally positive elements.

Let $\{\vartheta_p\}_{p \in M}$ be a $\mathbb{C}[t^{\pm 1}]$-module basis for $\mathcal{A}_t^{\mathrm{up}}$ (or $\{\vartheta_p\}_{p \in N}$ for $\mathcal{X}_t^{\mathrm{up}}$).

- The **structure constants** $a_{p_1,p_2;p} \in \mathbb{C}[t^{\pm 1}]$ are defined by

$$\vartheta_{p_1}\vartheta_{p_2} = \sum_p a_{p_1,p_2;p}\vartheta_p$$

- $\{\vartheta_p\}$ is **strongly positive** if the structure constants $a_{p_1,p_2;p}$ have non-negative integer coefficients.

The Fock-Goncharov conjecture

Conjecture (Fock-Goncharov)

The atomic elements are indexed by M and form a basis for $\mathcal{A}_t^{\mathrm{up}}$ (respectively, they are indexed by N and form a basis for $\mathcal{X}_t^{\mathrm{up}}$). For $\mathcal{A}_t^{\mathrm{up}}$ this basis includes all the quantum cluster monomials.

Note: atomic implies strong and universal positivity.

The Fock-Goncharov conjecture

Conjecture (Fock-Goncharov)

The atomic elements are indexed by M and form a basis for $\mathcal{A}_t^{\mathrm{up}}$ (respectively, they are indexed by N and form a basis for $\mathcal{X}_t^{\mathrm{up}}$). For $\mathcal{A}_t^{\mathrm{up}}$ this basis includes all the quantum cluster monomials.

Note: atomic implies strong and universal positivity.

This is not quite right:

- Lee-Li-Zelevinsky: The atomic elements are often linearly dependent.
- Gross-Hacking-Keel: $\mathcal{X}^{\mathrm{up}}$ is often just $\mathbb{C}$ and so cannot have a basis indexed by N.

Modified Fock-Goncharov conjectures

Corrections to the Fock-Goncharov conjecture:

- Need more charts, not just the clusters.
- Sometimes need to work with a formal completion:

$$\widehat{\mathcal{A}}_t := \mathbb{C}_t^{\Lambda}[M] \otimes_{\mathbb{C}_t^{\Lambda}[M^{\oplus}]} \mathbb{C}_t^{\Lambda}[\![M^{\oplus}]\!], \qquad \widehat{\mathcal{X}}_t := \mathbb{C}_t^{B}[N] \otimes_{\mathbb{C}_t^{B}[N^{\oplus}]} \mathbb{C}_t^{B}[\![N^{\oplus}]\!].$$

  Then the basis should only be a “topological basis.”

Modified Fock-Goncharov conjectures

Corrections to the Fock-Goncharov conjecture:

- Need more charts, not just the clusters.
- Sometimes need to work with a formal completion:

$$\widehat{\mathcal{A}}_t := \mathbb{C}_t^{\Lambda}[M] \otimes_{\mathbb{C}_t^{\Lambda}[M^{\oplus}]} \mathbb{C}_t^{\Lambda}[\![M^{\oplus}]\!], \qquad \widehat{\mathcal{X}}_t := \mathbb{C}_t^{B}[N] \otimes_{\mathbb{C}_t^{B}[N^{\oplus}]} \mathbb{C}_t^{B}[\![N^{\oplus}]\!].$$

  Then the basis should only be a "topological basis."

Theorem (Davison-M)

Subject to the above modifications, the Fock-Goncharov conjectures are true.

- In the classical limit, we recover [Gross-Hacking-Keel-Kontsevich].
- Get bases for $\mathcal{A}_t^{\mathrm{up}}$ and $\mathcal{X}_t^{\mathrm{up}}$ in nice situations. Always get bases for some other algebras $\mathcal{A}_t^{\mathrm{mid}}$ and $\mathcal{X}_t^{\mathrm{mid}}$ contained in $\mathcal{A}_t^{\mathrm{up}}$ and $\mathcal{X}_t^{\mathrm{up}}$.

Plethystic exponentials

Define

$$\mathrm{Exp}\left(\sum_{\substack{0\neq m\in M^{\oplus}\\ r\in\mathbb{Z}}} a_{mr} z^m (-t)^r\right) = \prod_{\substack{0\neq m\in M^{\oplus}\\ r\in\mathbb{Z}}} (1 - z^m(-t)^r)^{-a_{mr}}.$$

Motivation: If V is an $(M^{\oplus}\oplus\mathbb{Z})$-graded vector space, define

$$\chi(V) := \sum_{m,r} \dim(V_{m,r}) t^r z^m \in \mathbb{Z}[t^{\pm 1}][M^{\oplus}].$$

Then

$$\chi(\mathbb{S}\mathrm{ym}^*(V)) = \mathrm{Exp}(\chi(V)).$$

Plethystic exponentials

Now define

$$\mathbb{E}(f) := \mathrm{Exp}\left(f(t + t^3 + t^5 + t^7 + \ldots)\right).$$

I.e.,

$$\mathbb{E}(\chi(V)) = \chi(\mathbb{S}\mathrm{ym}^*(V \otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*, \mathbb{Q})[-1])).$$

Plethystic exponentials

Now define

$$\mathbb{E}(f) := \mathrm{Exp}\left(f(t + t^3 + t^5 + t^7 + \dots)\right).$$

I.e.,

$$\mathbb{E}(\chi(V)) = \chi(\mathbb{S}\mathrm{ym}^*(V \otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*, \mathbb{Q})[-1])).$$

In particular:

$$\mathbb{E}(-z^v) = \Psi_t(z^v)$$

where Ψ_t is the "quantum exponential." Quantum mutation can be defined as conjugation by a quantum exponential.

Simplifying assumption

For convenience, I'm going to restrict to constructing theta bases for $\widehat{\mathcal{A}}_t$, but the construction for $\widehat{\mathcal{X}}_t$ is similar.

Scattering diagrams

- A **scattering diagram** $\mathfrak{D}$ is a collection of codimension one walls $\mathfrak{d}$ in $M_{\mathbb{R}}$ with attached elements $g \in \mathbb{C}_t^{\Lambda}[\![M^{\oplus}]\!]$.

Scattering diagrams

- A **scattering diagram** $\mathfrak{D}$ is a collection of codimension one walls $\mathfrak{d}$ in $M_{\mathbb{R}}$ with attached elements $g \in \mathbb{C}_t^{\Lambda}[\![M^{\oplus}]\!]$.
- Path $\gamma \subset M_{\mathbb{R}} \rightsquigarrow$ **path-ordered product** $\theta_{\gamma,\mathfrak{D}} : \mathbb{C}_t^{\Lambda}[\![M^{\oplus}]\!] \xrightarrow{\sim} \mathbb{C}_t^{\Lambda}[\![M^{\oplus}]\!]$:
    - Whenever γ crosses a wall, conjugate by the function g attached to the wall (or its inverse).
    - $\mathfrak{D}$ is called **consistent** if $\theta_{\gamma,\mathfrak{D}}$ only depends on the endpoints of γ.

Scattering diagrams

- A **scattering diagram** $\mathfrak{D}$ is a collection of codimension one walls $\mathfrak{d}$ in $M_{\mathbb{R}}$ with attached elements $g \in \mathbb{C}_t^{\wedge}[\![M^{\oplus}]\!]$.
- Path $\gamma \subset M_{\mathbb{R}} \rightsquigarrow$ **path-ordered product** $\theta_{\gamma,\mathfrak{D}} : \mathbb{C}_t^{\wedge}[\![M^{\oplus}]\!] \xrightarrow{\sim} \mathbb{C}_t^{\wedge}[\![M^{\oplus}]\!]$:
  - Whenever γ crosses a wall, conjugate by the function g attached to the wall (or its inverse).
  - $\mathfrak{D}$ is called **consistent** if $\theta_{\gamma,\mathfrak{D}}$ only depends on the endpoints of γ.
- Let

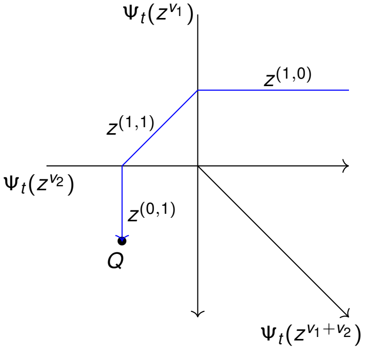
$$\mathfrak{D}_{\mathrm{in}}^{\mathcal{A}} := \{(e_i^{\perp}, \Psi_t(z^{v_i})) | i \in I \setminus F\}.$$

- This canonically determines a consistent scattering diagram $\mathfrak{D}^{\mathcal{A}} = \mathrm{Scat}(\mathfrak{D}_{\mathrm{in}}^{\mathcal{A}})$.

Broken lines

Broken line with ends (p, Q), $p \in M$, Q generic in $M_{\mathbb{R}}$ — a piecewise-straight path $\gamma : (-\infty, 0] \to M_{\mathbb{R}}$, bending only at walls, with a monomial $a_i z^{p_i} \in \mathbb{C}_t[M]$ attached to each straight segment, such that:

- The first attached monomial is z^p,
- $\gamma(0) = Q$,
- $p_i = -\gamma_i'$
- $a_{i+1} z^{p_{i+1}}$ is a term in $\theta_{\mathfrak{d}_i}(a_i z^{p_i})$.



Theta functions

For $p \in M$, $Q \in M_{\mathbb{R}}$, define

$$\vartheta_{p,Q} := \sum_{\mathrm{Ends}(\gamma)=(p,Q)} a_\gamma z^{m_\gamma},$$

$a_\gamma z^{m_\gamma} :=$ monomial attached to the last straight segment of γ.

Theta functions

For $p \in M$, $Q \in M_{\mathbb{R}}$, define

$$\vartheta_{p,Q} := \sum_{\mathrm{Ends}(\gamma)=(p,Q)} a_\gamma z^{m_\gamma},$$

$a_\gamma z^{m_\gamma}$:= monomial attached to the last straight segment of γ.

Lemma (Carl-Pumperla-Siebert, M)

For consistent scattering diagrams, different choices of Q are related by path-ordered product.

Theta functions

For $p \in M$, $Q \in M_{\mathbb{R}}$, define

$$\vartheta_{p,Q} := \sum_{\mathrm{Ends}(\gamma)=(p,Q)} a_\gamma z^{m_\gamma},$$

$a_\gamma z^{m_\gamma} :=$ monomial attached to the last straight segment of γ.

Lemma (Carl-Pumperla-Siebert, M)

For consistent scattering diagrams, different choices of Q are related by path-ordered product.

If $\langle e_i, Q\rangle > 0$ for all $i \in I \setminus F$, then $\vartheta_{p,Q}$ gives ϑ_p in the initial cluster $\mathcal{A}_t^S$.

Theta functions

For $p \in M$, $Q \in M_{\mathbb{R}}$, define

$$\vartheta_{p,Q} := \sum_{\mathrm{Ends}(\gamma)=(p,Q)} a_\gamma z^{m_\gamma},$$

$a_\gamma z^{m_\gamma}$:= monomial attached to the last straight segment of γ.

Lemma (Carl-Pumperla-Siebert, M)

For consistent scattering diagrams, different choices of Q are related by path-ordered product.

If $\langle e_i, Q\rangle > 0$ for all $i \in I \setminus F$, then $\vartheta_{p,Q}$ gives ϑ_p in the initial cluster $\mathcal{A}_t^S$. Other Q's give expressions in the other clusters and in more general coordinate charts.

Theta functions

For $p \in M$, $Q \in M_{\mathbb{R}}$, define

$$\vartheta_{p,Q} := \sum_{\mathrm{Ends}(\gamma)=(p,Q)} a_\gamma z^{m_\gamma},$$

$a_\gamma z^{m_\gamma} :=$ monomial attached to the last straight segment of γ.

Lemma (Carl-Pumperla-Siebert, M)

For consistent scattering diagrams, different choices of Q are related by path-ordered product.

If $\langle e_i, Q\rangle > 0$ for all $i \in I \setminus F$, then $\vartheta_{p,Q}$ gives ϑ_p in the initial cluster $\mathcal{A}_t^S$. Other Q's give expressions in the other clusters and in more general coordinate charts.

Universally positive with respect to the scattering atlas — $\vartheta_{p,Q}$ has positive coefficients for each Q.

Positivity of the scattering diagram

Theorem (Davison-M)

Up to equivalence, every scattering function of $\mathfrak{D}^{\mathcal{A}}$ has the form $\mathbb{E}(-p(t)z^{v})$ for some $p(t) \in \mathbb{Z}_{\geq 0}[t^{\pm 1}]$.

Theorem $\Longrightarrow$ Positivity of broken lines

$\Longrightarrow$ Universal and strong positivity, and atomicity.

Stability scattering diagrams

- Let Q be a quiver, $N = \mathbb{Z}^{Q_0}$, $B =$ adjacency matrix for Q.

Stability scattering diagrams

- Let Q be a quiver, $N = \mathbb{Z}^{Q_0}$, $B =$ adjacency matrix for Q.
- $\zeta \in M_{\mathbb{R}} \rightsquigarrow$ stability conditions for rep(Q).

Stability scattering diagrams

- Let Q be a quiver, $N = \mathbb{Z}^{Q_0}$, $B =$ adjacency matrix for Q.
- $\zeta \in M_{\mathbb{R}} \rightsquigarrow$ stability conditions for rep(Q).
- Consistent $\mathfrak{D}^{\mathrm{Stab}}$ with scattering function g at generic $\zeta \in M_{\mathbb{R}}$ given by:

$$g^{-1} = \chi\left(\mathrm{H}_c(\mathfrak{M}^{\zeta\text{-sst}}(Q), \mathbb{Q})^*_{\mathrm{vir}}\right). \qquad \text{[Bridgeland]}$$

Stability scattering diagrams

- Let Q be a quiver, $N = \mathbb{Z}^{Q_0}$, $B =$ adjacency matrix for Q.
- $\zeta \in M_{\mathbb{R}} \rightsquigarrow$ stability conditions for rep(Q).
- Consistent $\mathfrak{D}^{\mathrm{Stab}}$ with scattering function g at generic $\zeta \in M_{\mathbb{R}}$ given by:

$$g^{-1} = \chi\left(\mathrm{H}_c(\mathfrak{M}^{\zeta\text{-sst}}(Q), \mathbb{Q})^*_{\mathrm{vir}}\right). \qquad \text{[Bridgeland]}$$

- $\mathfrak{D}^{\mathrm{Stab}} = \mathfrak{D}^{\mathcal{A}}$ iff $Q \setminus \{\text{loops}\}$ is acyclic.
- Otherwise need Q with "genteel" potential (we avoid this).

Stability scattering diagrams

- Let Q be a quiver, $N = \mathbb{Z}^{Q_0}$, $B =$ adjacency matrix for Q.
- $\zeta \in M_{\mathbb{R}} \rightsquigarrow$ stability conditions for rep(Q).
- Consistent $\mathfrak{D}^{\mathrm{Stab}}$ with scattering function g at generic $\zeta \in M_{\mathbb{R}}$ given by:

$$g^{-1} = \chi\left(\mathrm{H}_c(\mathfrak{M}^{\zeta\text{-sst}}(Q), \mathbb{Q})^*_{\mathrm{vir}}\right). \qquad \text{[Bridgeland]}$$

- $\mathfrak{D}^{\mathrm{Stab}} = \mathfrak{D}^{\mathcal{A}}$ iff $Q \setminus \{\text{loops}\}$ is acyclic.
- Otherwise need Q with "genteel" potential (we avoid this).
- Davison-Meinhardt: $\mathrm{H}_c(\mathfrak{M}_n^{\zeta\text{-sst}}(Q), \mathbb{Q})^*_{\mathrm{vir}} = \mathbb{S}\mathrm{ym}(\mathrm{BPS}_n^{\zeta}(Q) \otimes H(\mathrm{pt}/\mathbb{C}^*, \mathbb{Q})_{\mathrm{vir}})$.
- So $g = \sum_m \mathbb{E}(-p_m(t) z^m)$ for some positive p_m's.

Proof of positivity of $\mathfrak{D}^{\mathcal{A}}$

- Use tricks for manipulating scattering diagrams and induction to reduce to the case where $\mathfrak{D}_{\text{in}}$ has only two walls [GHKK]:

$$\mathfrak{D}_{\text{in}} = \{(e_1^{\perp}, \mathbb{E}(-t^{m_1} z^{v_1})), (e_1^{\perp}, \mathbb{E}(-t^{m_2} z^{v_2}))\}.$$

Here, $m_1, m_2 \in \mathbb{Z}$, but after a substitution we can assume they're in $\{0, -1\}$. Also assume $\Lambda(v_1, v_2) \geq 0$.

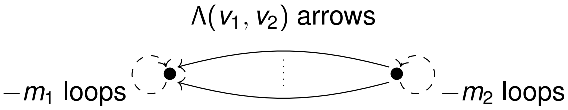
Proof of positivity of $\mathfrak{D}^{\mathcal{A}}$

- Use tricks for manipulating scattering diagrams and induction to reduce to the case where $\mathfrak{D}_{\text{in}}$ has only two walls [GHKK]:

$$\mathfrak{D}_{\text{in}} = \{(e_1^{\perp}, \mathbb{E}(-t^{m_1} z^{v_1})), (e_1^{\perp}, \mathbb{E}(-t^{m_2} z^{v_2}))\}.$$

  Here, $m_1, m_2 \in \mathbb{Z}$, but after a substitution we can assume they're in $\{0, -1\}$. Also assume $\Lambda(v_1, v_2) \geq 0$.

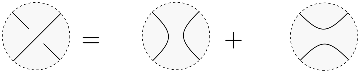
- Show $\mathrm{Scat}(\mathfrak{D}_{\text{in}}) = \mathfrak{D}^{\mathrm{Stab}}$ for Q as below:



□

Skein algebras

- Let $\Sigma = (\mathbf{S}, \mathbf{M})$ be a closed surface with boundary **S**, and let **M** be a finite collection of marked points. Marked points in $\mathbf{S} \setminus \partial\mathbf{S}$ are called punctures.
- Consider the "skein algebra" consisting of isotopy classes of links in **S** with boundary in **M**, subject to relations:
  - Contractible arcs are equivalent to 0;
  - Contractible loops are equivalent to -2.
  - A loop around a puncture is equivalent to 2;
  - The skein relation:

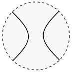
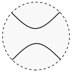


- Theorem [Fock-Goncharov, Fomin-Shapiro-Thurston]: This skein algebra $\mathrm{Sk}(\Sigma)$ has a cluster structure such that:
  - (tagged) triangulations corresponding to clusters;
  - arcs correspond to cluster variables.

Quantum skein algebra

In unpunctured cases, Muller describes a quantum analog with these modifications to the relations from before:

- Contractible loops are equivalent to $-t - t^{-1}$;
- The Kaufmann skein relation:

 $= t$  $+t^{-1}$ 

The resulting algebra $\mathsf{Sk}_t(\Sigma)$ is a quantum cluster algebra.

Bracelet Bases

- Musiker-Schiffler-Williams construct “bracelet bases” for $\mathsf{Sk}(\Sigma)$. Consists of disjoint unions of weighted arcs and loops (weighted loops defined using Chebyshev polynomials of the first kind).

Bracelet Bases

- Musiker-Schiffler-Williams construct “bracelet bases” for $\mathsf{Sk}(\Sigma)$. Consists of disjoint unions of weighted arcs and loops (weighted loops defined using Chebyshev polynomials of the first kind).
- Fock-Goncharov also considered parts of these bases (from the moduli of decorated SL_2-local systems viewpoint). Conjectured atomicity.

Bracelet Bases

- Musiker-Schiffler-Williams construct "bracelet bases" for $\mathsf{Sk}(\Sigma)$. Consists of disjoint unions of weighted arcs and loops (weighted loops defined using Chebyshev polynomials of the first kind).
- Fock-Goncharov also considered parts of these bases (from the moduli of decorated SL_2-local systems viewpoint). Conjectured atomicity.
- D. Thurston constructs quantum bracelet bases in unpunctured cases. Conjectures their strong positivity.

Bracelet Bases

- Musiker-Schiffler-Williams construct "bracelet bases" for $\mathrm{Sk}(\Sigma)$. Consists of disjoint unions of weighted arcs and loops (weighted loops defined using Chebyshev polynomials of the first kind).
- Fock-Goncharov also considered parts of these bases (from the moduli of decorated SL_2-local systems viewpoint). Conjectured atomicity.
- D. Thurston constructs quantum bracelet bases in unpunctured cases. Conjectures their strong positivity.
- Yurikusa: The g-vector cone is dense (in most cases). So the cluster atlas and the scattering atlas agree, and the theta bases satisfy the original Fock-Goncharov atomic bases conjectures (without needing the modifications I discussed).

Bracelets = Thetas

Theorem (Qin-M)

The bracelet bases agree with the theta bases. In unpunctured cases, the quantum bracelet bases agree with the quantum theta bases.

Very rough outline (using arguments involving broken lines, positivity, atomicity, modularity, pointedness, bar-invariance, etc.):

- Annulus case (the Kronecker quiver) — explicit check;
- Unpunctured surface with one boundary marking case;
- Gluing lemma — uses positivity and atomicity;
- Disjoint unions $\leftrightsquigarrow$ products.

Bracelets = Thetas

Theorem (Qin-M)

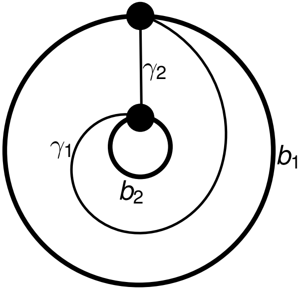
The bracelet bases agree with the theta bases. In unpunctured cases, the quantum bracelet bases agree with the quantum theta bases.

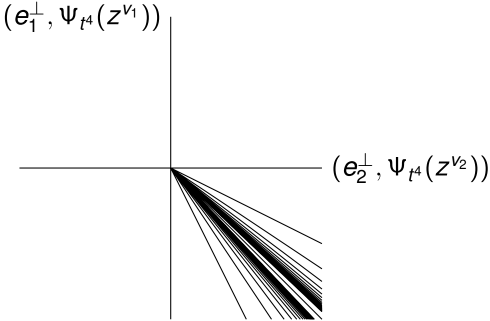
Very rough outline (using arguments involving broken lines, positivity, atomicity, modularity, pointedness, bar-invariance, etc.):

- Annulus case (the Kronecker quiver) — explicit check;
- Unpunctured surface with one boundary marking case;
- Gluing lemma — uses positivity and atomicity;
- Disjoint unions $\leftrightsquigarrow$ products.

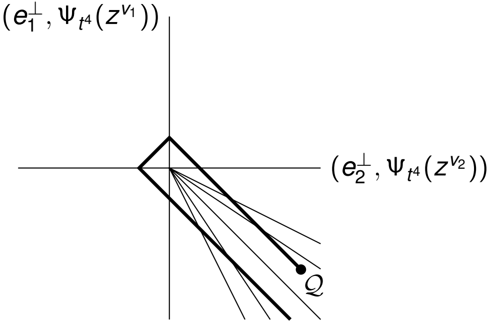
Hope to get analogous results for the Fock-Goncharov $\mathcal{X}$-spaces (moduli of framed PGL_2-local systems) and their quantum versions (Allegretti-Kim).

Annulus case





Annulus case



For $p \in \mathbb{Z}_{>0}(1,-1)$, $\vartheta_{p,Q} = z^p + ... + z^{-p} \rightsquigarrow$ Chebyshev relation.