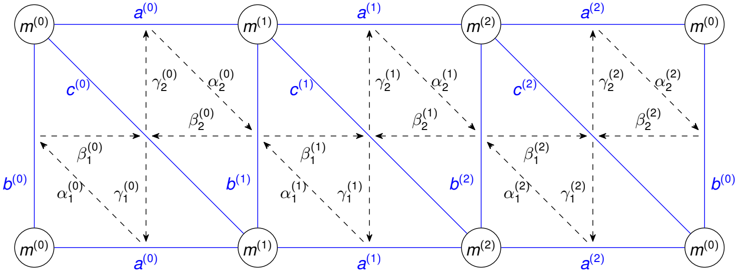
Stability scattering diagrams and quiver coverings

Travis Mandel

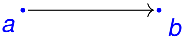
Based on joint work with Fan Qin and Qiyue Chen



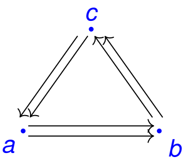
Quivers

- A (finite) **quiver** is a finite set of vertices Q^0 and arrows Q^1 (ordered pairs of vertices).
- **Examples**:

The A_2-quiver



The Markov quiver:



Quiver representations

- **Notation**: Fix a quiver Q. Let $r = |Q^0|$. Let

$$N := \mathbb{Z}^r, \qquad N^{\oplus} := \mathbb{N}^r \subset N.$$

- Given $n = (a_1, \ldots, a_r) \in N^{\oplus}$, an n-dimensional representation of Q is:
  - For each $i \in Q^0$, an a_i-dimensional $\mathbb{C}$-vector space V_i plus
  - for each arrow from i to j in Q^1, an element of $\mathrm{Hom}(V_i, V_j)$.

Quiver representations

- **Notation**: Fix a quiver Q. Let $r = |Q^0|$. Let

$$N := \mathbb{Z}^r, \qquad N^{\oplus} := \mathbb{N}^r \subset N.$$

- Given $n = (a_1, \ldots, a_r) \in N^{\oplus}$, an n-dimensional representation of Q is:
  - For each $i \in Q^0$, an a_i-dimensional $\mathbb{C}$-vector space V_i plus
  - for each arrow from i to j in Q^1, an element of $\mathrm{Hom}(V_i, V_j)$.
- **Example:** a $(3, 2)$-dimensional representation of

$$a \bullet \longrightarrow \bullet\, b$$

  is a linear transformation

$$\mathbb{C}^3 \longrightarrow \mathbb{C}^2$$

Moduli of Q-representations

- Let $\mathrm{Rep}(Q)$ be the category of representations of Q.

Moduli of Q-representations

- Let $\mathrm{Rep}(Q)$ be the category of representations of Q.
- For each $n \in N^{\oplus}$, let $\mathcal{M}_n(Q)$ be the moduli space of $E \in \mathrm{Rep}(Q)$ with $\dim(E) = n = (d_1, ..., d_r)$.

Moduli of Q-representations

- Let $\mathrm{Rep}(Q)$ be the category of representations of Q.
- For each $n \in N^{\oplus}$, let $\mathcal{M}_n(Q)$ be the moduli space of $E \in \mathrm{Rep}(Q)$ with $\dim(E) = n = (d_1, ..., d_r)$.
- So

$$\mathcal{M}_n(Q) = \prod_{a \in Q^1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{d_{s(a)}}, \mathbb{C}^{d_{t(a)}}) / \prod_{v \in Q^0} \mathrm{GL}_{\mathbb{C}}(d_v)$$

Semistable representations

- Let $M_{\mathbb{R}} = \mathrm{Hom}(N, \mathbb{R}) \cong \mathbb{R}^r$, and let $\theta \in M_{\mathbb{R}}$.

Semistable representations

- Let $M_{\mathbb{R}} = \mathrm{Hom}(N, \mathbb{R}) \cong \mathbb{R}^r$, and let $\theta \in M_{\mathbb{R}}$.
- Say $E \in \mathrm{Rep}(Q)$ is θ-**semistable** (resp. **stable**) if:
  - $\theta(\dim(E)) = 0$, and
  - $\theta(B) \leq 0$ (resp. $\theta(B) < 0$) for all sub-representations $B \subset E$.

Semistable representations

- Let $M_{\mathbb{R}} = \mathrm{Hom}(N, \mathbb{R}) \cong \mathbb{R}^r$, and let $\theta \in M_{\mathbb{R}}$.
- Say $E \in \mathrm{Rep}(Q)$ is θ-**semistable** (resp. **stable**) if:
  - $\theta(\dim(E)) = 0$, and
  - $\theta(B) \leq 0$ (resp. $\theta(B) < 0$) for all sub-representations $B \subset E$.
- $\mathcal{M}_n^{\theta\text{-sst}}(Q) =$ space of θ-semistable $E \in \mathrm{Rep}(Q)$ with $\mathrm{Rep}(Q) = n$.

Semistable representations

- Let $M_{\mathbb{R}} = \mathrm{Hom}(N, \mathbb{R}) \cong \mathbb{R}^r$, and let $\theta \in M_{\mathbb{R}}$.
- Say $E \in \mathrm{Rep}(Q)$ is θ-**semistable** (resp. **stable**) if:
  - $\theta(\dim(E)) = 0$, and
  - $\theta(B) \leq 0$ (resp. $\theta(B) < 0$) for all sub-representations $B \subset E$.
- $\mathcal{M}_n^{\theta\text{-sst}}(Q) =$ space of θ-semistable $E \in \mathrm{Rep}(Q)$ with $\mathrm{Rep}(Q) = n$.
- $\mathcal{M}_n^{\theta\text{-st}}(Q) =$ open subspace of θ-stable objects.

Semistable representations

- Let $M_{\mathbb{R}} = \text{Hom}(N, \mathbb{R}) \cong \mathbb{R}^r$, and let $\theta \in M_{\mathbb{R}}$.
- Say $E \in \text{Rep}(Q)$ is θ-**semistable** (resp. **stable**) if:
  - $\theta(\dim(E)) = 0$, and
  - $\theta(B) \leq 0$ (resp. $\theta(B) < 0$) for all sub-representations $B \subset E$.
- $\mathcal{M}_n^{\theta\text{-sst}}(Q) =$ space of θ-semistable $E \in \text{Rep}(Q)$ with $\text{Rep}(Q) = n$.
- $\mathcal{M}_n^{\theta\text{-st}}(Q) =$ open subspace of θ-stable objects.
- If $\mathcal{M}_n^{\theta\text{-st}} \neq \emptyset$, let $\chi_{n,\theta} \in \mathbb{Z}$ be the topological Euler characteristic of $\mathcal{M}_n^{\theta\text{-sst}}(Q)$. Otherwise, $\chi_{n,\theta} := 0$.

Simple representations

- For each $i \in Q^0$, let $e_i \in N^{\oplus}$ be the corresponding basis vector $(0, ..., 0, 1, 0, ..., 0)$.
- Up to isomorphism, there is a *unique* representation S_i of Q with $\mathtt{dim}(S_i) = e_i$.
- S_i is a **simple representation**: it has no subrepresentations.

Simple representations

- For each $i \in Q^0$, let $e_i \in N^{\oplus}$ be the corresponding basis vector $(0, ..., 0, 1, 0, ..., 0)$.
- Up to isomorphism, there is a *unique* representation S_i of Q with $\dim(S_i) = e_i$.
- S_i is a **simple representation**: it has no subrepresentations.
- So $\mathcal{M}_{e_i}^{\theta\text{-sst}}(Q) = \{pt\}$ for all $\theta \in e_i^{\perp}$.

Simple representations

- For each $i \in Q^0$, let $e_i \in N^{\oplus}$ be the corresponding basis vector $(0, ..., 0, 1, 0, ..., 0)$.
- Up to isomorphism, there is a *unique* representation S_i of Q with $\mathrm{dim}(S_i) = e_i$.
- S_i is a **simple representation**: it has no subrepresentations.
- So $\mathcal{M}_{e_i}^{\theta\text{-sst}}(Q) = \{pt\}$ for all $\theta \in e_i^{\perp}$.
- If Q is **acyclic** (has no oriented cycles), these are the only simple objects.
- But if Q has an oriented cycle, then there are additional simple objects.

Scattering diagrams

- Consider the formal power series ring $\Bbbk[\![N^{\oplus}]\!] = \Bbbk[\![x_1, x_2, \ldots, x_r]\!]$.
  - For $n = (a_1, \ldots, a_r)$, denote $x^n = x_1^{a_1} \cdots x_r^{a_r}$.
- Let $N^+ := N^{\oplus} \setminus \{0\}$.

Scattering diagrams

- Consider the formal power series ring $\Bbbk[\![N^{\oplus}]\!] = \Bbbk[\![x_1, x_2, \ldots, x_r]\!]$.
  - For $n = (a_1, \ldots, a_r)$, denote $x^n = x_1^{a_1} \cdots x_r^{a_r}$.
- Let $N^+ := N^{\oplus} \setminus \{0\}$.

- A **wall** in $M_{\mathbb{R}} \cong \mathbb{R}^r$ is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$ where
  - $\mathfrak{d}$ is a convex $(r-1)$-dimensional integral polyhedral cone such that $\mathfrak{d}^{\perp} \subset N_{\mathbb{R}}$ intersects N^+.
  - Let $n_{\mathfrak{d}} \in N^+$ be a primitive vector in $\mathfrak{d}^{\perp}$.

Scattering diagrams

- Consider the formal power series ring $\Bbbk[\![N^{\oplus}]\!] = \Bbbk[\![x_1, x_2, \ldots, x_r]\!]$.
  - For $n = (a_1, \ldots, a_r)$, denote $x^n = x_1^{a_1} \cdots x_r^{a_r}$.
- Let $N^+ := N^{\oplus} \setminus \{0\}$.

- A **wall** in $M_{\mathbb{R}} \cong \mathbb{R}^r$ is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$ where
  - $\mathfrak{d}$ is a convex $(r-1)$-dimensional integral polyhedral cone such that $\mathfrak{d}^{\perp} \subset N_{\mathbb{R}}$ intersects N^+.
  - Let $n_{\mathfrak{d}} \in N^+$ be a primitive vector in $\mathfrak{d}^{\perp}$.
  - $f_{\mathfrak{d}} = 1 + c_1 x^{n_{\mathfrak{d}}} + c_2 x^{2n_{\mathfrak{d}}} + \ldots \in \Bbbk[\![x^{n_{\mathfrak{d}}}]\!]$.

Scattering diagrams

- Consider the formal power series ring $\Bbbk[\![N^{\oplus}]\!] = \Bbbk[\![x_1, x_2, \ldots, x_r]\!]$.
  - For $n = (a_1, \ldots, a_r)$, denote $x^n = x_1^{a_1} \cdots x_r^{a_r}$.
- Let $N^+ := N^{\oplus} \setminus \{0\}$.
- A **wall** in $M_{\mathbb{R}} \cong \mathbb{R}^r$ is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$ where
  - $\mathfrak{d}$ is a convex $(r-1)$-dimensional integral polyhedral cone such that $\mathfrak{d}^{\perp} \subset N_{\mathbb{R}}$ intersects N^+.
  - Let $n_{\mathfrak{d}} \in N^+$ be a primitive vector in $\mathfrak{d}^{\perp}$.
  - $f_{\mathfrak{d}} = 1 + c_1 x^{n_{\mathfrak{d}}} + c_2 x^{2n_{\mathfrak{d}}} + \ldots \in \Bbbk[\![x^{n_{\mathfrak{d}}}]\!]$.
- A **scattering diagram** $\mathfrak{D}$ is a set of walls.

Scattering diagrams

- Consider the formal power series ring $\mathbb{k}[\![N^{\oplus}]\!] = \mathbb{k}[\![x_1, x_2, \ldots, x_r]\!]$.
  - For $n = (a_1, \ldots, a_r)$, denote $x^n = x_1^{a_1} \cdots x_r^{a_r}$.
- Let $N^+ := N^{\oplus} \setminus \{0\}$.
- A **wall** in $M_{\mathbb{R}} \cong \mathbb{R}^r$ is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$ where
  - $\mathfrak{d}$ is a convex $(r-1)$-dimensional integral polyhedral cone such that $\mathfrak{d}^{\perp} \subset N_{\mathbb{R}}$ intersects N^+.
  - Let $n_{\mathfrak{d}} \in N^+$ be a primitive vector in $\mathfrak{d}^{\perp}$.
  - $f_{\mathfrak{d}} = 1 + c_1 x^{n_{\mathfrak{d}}} + c_2 x^{2n_{\mathfrak{d}}} + \ldots \in \mathbb{k}[\![x^{n_{\mathfrak{d}}}]\!]$.
- A **scattering diagram** $\mathfrak{D}$ is a set of walls.
- For each generic $\theta \in M_{\mathbb{R}}$, let $f_\theta = \prod_{\mathfrak{d} \ni \theta} f_{\mathfrak{d}}$.
  - Up to "equivalence," $\mathfrak{D}$ can be determined by specifying f_θ for all generic $\theta \in M_{\mathbb{R}}$.

The stability scattering diagram

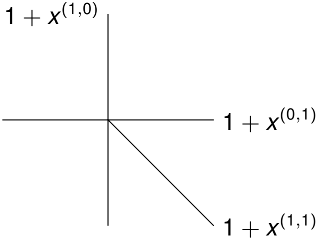
- Fix generic $\theta \in M_{\mathbb{R}}$.
- Let n be the primitive element of $\theta^{\perp} \cap N^{\oplus}$.
- We define

$$f_\theta = \sum_{k=0}^{\infty} \chi_{kn,\theta} x^{kn}.$$

- The corresponding scattering diagram $\mathfrak{D}(Q)$ is called the **stability scattering diagram**.

The stability scattering diagram for the A_2-quiver

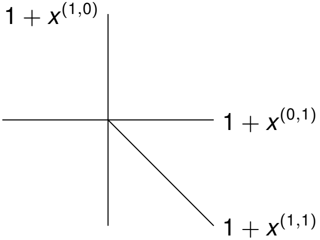
$\mathfrak{D}(\bullet \longrightarrow \bullet)$ is as follows:



- A_2 has only three nontrivial stable representations (each giving $\chi_{n,\theta} = 1$):
  1. $S_1 = \mathbb{C} \to 0$ is stable for $\theta \in e_1^{\perp} \setminus \{0\}$.
  2. $S_2 = 0 \to \mathbb{C}$ is stable for $\theta \in e_2^{\perp} \setminus \{0\}$
  3. $\mathbb{C} \xrightarrow{\text{Id}} \mathbb{C}$ is stable for $\theta \in \mathbb{R}_{>0}(1,-1)$.

The stability scattering diagram for the A_2-quiver

$\mathfrak{D}(\bullet \longrightarrow \bullet)$ is as follows:



- A_2 has only three nontrivial stable representations (each giving $\chi_{n,\theta} = 1$):
  1. $S_1 = \mathbb{C} \to 0$ is stable for $\theta \in e_1^{\perp} \setminus \{0\}$.
  2. $S_2 = 0 \to \mathbb{C}$ is stable for $\theta \in e_2^{\perp} \setminus \{0\}$
  3. $\mathbb{C} \xrightarrow{\mathrm{Id}} \mathbb{C}$ is stable for $\theta \in \mathbb{R}_{>0}(1, -1)$.
- $\mathbb{C} \xrightarrow{\mathrm{Id}} \mathbb{C}$ this is NOT stable for $\theta \in \mathbb{R}_{<0}(1, -1)$:
  - Because $0 \longrightarrow \mathbb{C}$ is a suprepresentation of dimension $(0, 1)$.
  - For $r \in \mathbb{R}_{<0}$, $(r, -r) \cdot (0, 1) = -r > 0$.

The Kronecker Quiver

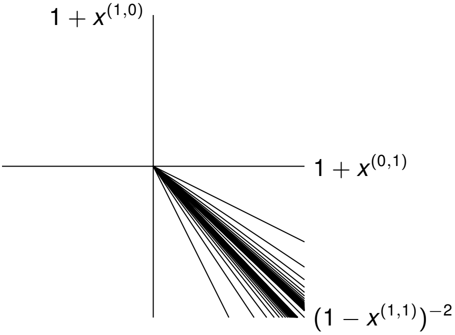


Figure: The scattering diagram for the Kronecker Quiver $K_2 : \bullet \implies \bullet$

Higher Kronecker quivers

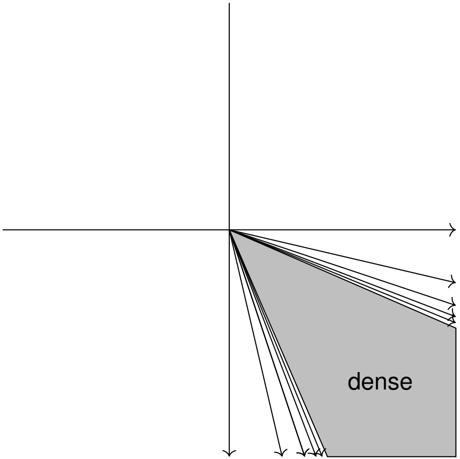
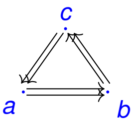


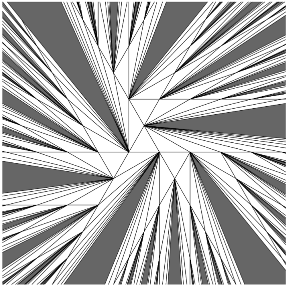
Figure: Sketch of the scattering diagram for the n-Kronecker quiver K_n, $n \geq 3$.

The Markov Quiver

- Recall the Markov Quiver:



- This scattering diagram lives in three dimensions. A slice is shown below.
- (Image from Fock and Goncharov, Cluster Poisson varieties at infinity (Selecta Math). arXiv:1104.0407)



Wall-crossing

- We have a skew-symmetric form on N defined by

$$B(e_i, e_j) = \#(\text{arrows } i \to j) - \#(\text{arrows } j \to i).$$

Wall-crossing

- We have a skew-symmetric form on N defined by

$$B(e_i, e_j) = \#(\text{arrows } i \to j) - \#(\text{arrows } j \to i).$$

- When we cross a wall $\mathfrak{d}$ from the positive to negative side, we apply an automorphism of $\Bbbk[\![N^{\oplus}]\!]$:

$$\phi_{\mathfrak{d}} : x^n \mapsto x^n f_{\mathfrak{d}}^{B(n_{\mathfrak{d}}, n)}.$$

Wall-crossing

- We have a skew-symmetric form on N defined by

$$B(e_i, e_j) = \#(\text{arrows } i \to j) - \#(\text{arrows } j \to i).$$

- When we cross a wall $\mathfrak{d}$ from the positive to negative side, we apply an automorphism of $\Bbbk[\![N^{\oplus}]\!]$:

$$\phi_{\mathfrak{d}} : x^n \mapsto x^n f_{\mathfrak{d}}^{B(n_{\mathfrak{d}}, n)}.$$

- Given a path γ, define $\phi_\gamma =$ the composition of $\phi_{\mathfrak{d}}$ for all $\mathfrak{d}$'s crossed by γ, in order.

- **Theorem**: $\mathfrak{D}(Q)$ is consistent, meaning ϕ_γ is determined by the endpoints of γ.
  - In different contexts, due to Reineke, Gross-Pandharipande, Kontsevich-Soibelman, Bridgeland.

Connection to cluster algebras

- Call $(\mathfrak{d}, f_\mathfrak{d})$ **incoming** if $\mathfrak{d}$ is a full hyperplane.
- The **cluster scattering diagram** $\mathfrak{D}^{\mathrm{cl}}(Q)$ is the consistent scattering diagram whose only incoming walls are the $e_i^\perp$ associated to the simple objects S_i, $i \in Q^0$.

Connection to cluster algebras

- Call $(\mathfrak{d}, f_{\mathfrak{d}})$ **incoming** if $\mathfrak{d}$ is a full hyperplane.
- The **cluster scattering diagram** $\mathfrak{D}^{\mathrm{cl}}(Q)$ is the consistent scattering diagram whose only incoming walls are the $e_i^{\perp}$ associated to the simple objects S_i, $i \in Q^0$.
- **Fact**: The incoming walls totally determine $\mathfrak{D}$ (up to equivalence).

Connection to cluster algebras

- Call $(\mathfrak{d}, f_{\mathfrak{d}})$ **incoming** if $\mathfrak{d}$ is a full hyperplane.
- The **cluster scattering diagram** $\mathfrak{D}^{\mathrm{cl}}(Q)$ is the consistent scattering diagram whose only incoming walls are the $e_i^{\perp}$ associated to the simple objects S_i, $i \in Q^0$.

- **Fact**: The incoming walls totally determine $\mathfrak{D}$ (up to equivalence).
- The cluster scattering diagrams are very useful for understanding cluster algebras [Gross-Hacking-Keel-Kontsevich].

Connection to cluster algebras

- Call $(\mathfrak{d}, f_{\mathfrak{d}})$ **incoming** if $\mathfrak{d}$ is a full hyperplane.
- The **cluster scattering diagram** $\mathfrak{D}^{\mathrm{cl}}(Q)$ is the consistent scattering diagram whose only incoming walls are the $e_i^{\perp}$ associated to the simple objects S_i, $i \in Q^0$.

- **Fact**: The incoming walls totally determine $\mathfrak{D}$ (up to equivalence).
- The cluster scattering diagrams are very useful for understanding cluster algebras [Gross-Hacking-Keel-Kontsevich].

- "Quantum" scattering diagrams can be used to construct quantum theta function bases for quantum cluster algebras [Davison-M].
  - We showed the coefficients of the quantum scattering functions are always positive integers.
  - Combines cluster scattering diagram techniques with results of Davison-Meinhardt related to quantum stability scattering diagrams.

Connection to cluster algebras

- Call $(\mathfrak{d}, f_{\mathfrak{d}})$ **incoming** if $\mathfrak{d}$ is a full hyperplane.
- The **cluster scattering diagram** $\mathfrak{D}^{\mathrm{cl}}(Q)$ is the consistent scattering diagram whose only incoming walls are the $e_i^{\perp}$ associated to the simple objects S_i, $i \in Q^0$.
- **Fact**: The incoming walls totally determine $\mathfrak{D}$ (up to equivalence).
- The cluster scattering diagrams are very useful for understanding cluster algebras [Gross-Hacking-Keel-Kontsevich].
- "Quantum" scattering diagrams can be used to construct quantum theta function bases for quantum cluster algebras [Davison-M].
  - We showed the coefficients of the quantum scattering functions are always positive integers.
  - Combines cluster scattering diagram techniques with results of Davison-Meinhardt related to quantum stability scattering diagrams.

Another perspective: the path algebra

- : **Issue:** $\mathfrak{D}(Q) = \mathfrak{D}^{\mathrm{cl}}(Q)$ if and only if Q is acyclic. We can sometimes fix this by introducing a "potential."

Another perspective: the path algebra

- : **Issue:** $\mathfrak{D}(Q) = \mathfrak{D}^{\mathrm{cl}}(Q)$ if and only if Q is acyclic. We can sometimes fix this by introducing a "potential."

- The **path algebra** $\mathbb{C}Q$ of Q consists of finite $\mathbb{C}$-linear combinations of paths $a_1 \cdots a_k$ in Q (each $a_i \in Q^1$);

Another perspective: the path algebra

- : **Issue:** $\mathfrak{D}(Q) = \mathfrak{D}^{\text{cl}}(Q)$ if and only if Q is acyclic. We can sometimes fix this by introducing a "potential."

- The **path algebra** $\mathbb{C}Q$ of Q consists of finite $\mathbb{C}$-linear combinations of paths $a_1 \cdots a_k$ in Q (each $a_i \in Q^1$);
- Multiplication is concatenation of paths (read from right to left);

Another perspective: the path algebra

- : **Issue:** $\mathfrak{D}(Q) = \mathfrak{D}^{\mathrm{cl}}(Q)$ if and only if Q is acyclic. We can sometimes fix this by introducing a "potential."

- The **path algebra** $\mathbb{C}Q$ of Q consists of finite $\mathbb{C}$-linear combinations of paths $a_1 \cdots a_k$ in Q (each $a_i \in Q^1$);
- Multiplication is concatenation of paths (read from right to left);
- Subject to the relations $pq = 0$ unless p starts where q ends.

Another perspective: the path algebra

- : **Issue:** $\mathfrak{D}(Q) = \mathfrak{D}^{\mathrm{cl}}(Q)$ if and only if Q is acyclic. We can sometimes fix this by introducing a "potential."

- The **path algebra** $\mathbb{C}Q$ of Q consists of finite $\mathbb{C}$-linear combinations of paths $a_1 \cdots a_k$ in Q (each $a_i \in Q^1$);
- Multiplication is concatenation of paths (read from right to left);
- Subject to the relations $pq = 0$ unless p starts where q ends.
- For each $i \in Q^0$ we have an idempotent lazy path e_i.

Modules over the path algebra

- Representations of Q can equivalently be understood as (left) modules over $\mathbb{C}Q$.
- A Q-rep (V_i, ρ_α) determines a module

$$M = \bigoplus_i V_i, \qquad \alpha \cdot v = \rho_\alpha(v_i).$$

- Conversely, a module M yields a representation

$$V_i = e_i M, \qquad \rho_\alpha(v_i) = \alpha \cdot v_i.$$

Quiver potentials

- A potential $W \in \mathbb{C}Q$ is a linear combination of closed paths in Q.

Quiver potentials

- A potential $W \in \mathbb{C}Q$ is a linear combination of closed paths in Q.
- Given a closed path $w = a_1 \cdots a_k \in \mathbb{C}Q$ and an arrow $\alpha \in Q^1$, define

$$\partial_\alpha(w) = \sum_{i : a_i = \alpha} a_{i+1} \cdots a_k a_1 \cdots a_{i-1}.$$

Quiver potentials

- A potential $W \in \mathbb{C}Q$ is a linear combination of closed paths in Q.
- Given a closed path $w = a_1 \cdots a_k \in \mathbb{C}Q$ and an arrow $\alpha \in Q^1$, define

$$\partial_\alpha(w) = \sum_{i:a_i=\alpha} a_{i+1} \cdots a_k a_1 \cdots a_{i-1}.$$

- Let $\langle \partial W \rangle$ be the two sided ideal of $\mathbb{C}Q$ generated by $\partial_\alpha(W)$ for $\alpha \in Q^1$.
- Define the Jacobian algebra

$$J = \mathbb{C}Q / \langle \partial W \rangle$$

Quiver potentials

- A potential $W \in \mathbb{C}Q$ is a linear combination of closed paths in Q.
- Given a closed path $w = a_1 \cdots a_k \in \mathbb{C}Q$ and an arrow $\alpha \in Q^1$, define

$$\partial_\alpha(w) = \sum_{i : a_i = \alpha} a_{i+1} \cdots a_k a_1 \cdots a_{i-1}.$$

- Let $\langle \partial W \rangle$ be the two sided ideal of $\mathbb{C}Q$ generated by $\partial_\alpha(W)$ for $\alpha \in Q^1$.
- Define the Jacobian algebra

$$J = \mathbb{C}Q / \langle \partial W \rangle$$

- Let $\mathrm{Rep}(Q, W)$ be the category of J-modules.

More stability scattering diagrams

- Fact: All the constructions we saw for $\mathrm{Rep}(Q)$ also work with $\mathrm{Rep}(Q, W)$.
- In many non-acyclic cases now, $\mathfrak{D}(Q, W) = \mathfrak{D}^{\mathrm{cl}}(Q)$ as long as W is "non-degenerate."
- For the Markov quiver with certain non-degenerate potentials, $\mathfrak{D}(Q, W)$ has one more wall than $\mathfrak{D}(Q)$ [Chen-M-Qin].

Coverings

- Let G be a finite group acting freely on Q.
- Let $\overline{Q} = Q/G$.

Coverings

- Let G be a finite group acting freely on Q.
- Let $\overline{Q} = Q/G$.
- There's a homomorphism

$$\sigma : \mathbb{C}\overline{Q} \to \mathbb{C}Q \qquad p \mapsto \sum (\text{all lifts of } p)$$

Coverings

- Let G be a finite group acting freely on Q.
- Let $\overline{Q} = Q/G$.
- There's a homomorphism

$$\sigma : \mathbb{C}\overline{Q} \to \mathbb{C}Q \qquad p \mapsto \sum (\text{all lifts of } p)$$

- For potentials $\overline{W} \in \mathbb{C}\overline{Q}$, $W = \sigma(W) \in \mathbb{C}Q$, σ descends to $\sigma : \overline{J} \to J$.

Coverings

- Let G be a finite group acting freely on Q.
- Let $\overline{Q} = Q/G$.
- There's a homomorphism

$$\sigma : \mathbb{C}\overline{Q} \to \mathbb{C}Q \qquad p \mapsto \sum (\text{all lifts of } p)$$

- For potentials $\overline{W} \in \mathbb{C}\overline{Q}$, $W = \sigma(W) \in \mathbb{C}Q$, σ descends to $\sigma : \overline{J} \to J$.
- Induces $\sigma^* : \mathrm{Rep}(\overline{Q}, \overline{W}) \to \mathrm{Rep}(Q, W)$.

Coverings

- Let G be a finite group acting freely on Q.
- Let $\overline{Q} = Q/G$.
- There's a homomorphism

$$\sigma : \mathbb{C}\overline{Q} \to \mathbb{C}Q \qquad p \mapsto \sum (\text{all lifts of } p)$$

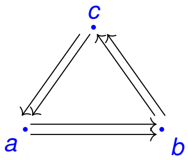
- For potentials $\overline{W} \in \mathbb{C}\overline{Q}$, $W = \sigma(W) \in \mathbb{C}Q$, σ descends to $\sigma : \overline{J} \to J$.
- Induces $\sigma^* : \mathrm{Rep}(\overline{Q}, \overline{W}) \to \mathrm{Rep}(Q, W)$.

Theorem (Chen-M-Qin)

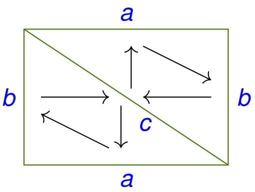
If (Q, W) admits "nice gradings" and J is finite-dimensional, then we can identify $\mathfrak{D}(\overline{Q}, \overline{W})$ with a "slice" of $\mathfrak{D}(Q, W)$.

The Markov quiver

- We were especially interested in the Markov quiver.



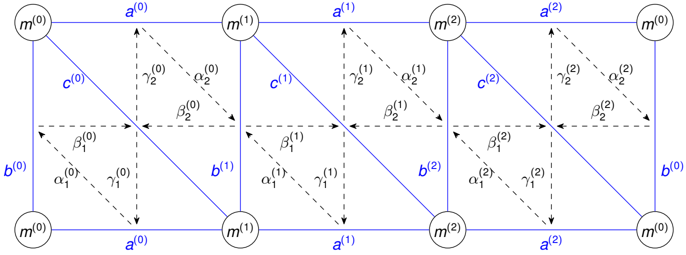
- When studying cluster algebras from surfaces, this quiver is associated to the 1-punctured torus:



- There's also a certain potential W associated to this Q [Labardini-Fragoso].

The 1-punctured torus from the 3-punctured torus

- The 3-punctured torus gives a covering of the 1-punctured torus.



- Our result applies to this covering.
- In other work with Qin, we showed that the “bracelets basis” equals the “theta basis” *except* for the 1-punctured torus.
- Our result here implies that the result holds for the 1-punctured torus as well if we use the stability scattering diagram instead of the cluster scattering diagram.