Introduction to toric varieties and algebraic geometry

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Chapter 1

Introduction

A **toric variety** is a (normal) algebraic variety X containing an algebraic torus $(\mathbb{C}^*)^r$ as a dense open subset such that the torus action extends to all of X. Toric varieties are spaces of fundamental importance in algebraic geometry. They are important objects of study in their own right, useful test cases for developing new theories, and common ambient spaces or local models for other spaces of interest. Their geometry is typically understood entirely in terms of the combinatorics of lattice cones and polytopes. These notes begin by reviewing relevant concepts from algebraic geometry as in [Har77, Ch. I-II]. Then, mostly following [Ful93], we proceed to learn about toric varieties, including their constructions from fans and polytopes, torus orbits, line bundles, blowups, resolution of singularities, homogeneous coordinates, moment maps, Chow groups, etc.

Chapter 2

Background on varieties and schemes

2.1 Affine varieties

Let $\Bbbk$ be a fixed algebraically closed field of characteristic 0. Define affine n-space over $\Bbbk$, denoted $\mathbb{A}^n_{\Bbbk}$ or simply $\mathbb{A}^n$ if $\Bbbk$ is understood, to be the set of all n-tuples of elements of $\Bbbk$.

Let $A := \Bbbk[x_1, \ldots, x_n]$ be the polynomial ring in n variables over $\Bbbk$. Note that elements of A can be evaluated at points in $\mathbb{A}^n$. Given a subset $T \subset A$, define the zero set of T to be the common zeroes of all elements of T, i.e.,

$$Z(T) := \{P \in \mathbb{A}^n | f(P) = 0 \text{ for all } f \in T\}.$$

Note that if $\mathfrak{a}$ is the ideal generated by T, then $Z(\mathfrak{a}) = Z(T)$. Since A is Noetherian, every ideal $\mathfrak{a}$ is finitely generated, so every $Z(T)$ can be expressed as the zero set of a finite set of polynomials $f_1, \ldots, f_r$.

The **Zariski topology** on $\mathbb{A}^n$ is defined by specifying that a set V is closed if there exists some $T \subset A$ such that $V = Z(T)$.

A non-empty subset Y of any topological space X is said to be **irreducible** if it cannot be expressed as a union $Y = Y_1 \cup Y_2$ for two proper subsets Y_1, Y_2, each one of which is closed in Y.

Example 2.1.1. In $\mathbb{A}^2$ with the Zariski topology, the x- and y-axes are both irreducible, but their union is not.

Definition 2.1.2. An **affine (algebraic) variety** is an irreducible Zariski-closed subset of $\mathbb{A}^n$. An open subset of an affine variety is a **quasi-affine variety**.

Given any subset $Y \subset \mathbb{A}^n$, define the ideal of Y by

$$I(Y) = \{f \in A | f(P) = 0 \text{ for all } P \in Y\}.$$

Recall that the radical of an ideal $\mathfrak{a} \subset A$ (for any ring A) is the set

$$\sqrt{\mathfrak{a}} := \{f \in A | f^r \in \mathfrak{a} \text{ for some } r \in \mathbb{Z}_{\geq 1}\}.$$

An ideal $\mathfrak{a}$ is said to be **radical** if $\sqrt{\mathfrak{a}} = \mathfrak{a}$.

Theorem 2.1.3. *There is a one-to-one inclusion-reversing correspondence between Zariski-closed sets $Y \subset \mathbb{A}^n$ and radical ideals in A, given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. Furthermore, Y is irreducible if and only if $I(Y)$ is prime, so the correspondence restricts to a bijection between affine varieties and prime ideals.*

Example 2.1.4. Points $(a_1, \ldots, a_n) \subset \mathbb{A}^n$ are in bijection with the maximal ideals of A:

$$I((a_1, \ldots, a_n)) = \langle x - a_1, \ldots, x - a_n \rangle.$$

Example 2.1.5. In A, $\langle x \rangle$ and $\langle y \rangle$ are prime ideals corresponding to the irreducible varieties $\{x = 0\}$ and $\{y = 0\}$, respectively. On the other hand, $\langle xy \rangle$ is not prime, and the corresponding variety $\{x = 0\} \cup \{y = 0\}$ is reducible.

Definition 2.1.6. If $Y \subset \mathbb{A}^n$ is Zariski-closed, we define the coordinate ring $A(Y)$ of Y to be $A/I(Y)$.

One should think of $A(Y)$ as the ring of polynomial functions (or **regular functions**) on Y. Note that irreducible Zariski-closed subsets of Y correspond to prime ideals of $A(Y)$, with points in Y corresponding to maximal ideals in $A(Y)$.

Also, note that Y is an affine variety (i.e., $I(Y)$ is a prime ideal) if and only if $A(Y)$ is an integral domain. Thus, we have a bijection between affine varieties and finitely generated $\Bbbk$-algebras which are integral domains.

An element a of a ring R is **nilpotent** if $a^k = 0$ for some $k \in \mathbb{Z}_{\geq 0}$. A ring R is called **reduced** if it has no nonzero nilpotent elements. Note that $A/\mathfrak{a}$ is reduced if and only if $\mathfrak{a}$ is radical. Thus, we have a bijection between Zariski-closed subsets of $\mathbb{A}^n$ and *reduced* finitely generated $\Bbbk$-algebras.

Given a reduced finitely generated $\Bbbk$-algebra R (possible an integral domain), we denote the corresponding Zariski-closed set (affine variety if R is an integral domain) by $\operatorname{Spec} R$. Note that points of $\operatorname{Spec} R$ correspond to maximal ideals of R, while more general irreducible Zariski-closed subsets of $\operatorname{Spec} R$ correspond to prime ideals in R.

Definition 2.1.7. Given a commutative ring R, call $S \subset R$ a **multiplicative set** if it is closed under multiplication and contains 1. If R is an integral domain[1] and $0 \notin S$, define the **localization** $S^{-1}R$ by

$$S^{-1}R := \left\{ \frac{f}{g} \,\middle|\, f \in R, g \in S \right\}$$

interpreted as a subring of the field of fractions of R.

Let $Y = \operatorname{Spec} R$ be an affine variety with coordinate ring $A(Y) = R = A/I(Y)$ (so R is an integral domain). Let $f \in R \setminus \{0\}$, and let $V = Z(f) \subset Y$. Let $U := Y \setminus V$ be the quasi-affine variety given by the complement of V. Denote $S_f := \{1, f, f^2, f^3, \ldots\}$, and consider the localization

$$R_f := S_f^{-1} R.$$

That is, we allow division by powers of f, or equivalently, by functions which do not vanish on U. We call R_U the **coordinate ring** of U (or the ring of regular functions on U) and define

$$\operatorname{Spec} R_f := U.$$

[1] See https://en.wikipedia.org/wiki/Localization_(commutative_algebra) for the definition of $S^{-1}R$ when R is not an integral domain. Briefly, in this general setting, $S^{-1}R$ is defined as the set of ordered pairs (r, s) with $r \in R$ and $s \in S$, modulo the equivalence relation $(r_1, s_1) \sim (r_2, s_2)$ whenever there exists a $t \in S$ such that $t(r_1 s_2 - r_2 s_1) = 0$.

Note again that points in U correspond to maximal ideals in R_U, and more general irreducible closed subsets of U correspond to prime ideals in R_U.

More generally, given any commutative ring R, there is an associated "affine scheme" $\operatorname{Spec} R$. This will be explained in the next couple subsections.

2.2 Sheaves

Let X be a topological space, and let $\mathcal{C}$ be any category. A **presheaf** $\mathcal{F}$ on X consists of the following data:

1. for every open set $U \subset X$, an object $\mathcal{F}(U) \in \mathcal{C}$, and
2. for every inclusion $V \subset U$ of open subsets of X, a morphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathcal{C}$,

subject to the conditions

(a) ρ_{UU} is the identity map on $\mathcal{F}(U)$, and

(b) if $W \subset V \subset U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

For another perspective, let $\mathfrak{Top}(X)$ be the category of open subsets of X with inclusions as the morphisms. Then a presheaf on X is a contravariant functor $\mathcal{F} : \mathfrak{Top}(X) \to \mathcal{C}$.

For us, $\mathcal{C}$ will typically be a category of Abelian groups, commutative rings, R-modules for some commutative ring R, or some other category where objects are Abelian groups with some additional structure. For $s \in \mathcal{F}(U)$ and $V \subset U$, we may denote $\rho_{UV}(s)$ by $s|_V$. A presheaf $\mathcal{F}$ is said to be a **sheaf** if it satisfies the following additional condition (the gluing condition):

3. for U an open set and $\{V_i\}$ an open covering of U, suppose we have elements $s_i \in \mathcal{F}(V_i)$ for each i satisfying the property that, for each i, j, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$. Then there is a unique element $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for each i.

The uniqueness of the gluing above is then equivalent to the statement that whenever $\{V_i\}$ is an open covering for an open set U and $s \in \mathcal{F}(U)$ satisfies $s|_{V_i} = 0$ for all i, we necessarily have $s = 0$. Another consequence of this is that $\mathcal{F}(\emptyset) = 0$ ([Har77] "incorrectly" takes this as part of the definition of a presheaf).

Viewing a (pre)sheaf as a contravariant functor, a morphism of (pre)sheaves is then a natural transformation between functors. More concretely, a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of (pre)sheaves on X is a morphism $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set $U \subset X$ such that, whenever $V \subset U$ is an inclusion of open subsets, we have

$$\rho^{\mathcal{G}}_{UV} \circ \varphi(U) = \varphi(V) \circ \rho^{\mathcal{F}}_{UV}.$$

If $f : X \to Y$ is a continuous map and $\mathcal{F}$ is a sheaf on X, then the **direct image** sheaf $f_*\mathcal{F}$ is the sheaf on Y defined as follows: for $U \subset Y$ open,

$$f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U)).$$

Similarly, for $f : X \to Y$ continuous and $\mathcal{G}$ a sheaf on Y, the inverse image presheaf $(f^{-1}\mathcal{G})^-$ on X is defined by

$$(f^{-1}\mathcal{G})^-(U) := \varinjlim_{V \supset f(U)} \mathcal{G}(V) \tag{2.1}$$

for all open subsets $U \subset X$. Here, the direct limit[2] is taken over all open subsets $V \subset Y$ which contain $f(U)$.[3]

In particular, if x is a point in Y and $i : x \hookrightarrow Y$ is the inclusion map, then

$$\mathcal{G}_x := (i^{-1}\mathcal{G})^- = \varinjlim_{V \supset x} \mathcal{G}(V)$$

is called the **stalk** of $\mathcal{G}$ at x. In general, elements of $\mathcal{G}(V)$ (for any sheaf $\mathcal{G}$ and open set V) are called **sections**, while elements of $\mathcal{G}_x$ are called **germs**. For each open V and point $x \in V$, there is a map $\mathcal{G}(V) \to \mathcal{G}_x$, $g \mapsto g_x$. Here, g_x is called the germ of g at x.

2.2.1 Sheafification

Many operations in the category $\mathcal{C}$ (e.g., taking kernels, cokernels, or images) carry over in a naive way to give operations on presheaves valued in $\mathcal{C}$. For example, if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then the presheaf kernel/cokernel/image of φ is the presheaf given on open sets U by $U \mapsto \ker(\varphi(U))$, $U \mapsto \operatorname{coker}(\varphi(U))$, or $U \mapsto \operatorname{Im}\varphi(U)$, respectively. If φ is a morphism of sheaves, then the presheaf kernel of φ is also a sheaf. However, the presheaf cokernel and image of φ are in general not sheaves — they do not always satisfy the gluing condition. This issue is rectified through "sheafification."

Proposition 2.2.1 ([Har77], Proposition-Definition 1.2)**.** *Given a presheaf $\mathcal{F}$, there is a sheaf $\mathcal{F}^+$ (commonly called the* ***sheafification*** *of $\mathcal{F}$ or the* ***sheaf associated to the presheaf*** *$\mathcal{F}$) and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$, such that for any sheaf $\mathcal{G}$ and any morphism $\varphi : \mathcal{F} \to \mathcal{G}$, there is a unique morphism $\psi : \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi = \psi \circ \theta$. The pair $(\mathcal{F}^+, \theta)$ is unique up to unique isomorphism.*

The proof is short and constructive: $\mathcal{F}^+(U)$ is taken to be the set of functions $s : U \to \bigsqcup_{P \in U} \mathcal{F}_P$ to the stalks of $\mathcal{F}$ over points P of U, such that

- $s(P) \in \mathcal{F}_P$ for each $P \in U$, and
- for each $P \in U$, there is a neighborhood V of P contained in U and an element $t \in \mathcal{F}(V)$ such that, for all $Q \in V$, the germ t_Q of t at Q is equal to $s(Q)$.

If $\mathcal{F}$ is already a sheaf, then $\theta : \mathcal{F} \to \mathcal{F}^+$ is an isomorphism.

One now defines the kernel/cokernel/image of a morphism of sheaves to be the sheaf assoicated to the presheaf kernel/cokernel/image. Similarly, for $f : X \to Y$ continuous and $\mathcal{G}$ a sheaf on Y, the **inverse image sheaf** $f^{-1}\mathcal{G}$ on X is the sheaf associated to the presheaf $(f^{-1}\mathcal{G})^-$ from (2.1).

[2]See here for the definition of direct limits in terms of their universal property (the approach I badly sketched in class), and see here for a more concrete approach that works for categories of sets with some additional algebraic structure (e.g., the category of commutative rings).

[3]One point of confusion in class was whether this should be a direct limit or an inverse limit. It is in fact a direct limit in the category $\mathcal{C}$. This is a bit confusing because we index the objects in the limit using the open sets $V \in \mathfrak{Top}(Y)$, and because sheaves are contravariant, the direction of the arrows in $\mathfrak{Top}(Y)$ is the opposite of that in $\mathcal{C}$. Another confusing point is that I used the sheaf of analytic functions as an example in class. This is a good example, but be careful not to confuse the stalks of this sheaf (i.e., power series centered at a point and with positive radius of convergence) with the ring of *formal* power series $\Bbbk[\![x]\!]$ (i.e., power series which do not necessarily converge) — the latter can be defined as the inverse limit $\varprojlim_n \Bbbk[x]/\langle x^n \rangle$.

2.3 Schemes

A **ringed space** is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X.

Given a commutative ring R, the **spectrum of** R, denoted $\operatorname{Spec} R$, is the locally[4] ringed space $(X, \mathcal{O}_X)$ given as follows:

- The underlying set for the topological space X is the set of prime ideals of R.
  - Recall that in the setting of varieties, i.e., where R is a finitely generated integral $\Bbbk$-algebra, the prime ideals correspond to the affine subvarieties (irreducible Zariski-closed subsets);
  - What we normally think of as points (the "**geometric points**") correspond to maximal ideals.
- We then equip X with the **Zariski topology**, defined as follows. If $\mathfrak{a}$ is any ideal of R, define $V(\mathfrak{a}) \subset \operatorname{Spec} R$ to be the set of prime ideals which contain $\mathfrak{a}$. A set is then defined to be **closed** if it is equal to $V(\mathfrak{a})$ for some ideal $\mathfrak{a}$.
  - Think of R as the coordinate ring on $\operatorname{Spec} R$. Then $V(\mathfrak{a})$ should be viewed as the common zero locus of the elements of $\mathfrak{a}$.
  - For example: $R = \Bbbk[x, y]$, $\mathfrak{a} = \langle x \rangle$. Then $V(\mathfrak{a})$ includes $\langle x \rangle$ (the prime ideal corresponding to the locus $x = 0 \subset \mathbb{A}^2_{\Bbbk}$) and also the maximal ideals $\langle x, y - b \rangle$ for $b \in \Bbbk$, i.e., the maximal ideals corresponding to the points $(0, b)$.
- If R is an integral domain, $\mathfrak{a} \subset R$ is an ideal, and $U := X \setminus V(\mathfrak{a})$, define

$$\mathcal{O}_X(U) := S_U^{-1} R$$

where

$$S_U = \{f \in R | V(\langle f \rangle) \subset V(\mathfrak{a})\}.$$

That is, we allow division by elements whose zero sets are contained in the locus $V(\mathfrak{a})$ which we are taking the complement of.

  - When R is not an integral domain this is a bit more delicate. Given $f \in R$, let $D_f \subset \operatorname{Spec} R$ be the set of prime ideals which do not contain f (interpret this as points where f does not vanish). Then $\mathcal{O}_X(D_f) = S^{-1}R$ where $S^{-1} = \{1, f, f^2, f^3, \ldots\}$. The sets D_f are called the *distinguished open sets* and form a basis for the Zariski topology. Then

$$\mathcal{O}_X\left(\bigcup_{i \in I} D_{f_i}\right) := \{(f_i)_{i \in I} | f_i|_{D_i \cap D_j} = f_j|_{D_i \cap D_j} \text{ for all } i, j \in I\}. \tag{2.2}$$

  - [Har77] uses a different approach which is more technical, but it has the advantage of being closer to the intuition where elements of $\mathcal{O}(U)$ are like functions taking values at the points of U. Briefly, if $f \in R$, then the "value" of f at $\mathfrak{p}$ is the projection of f to $R/\mathfrak{p}$ (which equals $\Bbbk$ if $\mathfrak{p}$ is maximal and R is a reduced finitely generated $\Bbbk$-algebra). For schemes

[4] A *locally* ringed space is a ringed space where the stalks are local rings (i.e., they have a unique maximal ideal). The stalk of $\operatorname{Spec} R$ at $\mathfrak{p}$ is the localization $R_{\mathfrak{p}} := S_{\mathfrak{p}}^{-1} R$ for $S_{\mathfrak{p}} := R \setminus \mathfrak{p}$. The unique maximal ideal of $R_{\mathfrak{p}}$ is the image of $\mathfrak{p}$. These technical details are important in general, but they will not be explicitly important for us.

though, one should actually retain more information, instead taking the element induced by f in the local ring $R_{\mathfrak{p}} := S_{\mathfrak{p}}^{-1}R$ where $S_{\mathfrak{p}} := R \setminus \mathfrak{p}$. Then elements of $\mathcal{O}(U)$ are functions $U \to \bigsqcup_{\mathfrak{p}\in U} R_{\mathfrak{p}}$ satisfying the natural gluing condition (i.e. being induced by elements on affine open neighborhoods). Elements of $\mathcal{O}_X(U)$ are called **regular functions** on U.

Definition 2.3.1. A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ together with a map $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y.

An additional condition is imposed when defining morphisms of *locally* ringed spaces, but this is technical and will not really help us when trying to understand toric varieties, so I'll defer this to a footnote.[5]

Homomorphisms of rings $\varphi : A \to B$ correspond to morphisms

$$(f, f^\sharp) : \operatorname{Spec} B \to \operatorname{Spec} A$$

of locally ringed spaces:

- Given $\mathfrak{p} \in \operatorname{Spec} B$, $f(\mathfrak{p}) := \varphi^{-1}(\mathfrak{p})$;
- φ induces maps between the relevant localizations of A and B to give the map $f^\sharp$ of sheaves.
  - Suppose R is an integral domain. Then $f^\sharp : \mathcal{O}_{\operatorname{Spec} A}(U) \to \mathcal{O}_{\operatorname{Spec} B}(f^{-1}(U))$ takes an element $\frac{g}{h}$ with $g \in R$ and $h \in S_U$ to $\frac{\varphi(g)}{\varphi(h)}$.
    - ∗ Note that $h \in S_U$ means $V(\langle h\rangle) \subset X \setminus U$. So if $\mathfrak{p} \subset B$ is a prime ideal containing $\varphi(h)$, then $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$ contains h, hence is in $V(\langle h\rangle) \subset X \setminus U$. That is, $f(V(\langle\varphi(h)\rangle)) \subset X \setminus U$, so $V(\langle\varphi(h)\rangle)Y \setminus f^{-1}(U)$. Thus, $h \in S_{f^{-1}(U)}$, so this is well-defined.
  - Intuitively (for varieties over $\Bbbk$ at least), if we think of elements of the sheaves as functions, then $f^\sharp$ takes a function $h : U \to \Bbbk$ to $h \circ f : f^{-1}(U) \to \Bbbk$ (similarly for the richer version where these functions take values in local rings — this is actually what [Har77] does).

Definition 2.3.2. An **affine scheme** is a locally ringed space which is isomorphic to $\operatorname{Spec} R$ for some R. A scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that each point of X has a neighborhood U such that $(U, \mathcal{O}_X|_U)$ is isomorphic to an affine scheme. The sheaf $\mathcal{O}_X$ is called the **structure sheaf** of the scheme.

If we can take each $(U, \mathcal{O}_X|_U)$ above to be a quasi-affine variety, then X is called an (algebraic) **variety**.

A scheme over a ring R, also called an R-scheme, is a scheme X equipped with a morphism $X \to \operatorname{Spec} R$. Note that this induces morphisms $R \to \mathcal{O}_X(U)$ for each open $U \subset X$, so the coordinate rings are R-algebras. A morphism between R-schemes $X \to \operatorname{Spec} R$ and $Y \to \operatorname{Spec} R$ is a morphism $X \to Y$ which commutes with the morphisms to $\operatorname{Spec} R$

For $\Bbbk$ an algebraically closed field,[6] the **geometric points** of a $\Bbbk$-scheme X (i.e., what we usually think of as points) are the morphisms $\operatorname{Spec} \Bbbk \to X$. I.e., for $U \subset X$ an affine open subset, the

[5] Morphisms of locally ringed spaces are required to induce "local homomorphisms" between stalks $f^\sharp_P : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,P}$ — a homomorphism $\varphi : A \to B$ between local rings A, B with maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$ is said to be "local" if $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

[6] If $\Bbbk$ is not algebraically closed, then the geometric points are defined as the morphisms $\operatorname{Spec} \overline{\Bbbk} \to X$ over $\operatorname{Spec} \Bbbk$, where $\overline{\Bbbk}$ is the algebraic or separable closure of $\Bbbk$.

geometric points of X in U correspond to the $\Bbbk$-algebra morphisms $\mathcal{O}_X(U) \to \Bbbk$ — such morphisms, in turn, correspond to the maximal ideals of $\mathcal{O}_X(U)$.

In particular, for R a $\Bbbk$-algebra and $X = \operatorname{Spec} R$, the geometric points of X correspond to the maximal ideals of R. We may denote the set of geometric points/maximal ideals of such a scheme by $\operatorname{mSpec} R$.

On the other hand, if R is an integral domain, then the zero ideal $\{0\}$ is the unique minimal prime ideal of R. Hence, $V(\{0\})$ is all of $\operatorname{Spec} R$ — i.e., this single point $\eta = \{0\} \in \operatorname{Spec} R$, called the **generic point**, is dense in $\operatorname{Spec} R$.

2.3.1 Some properties of schemes

Here we briefly review a few definitions as in [Har77, Ch. 2, §3].

A scheme is called **irreducible** if the underlying topological space is irreducible as defined in §2.1.

A scheme X is called **reduced** if the each stalk $\mathcal{O}_{X,P}$ has no nilpotent elements (equivalently, if $\mathcal{O}_X(U)$ has no nilpotent elements for each open set U). Intuitively, this means we do not have any "fat" points, or points with higher multiplicity (e.g., $\operatorname{Spec} \Bbbk[x]/x^2$).

A scheme X is **integral** if for every open set $U \subset X$, $\mathcal{O}_X(U)$ is an integral domain. By [Har77, Ch. 2, Prop. 3.1], a scheme is integral if and only if it is both reduced and irreducible.

A scheme is **Noetherian** if it can be covered by a finite number of affine open subset $\operatorname{Spec} A_i$ with each A_i a Noetherian ring. We note that algebraic varieties are Noetherian [Har77, Ch. 2, Example 3.2.1].

For varieties X and Y, a **rational** map $\varphi : X \dashrightarrow Y$ is a morphism from a nonempty open subset $U \subset X$ to Y. The term "rational" refers to the fact that φ can be expressed in coordinates using rational functions. A rational map φ is called **birational** if there exists a rational map $Y \dashrightarrow X$ which is inverse to φ. I.e., a birational map $X \dashrightarrow Y$ is an isomorphism between Zariski dense open subsets of X and Y. In terms of coordinates, $\varphi : X \dashrightarrow Y$ is birational if the induced map on function fields is an isomorphism.

A scheme is called **normal** if all of its local rings are integrally closed domains. Geometrically, an algebraic variety X is normal if and only every finite[7] birational morphism to X is an isomorphism. E.g., the cuspidal cubic $C = Z(y^3 - x^2) \subset \mathbb{A}^2$ is not normal because the map $\mathbb{A}^1 \to C \subset \mathbb{A}^2$, $t \mapsto (t^3, t^2)$ is finite and birational but is not an isomorphism. Similarly for nodal cubics.

2.4 Projective varieties and projective schemes

2.4.1 Projective varieties

Let $\mathbb{P}^n_{\Bbbk}$, or simply $\mathbb{P}^n$, denote **projective n-space**, i.e., the set of equivalence classes of $(n+1)$-tuples $(a_0, \ldots, a_n) \in \Bbbk^{n+1} \setminus \{(0, 0, \ldots, 0)\}$, modulo the equivalence relation

$$(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$$

for all $\lambda \in \Bbbk^*$. I.e., $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/\Bbbk^*$, where the $\Bbbk^*$ acts via multiplication on each component.

Equivalently, $\mathbb{P}^n_{\Bbbk}$ is the space of lines $\mathbb{A}^1_{\Bbbk}$ through the origin in $\mathbb{A}^{n+1}_{\Bbbk}$.

[7] Finite here (at least for $\Bbbk = \mathbb{C}$) can be taken to mean that the inverse image of a compact set is compact (i.e., the morphism is proper), and the inverse image of a point is finite (i.e., the morphism is quasi-finite).

Definition 2.4.1. A **graded ring** is a ring R, together with a decomposition

$$R = \bigoplus_{d\in\mathbb{Z}_{\geq 0}} R_d$$

of R (with its addition operation) into a direct sum of Abelian groups R_d, such that, for any $d, e \geq 0$, $R_d \cdot R_e \subset R_{d+e}$. The elements of R_d are called the **homogeneous** elements of degree d. An ideal I is **homogeneous** if it can be generated by homogeneous elements, or equivalently, if $I = \bigoplus_{d\in\mathbb{Z}_{\geq 0}}(I \cap R_d)$.

Remark 2.4.2. Definition 2.4.1 is really the definition of a $\mathbb{Z}_{\geq 0}$-graded ring, but in general, rings can be graded by any monoid K by simply replacing $\mathbb{Z}_{\geq 0}$ in Definition 2.4.1 by K. The definitions of homogeneous elements and ideals carry over similarly. We will likely consider K-graded rings for more general K later, but for now, assume all graded rings are $\mathbb{Z}_{\geq 0}$-graded.

Remark 2.4.3. The sum, product, intersection, and radical of homogeneous ideals are homogeneous. Suppose that, for any two homogeneous elements $f, g \in R$ with $fg \in I$, we have either $f \in I$ or $g \in I$; then I is prime.

Let $S := \Bbbk[x_0, x_1, \ldots, x_n]$. The degree of a monomial $ax_0^{k_0}x_1^{k_1}\cdots x_n^{k_n}$ is defined to be $\sum_{i=0}^n k_n$. We view S as a graded ring $S = \sum_{d\geq 0} S_d$ with S_d being the set of all linear combinations of degree d monomials.

An element $f \in S$ does not give a well-defined function on $\mathbb{P}^n$ (unless f is constant) because it does not respect the $\Bbbk^*$-scaling action. But if f is homogeneous of degree d, then

$$f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d f(a_0, \ldots, a_n),$$

so the **zero set** of f is well-defined.

If T is any set of homogeneous elements of S, then the **zero set** of T is

$$Z(T) = \{P \in \mathbb{P}^n | f(P) = 0 \text{ for all } f \in T\}.$$

If $I \subset S$ is a homogeneous ideal, then $Z(I) := Z(T)$ where T is the set of homogeneous elements in I. Since S is Noetherian, any set of homogeneous elements T has a finite subset $f_1, \ldots, f_r$ such that $Z(T) = Z(f_1, \ldots, f_r)$.

Definition 2.4.4. The **Zariski topology** on $\mathbb{P}^n$ is defined by saying that a set V is closed if $V = Z(T)$ for some set T of homogeneous elements of S.

A **projective (algebraic) variety** is an irreducible Zariski-closed subset of $\mathbb{P}^n$, with the induced topology. An open subset of a projective variety, with its induced topology, is a **quasi-projective variety**.

For any $Y \subset \mathbb{P}^n$, the **homogeneous ideal** of Y in S is the ideal $I(Y)$ generated by

$$\{f \in S | f \text{ is homogeneous and } f(P) = 0 \text{ for all } P \in Y\}.$$

If Y is Zariski closed, define the **homogeneous coordinate ring** of Y to be $S(Y) = S/I(Y)$.

The zero set of a nonzero linear homogeneous polynomial $f = \sum_{i=0}^n a_i x_i$ is called a **hyperplane**. Denote

$$H_i = Z(x_i) \subset \mathbb{P}^n$$

and let

$$U_i := \mathbb{P}^n \setminus H_i.$$

Note that a point $P = (a_0, a_1, \ldots, a_n) \in \mathbb{P}^n$ must have $a_i \neq 0$ for some i, i.e., $P \in U_i$ for some i. So $\{U_i\}_{i=0}^n$ is an open cover for $\mathbb{P}^n$.

Define

$$\varphi_i : U_i \to \mathbb{A}^n$$
$$(a_0, \ldots, a_n) \mapsto \left(\frac{a_0}{a_i}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \ldots, \frac{a_n}{a_i}\right)$$

Proposition 2.4.5 ([Har77], Ch. I, Prop. 2.2 and Cor. 2.3)**.** *The map φ_i is a homeomorphism of U_i with its induced topology to $\mathbb{A}^n$ with its Zariski topology. If Y is a (quasi-)projective variety, then Y is covered by the open sets $\{Y \cap U_i\}$, $i = 0, \ldots, n$, which are homeomorphic to (quasi-)affine varieties via the restrictions of the maps φ_i.*

For S a graded ring $\bigoplus_{d \geq 0} S_d$, let S_+ denote the ideal $\bigoplus_{d>0} S_d$.

Proposition 2.4.6 ([Har77], Ch. I, Exercise 2.4)**.** *There is a one-to-one inclusion-reversing correspondence between nonempty Zariski-closed subsets of $\mathbb{P}^n$ and homogeneous radical ideals of S not containing S_+, given by $Y \mapsto I(Y)$ and $I \mapsto Z(I)$. A closed set Y is irreducible if and only if $I(Y)$ is prime.*

Example 2.4.7. Consider $X = \mathbb{P}^2$, $S = \Bbbk[x_0, x_1, x_2]$. Let $X = \frac{x_1}{x_0}, Y = \frac{x_2}{x_0}$. Then $U_0 = \operatorname{Spec} \Bbbk[X, Y]$. Also,

$$U_1 = \operatorname{Spec} \Bbbk\left[\frac{x_0}{x_1}, \frac{x_2}{x_1}\right] = \operatorname{Spec} \Bbbk[X^{-1}, X^{-1}Y] \quad \text{and} \quad U_2 = \operatorname{Spec} \Bbbk\left[\frac{x_0}{x_2}, \frac{x_1}{x_2}\right] = \operatorname{Spec} \Bbbk[Y^{-1}, XY^{-1}].$$

2.4.2 Projective schemes

Let S be a graded ring $\bigoplus_{d \geq 0} S_d$. Let S_+ be the ideal $\bigoplus_{d>0} S_d$. Let

$$\operatorname{Proj} S$$

be the set of homogeneous prime ideals $\mathfrak{p}$ which do not contain all of S_+. For I a homogeneous ideal of S, define

$$V(I) := \{\mathfrak{p} \in \operatorname{Proj} S | \mathfrak{p} \supseteq I\}.$$

The **Zariski topology** on $\operatorname{Proj} S$ is defined by specifying that a set is closed if it equals $V(I)$ for some homogeneous ideal I.

We next define a sheaf of rings $\mathcal{O}_X$ on $X := \operatorname{Proj} S$ (see [Har77, pg. 76] for another approach). Given a homogeneous element $f \in S_+$, consider the distinguished open subset

$$D_+(f) := \operatorname{Proj} S \setminus V(f) = \{\mathfrak{p} \in \operatorname{Proj} S | f \notin \mathfrak{p}\}.$$

Recall that S_f denotes the localization of S by the set $\{1, f, f^2, f^3, \ldots\}$. Let $S_{(f)}$ denote the subring of S_f consisting of the elements of degree 0 (where the degree of an element $\frac{g}{h}$ is $\deg(g) - \deg(h)$). Then $\mathcal{O}_X|_{D_+(f)}$ is defined by specifying that

$$(D_+(f), \mathcal{O}_X|_{D_+(f)}) \cong \operatorname{Spec} S_{(f)}$$

where the isomorphism is an isomorphism of locally ringed spaces. The sets $D_+(f)$ form an open cover for X, and $\mathcal{O}_X$ is extended to the full space X as in (2.2).

Proposition 2.4.8 ([Har77], Ch. II, Prop. 2.5)**.** $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ *is a scheme.*

Remark 2.4.9. Note that we always have inclusions $S_0 \hookrightarrow S_{(f)}$. These induce projections $\operatorname{Spec} S_{(f)} \to \operatorname{Spec} S$, hence $\operatorname{Proj} S \to \operatorname{Spec} S_0$.

2.5 Sheaves of modules

2.5.1 Basic definitions

Here we recall the definition of a quasi-coherent sheaf as in [Har77, Ch. II, §5]. Let $(X, \mathcal{O}_X)$ be a ringed space. A **sheaf of $\mathcal{O}_X$-modules** is a sheaf $\mathcal{F}$ on X such that for each open $U \subset X$, $\mathcal{F}(U)$ is a module over $\mathcal{O}_X(U)$, and furthermore, the module structures (i.e., the actions of $\mathcal{O}_X(U)$ on $\mathcal{F}(U)$) are compatible with the restriction maps.

Given two sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, we may consider the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ defined as the sheafification (cf. §2.2.1) of the presheaf given on U by

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) := \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

The subscript $\mathcal{O}_X$ is often left off.

A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called **free** if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. It is called **locally free** if X can be covered by open sets U such that $\mathcal{F}|_U$ is free for each U. One may think of locally free sheaves as the algebro-geometric analog of a vector bundle. The number of $\mathcal{O}_X$-summands appearing here is the **rank** of $\mathcal{F}$. A locally free sheaf of rank one is called an **invertible sheaf** (because it can be tensored with another invertible sheaf to get just $\mathcal{O}_X$).

We say that $\mathcal{F}$ is **quasi-coherent** if it is locally presentable (locally the cokernel of a morphism of free modules); i.e., if there is an open cover $\{U_i\}$ such that for each i, we have an exact sequence

$$\mathcal{O}_X^{J_i}|_{U_i} \to \mathcal{O}_X^{I_i}|_{U_i} \to \mathcal{F}|_{U_i} \to 0.$$

Here, the index-sets J_i and I_i may be infinite. If $\mathcal{F}$ is finite-type and the I_i and F_i are all finite, then $\mathcal{F}$ is called **coherent**.

Equivalently, if X is a scheme, $\mathcal{F}$ is quasi-coherent if, for each pair of affine open subsets $V \subset U \subset X$, the natural homomorphism

$$\mathcal{O}(V) \otimes_{\mathcal{O}(U)} \mathcal{F}(U) \to \mathcal{F}(V), \quad f \otimes s \mapsto f \cdot s|_V \tag{2.3}$$

is an isomorphism (i.e., localization of $\mathcal{F}$ on affine open subsets U is via localization of the module $\mathcal{F}(U)$). Then $\mathcal{F}$ being coherent means additionally that for each affine open $U \subset X$, $\mathcal{F}(U)$ is a finitely-generated $\mathcal{O}_X(U)$-module. Morally, one may think of coherent sheaves as vector bundles on subschemes of X.

Suppose $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces. Recall the notions of direct image and inverse image from §2.2. For $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules on X, the **direct image** sheaf $f_*\mathcal{F}$ is naturally a sheaf of $\mathcal{O}_Y$-modules on Y by using $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ to induce the $\mathcal{O}_Y$-action.

In the reverse direction, if $\mathcal{G}$ is a sheaf of $\mathcal{O}_Y$-modules, then $f^{-1}(\mathcal{G})$ is an $f^{-1}(\mathcal{O}_Y)$-module. There is a morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, and one defines an $\mathcal{O}_X$-module $f^*\mathcal{G}$ via

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

A **sheaf of ideals** on X is a sheaf of modules $\mathcal{I}$ which is a subsheaf of $\mathcal{O}_X$. Given a closed subscheme $Y \subset X$, let $i : Y \hookrightarrow X$ denote the inclusion morphism. The sheaf $i_*\mathcal{O}_Y$ is quasi-coherent (and in fact, coherent). The **ideal sheaf** $\mathcal{I}_Y$ of Y is the sheaf of ideals given by the kernel of

$$i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Y.$$

In general, $\mathcal{I}_Y$ is quasi-coherent, and it is coherent if X is Noetherian. Explicitly, for $U = \operatorname{Spec} A$ an affine open subset of X and $Y \cap U$ the closed subscheme associated to the ideal $\mathfrak{a}$, we have $\mathcal{I}_Y(U) = \mathfrak{a}$ and $i_*\mathcal{O}_Y(U) = A/\mathfrak{a}$.

The sheaf associated to a graded module

A module M over a ring R determines a sheaf $\widetilde{M}$ on $X = \operatorname{Spec} R$ via $\widetilde{M}(U) = \mathcal{O}_X(U) \otimes_R M$. I.e., $\widetilde{M}$ is the sheaf whose stalk at $\mathfrak{p}$ is the localization $M_\mathfrak{p}$ of M at $\mathfrak{p}$. Note that (2.3) can be interpreted as saying that a sheaf $\mathcal{F}$ is quasi-coherent if its restriction to any affine open subset $U = \operatorname{Spec} R$ is $\widetilde{M}$ for $M = \mathcal{F}(U)$.

Similarly, if S is a graded ring, and if M is a graded S-module, then M determines is a quasi-coherent sheaf $\widetilde{M}$ on $X = \operatorname{Proj} S$. For each $\mathfrak{p} \in X$, let $M_{(\mathfrak{p})}$ be the group of elements of degree 0 in the localization $T^{-1}M := T^{-1}S \otimes_S M$ for $T = S \setminus \mathfrak{p}$. Similarly, for homogeneous $f \in S_+$, let $M_{(f)}$ be the group of degree-0 elements of $M_f := T^{-1}S \otimes_S M$ for $T = \{1, f, f^2, f^3, \ldots\}$. Then the stalk of $\widetilde{M}$ at $\mathfrak{p}$ is $M_{(\mathfrak{p})}$. Furthermore, the isomorphism $D_+(f) = \operatorname{Spec} S_{(f)}$ induces an identification $\widetilde{M}|_{D_+(f)} \cong \widetilde{M_{(f)}}$. Note that $\widetilde{M}$ is quasi-coherent (in fact, it is coherent if S is Noetherian and M is finitely generated). See [Har77, Prop. 5.11].

In particular, for each $n \in \mathbb{Z}$, let $S(n)$ denote the ring S with grading shifted, so that $S(n)_k = S_{n+k}$. There is an associated sheaf:

$$\mathcal{O}_X(n) := \widetilde{S(n)} \tag{2.4}$$

Example 2.5.1. Consider $X = \mathbb{P}^1 = \operatorname{Proj} S$ for $S = \Bbbk[x, y]$ with the standard grading ($\deg x = \deg y = 1$). Consider $D_+(y) = \operatorname{Spec} \Bbbk[\frac{x}{y}]$. We have

$$[\mathcal{O}_X(1)](D_+(y)) = S(1)_{(y)} = y \cdot \Bbbk\left[\frac{x}{y}\right].$$

Similarly, $[\mathcal{O}_X(1)](D_+(x)) = S(1)_{(x)} = x \cdot \Bbbk\left[\frac{y}{x}\right]$. The global sections of $\mathcal{O}_X(1)$ are then the (degree-1) elements of $\Bbbk[x^{\pm 1}, y^{\pm 1}]$ which lie in both $y \cdot \Bbbk[x/y]$ and $x \cdot \Bbbk[y/x]$ — thus, we find that $[\mathcal{O}_X(1)](X) \cong \Bbbk\langle x, y\rangle$.

More generally, for $X = \mathbb{P}^r = \operatorname{Proj} \Bbbk[x_0, \ldots, x_r]$, we can identify $[\mathcal{O}_X(n)](X)$ with the degree-n homogeneous elements of S. In particular, if $n < 0$, then $[\mathcal{O}_X(n)] = \{0\}$.

It turns out that all line bundles on $X = \mathbb{P}^r$ are isomorphic to $\mathcal{O}_X(n)$ for some $n \in \mathbb{Z}$.

More generally, if S is a graded ring generated by S_1 as an S_0-algebra, and if $X = \operatorname{Proj} S$, then $\mathcal{O}_X(n)$ is an invertible sheaf on X whose global sections can be identified with S_n. See [Har77, Ch. 2, Prop 5.15, Example 5.16.3].

For any sheaf $\mathcal{F}$ on $\operatorname{Proj} S$, one denotes $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_X(n)$.

2.5.2 The relative Proj construction

Here we follow parts of [Har77, Ch. II, §7]

Let X be a Noetherian scheme, and let $\mathcal{I}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules which furthermore has the structure of a sheaf of graded $\mathcal{O}_X$-algebras. So we have a decomposition $\mathcal{I} = \bigoplus_{d \geq 0} \mathcal{I}_d$, where $\mathcal{I}_d$ is the homogeneous part of degree d. Assume that $\mathcal{I}_0 = \mathcal{O}_X$, $\mathcal{I}_1$ is coherent, and $\mathcal{I}$ is locally generated by $\mathcal{I}_1$ as an $\mathcal{O}_X$-algebra (consequently, each $\mathcal{I}_d$ is also quasi-coherent).

For each affine open subset $U \subset X$, we consider $Y_U := \operatorname{Proj} \mathcal{I}(U)$ and the invertible sheaf $\mathcal{O}_{Y_U}(1)$ as in §(2.4). As in Remark 2.4.9, $\operatorname{Proj} \mathcal{I}(U)$ projects to $\operatorname{Spec} \mathcal{I}_0(U) = \operatorname{Spec} \mathcal{O}_X(U) = U$. One can show that these schemes Y_U, the morphisms to U, and the sheaves $\mathcal{O}_{Y_U}(1)$ naturally glue to yield a scheme $Y = \mathbf{Proj}\, \mathcal{I}$ with a projection to X

$$\pi : \mathbf{Proj}\, \mathcal{I} \to X$$

equipped with an invertible sheaf $\mathcal{O}_Y(1)$.

Relative Spec

We note that there is also a (simpler) relative Spec construction, where **Spec** of $\mathcal{I}$ (without any need for a grading) is defined analogously to **Proj** but using Spec in place of Proj. Note that the algebra-structure morphism $\mathcal{O}_X \to \mathcal{I}$ induces a morphism

$$f : \mathbf{Spec}\, \mathcal{I} \to X.$$

In fact, this f is an **affine morphism**, meaning that X admits an affine open cover $\{U_i\}$ such that each $f^{-1}(U_i)$ is an affine scheme. Furthermore, for every affine morphism $f : Y \to X$, $f_* \mathcal{O}_Y$ is quasi-coherent on X, and then $\mathbf{Spec}\, f_* \mathcal{O}_Y \cong Y$. Thus, there is an anti-equivalence of categories between the category of affine morphisms to X and the category of quasi-coherent sheaves of $\mathcal{O}_X$-algebras on X. See [Har77, Ch. 2, Exercise 5.17] for more details.

(Projective) vector bundles and locally free sheaves

Let $\mathcal{E}$ be a locally free coherent sheaf on X. Let $\mathcal{I} = \operatorname{Sym}(\mathcal{E})$ be the symmetric algebra[8] of $\mathcal{E}$,

$$\mathcal{I} = \operatorname{Sym}^\bullet \mathcal{E} = \bigoplus_{d \geq 0} \operatorname{Sym}^d(\mathcal{E}).$$

Applying **Spec** to $\mathcal{I}$ yields the vector bundle $\mathbf{V}(\mathcal{E})$ associated to[9] $\mathcal{E}$, equipped with the projection to X (see [Har77, Ch. 2, Exercise 5.18] for details). Applying **Proj** to $\mathcal{I}$ yields a projective bundle $\pi : \mathbb{P}(\mathcal{E}) \to X$.

Intuitively, each fiber of the vector bundle $\mathbf{Spec}\, \mathcal{I}$ looks like $\mathbb{A}^{n+1}_{\Bbbk}$ for $n + 1 = \operatorname{rank}(\mathcal{E})$ (for X a variety over $\Bbbk$). Then $\mathbb{P}(\mathcal{E})$ is obtained by taking the quotient by the $\Bbbk^*$-action on fibers to get a bundle whose fibers instead look like $\mathbb{P}^n_{\Bbbk}$.

[8] See [Har77, Ch. 2, Exercise 5.16] for details regarding tensor operations on sheaves. Basically, on the level of presheaves, for each open set U you can just take the symmetric algebra of $\mathcal{E}(U)$ as an $\mathcal{O}_X(U)$-module. Then sheafify.

[9] A slightly confusing point here is that $\mathcal{E}$ gives functions on $\mathbf{V}(\mathcal{E})$, not sections. Rather, the sections of $\mathbf{V}(\mathcal{E})$ are given by the dual sheaf $\mathcal{E}^\vee = \operatorname{Hom}(\mathcal{E}, \mathcal{O}_X)$. When one talks about, say, the tangent sheaf, they mean the sheaf of sections of the tangent bundle, so then the associated vector bundle is actually the dual to the tangent bundle, i.e., the cotangent bundle.

Example 2.5.2. Given an invertible sheaf $\mathcal{L}$ on X, the scheme $\mathbf{V}(\mathcal{L})$ is a line bundle over X. The scheme $\mathbb{P}(\mathcal{L} \oplus \mathcal{O}_X)$ is the "projectivization" of $\mathbf{V}(\mathcal{L})$, i.e., the $\mathbb{P}^1$-fibration obtained by adding a point at ∞ in each fiber of $\mathbf{V}(\mathcal{L})$.

Invertible sheaves and morphisms to projective space

Let X be a scheme over $\operatorname{Spec} \Bbbk$. Let $\mathcal{L}$ be an invertible sheaf on X, and let $s_0, \ldots, s_r \in \mathcal{L}(X)$. Let $S = \Bbbk[x_0, \ldots, x_r]$. For $U = \operatorname{Spec} A$ an affine open subset of X, we can (non-canonically) identify $\mathcal{L}|_U$ with $\mathcal{O}_X|_U$, and this induces a morphism $S \to A = \mathcal{O}_X(U)$ via $x_i \mapsto s_i|_U$. This yields a map $U \to \operatorname{Spec} S = \mathbb{A}^{r+1}_\Bbbk$. Let $Z_U \subset U$ be the locus mapped to $0 \in \mathbb{A}^{r+1}_\Bbbk$. Different identifications of $\mathcal{L}|_U$ with $\mathcal{O}_X|_U$ differ via multiplication by a unit u in A. This u must map to a unit in S, i.e., to an element of $\Bbbk^*$. Thus, Z_U is well-defined (canonically), as is the induced map $U \setminus Z_U \to (\mathbb{A}^{r+1} \setminus \{0\})/\Bbbk^* = \mathbb{P}^r_\Bbbk = \operatorname{Proj} S$. These maps glue to yield a map $X \setminus Z \to \operatorname{Proj} S = \mathbb{P}^r_\Bbbk$ for some locus Z (the union of the Z_U's).

Suppose that X is complete (i.e., X is proper over $\operatorname{Spec} \Bbbk$; i.e., X is compact). If the global sections $s_0, \ldots, s_r$ above generate $\mathcal{L}(X)$ over $\Bbbk$, the locus Z is called the base locus of $\mathcal{L}$. If Z is empty, $\mathcal{L}$ is said to be **basepoint-free**. In this case, the above construction yields a canonical map[10]

$$\varphi_{\mathcal{L}} : X \to \mathbb{P}H^0(X, \mathcal{L}). \tag{2.5}$$

A sheaf of $\mathcal{O}_X$-modules $\mathcal{L}$ on a scheme X is said to be **generated by global sections** if there is a family of global sections $\{s_i\}_{i \in I}$ in $\mathcal{L}(X)$ such that, for each $x \in X$, the images of s_i in the stalk $\mathcal{L}_x$ generate $\mathcal{L}_x$ as an $\mathcal{O}_x$-module. Equivalently, $\mathcal{L}$ is generated by global sections if and only if it can be written as a quotient of a free sheaf.

Proposition 2.5.3. *$\mathcal{L}$ is generated by global sections if and only if it is basepoint-free.*

Example 2.5.4. Suppose $X = \operatorname{Proj} S$ for S a graded ring generated by S_1 as an S_0-algebra. Then for $n \geq 0$, S_n gives global sections for the invertible sheaf $\mathcal{O}_X(n)$, and $\mathcal{O}_X(n)$ is generated by these global sections.

Definition 2.5.5. For X a proper scheme over $\operatorname{Spec} \Bbbk$, an invertible sheaf $\mathcal{L}$ on X is said to be **very ample** if it is basepoint-free and the map φ_L of (2.5) is a closed immersion (i.e., an isomorphism from X to a closed subscheme of $\mathbb{P}H^0(X, \mathcal{L})$). Equivalently, $\mathcal{L}$ is very ample if it is isomorphic to $i^*\mathcal{O}(1)$ for some closed immersion $i : X \to \mathbb{P}^r$ for some r. The sheaf $\mathcal{L}$ is called **ample** if $\mathcal{L}^k$ is very ample for some positive integer k.

If D is a Cartier divisor, one says that D is very ample (resp. ample) if $\mathcal{O}(D)$ is very ample (resp. ample).

Example 2.5.6. The sheaf $\mathcal{O}_{\mathbb{P}^r}(n)$ is very ample if and only if $n \in \mathbb{Z}_{\geq 1}$.

2.6 Blowups

If y is a nonsingular point in a variety X, then the blowup of X at y can intuitively be viewed as replacing the point y with $\mathbb{P}(T_yX) = (T_yX \setminus \{0\})/\Bbbk^* \cong \mathbb{P}^{\dim(X)-1}_\Bbbk$, where T_yX denotes the tangent

[10] $H^0(X, \mathcal{L})$ is the same thing as $\mathcal{L}(X)$.

space to X at y. More general, for Y a nonsingular subvariety of X, we understand the blowup of X along Y as removing Y and, in its place, gluing a copy of $\mathbb{P}(N_{Y/X})$, where $N_{Y/X}$ denotes the normal bundle to Y in X. I.e., each point y of Y is replaced with $(N_{Y/X,y} \setminus \{0\})/\Bbbk^* \cong \mathbb{P}_{\Bbbk}^{\dim(X)-\dim(Y)-1}$. The new locus is called the **exceptional locus**.

Let us write the general definition for the blowup $\mathrm{Bl}_{\mathcal{I}}(X)$ of a Noetherian scheme X with respect to a coherent sheaf of ideals $\mathcal{I}$. We will then go back and check how this relates to the more intuitive description given above. Let $\mathfrak{I} := \bigoplus_{d\geq 0} \mathcal{I}^d$, where $\mathcal{I}^d$ is the dth power of the ideal sheaf $\mathcal{I}$, and $\mathcal{I}^0 = \mathcal{O}_X$. Then

$$\mathrm{Bl}_{\mathcal{I}}(X) := \mathbf{Proj}\,\mathfrak{I}.$$

Note that $\mathrm{Bl}_{\mathcal{I}}(X)$ comes with a projection $\pi : \mathrm{Bl}_{\mathcal{I}}(X) \to X$.

If $\mathcal{I}$ is the ideal sheaf for a closed subscheme Y, we may denote this $\mathrm{Bl}_Y(X)$ and call it the blowup of X along Y or with center Y. In this case, note that $\mathcal{I}|_{X\setminus Y} = \mathcal{O}_X|_{X\setminus Y}$, so $\pi^{-1}(X \setminus Y) = X \setminus Y$.

Now suppose X is a nonsingular variety, and Y is a nonsingular subvariety with ideal sheaf $\mathcal{I}_Y = \mathcal{I}$. To check that $\mathrm{Bl}_Y(X)$ coincides with the geometric description we gave before, we just want to check that the exceptional locus $\widetilde{Y} := \pi^{-1}(Y)$ is isomorphic to $\mathbb{P}(N_{Y/X})$. For this we follow [Har77, Ch. 2, Thm. 8.24].

The conormal sheaf (dual to the normal sheaf) of Y is $\mathcal{I}_Y/\mathcal{I}_Y^2$ [Har77, Def. on pg 182, Ch. 2, §8], so the normal bundle to Y is the associated vector bundle $\mathbf{V}(\mathcal{I}_Y/\mathcal{I}_Y^2)$ as in §2.5.2. We have

$$\widetilde{Y} \cong \mathbf{Proj}(\bigoplus_{d\geq 0}(\mathcal{I}^d \otimes \mathcal{O}_X/\mathcal{I})) = \mathbf{Proj}(\bigoplus_{d\geq 0}\mathcal{I}^d/\mathcal{I}^{d+1}).$$

In this nonsingular setting, one can show that $\mathcal{I}/\mathcal{I}^2$ is indeed locally free, and furthermore, $\mathcal{I}^d/\mathcal{I}^{d+1} \cong \mathrm{Sym}^d(\mathcal{I}/\mathcal{I}^2)$. Thus,

$$\widetilde{Y} \cong \mathbf{Proj}(\mathrm{Sym}(\mathcal{I}/\mathcal{I}^2)) = \mathbb{P}(\mathcal{I}/\mathcal{I}^2) = \mathbb{P}(N_{Y/X})$$

as desired.

Example 2.6.1. Consider $X = \mathbb{A}^2$ and $y = (0,0) \in X$. Then $I = \Gamma(X, \mathcal{I}) = \langle x, y\rangle$. We can identify $\bigoplus_{d\geq 0} I^d$ with the subring of $\Bbbk[x,y,t]$ generated over $\Bbbk[x,y]$ by xt and yt (here, the power of t denotes the grading)—let us write this as $\Bbbk[x,y,xt,yt]$. Then

$$\mathrm{Bl}_y X \cong \mathrm{Proj}\,\Bbbk[x,y,xt,yt]$$

with grading given by the power of t.

Alternatively, let $A = \Bbbk[x,y]$. We can identify $\Bbbk[x,y,xt,yt]$ with $A[\widetilde{x},\widetilde{y}]/\langle x\widetilde{y} - \widetilde{x}y\rangle$ (via $\widetilde{x} = xt$ and $\widetilde{y} = yt$). We then identify $A[\widetilde{x},\widetilde{y}]$ with the ring of homogeneous coordinates on $\mathbb{P}^1_A \cong \mathbb{A}^2 \times \mathbb{P}^1$ (with $\widetilde{x}$ and $\widetilde{y}$ being the homogeneous coordinates on $\mathbb{P}^1$), and thus identify $\mathrm{Bl}_y X$ with the subvariety of $\mathbb{A}^2 \times \mathbb{P}^1$ defined by $x\widetilde{y} = \widetilde{x}y$. See [Har77, Ch. 1, §4, pg 28, and Ch. 2, Example 7.12.1].

2.7 Divisors

Consider a scheme X. We assume X is reasonably nice: Noetherian, integral, separated, and regular in codimension one [Har77, Ch. 2, §6, pg 130]. These properties hold for all toric varieties, so when we extend to the more generally defined Cartier divisors, we will continue to make these assumptions in order to keep things much simpler.

A **prime divisor** on X is a closed integral subscheme $Y \subset X$ of codimension[11] one. A **Weil divisor** D is an element of the free Abelian group $\operatorname{Div} X$ generated by the prime divisors of X; we write $D = \sum n_i Y_i$ where the $n_i \in \mathbb{Z}$ and Y_i are prime divisors, and only finitely many n_i are nonzero. D is called **effective** if each n_i is non-negative.

Let K be the function field of X, and let $K^* = K \setminus \{0\}$. Each prime divisor $Y \subset X$ determines a discrete valuation ν_Y on K which takes $f \in K^*$ to the order of 0 (or negative the order of pole) of f along Y. More precisely, letting η_Y denote the generic point of Y, $\mathcal{O}_{X,\eta_Y}$ is a discrete valuation ring with quotient field K, and the induced discrete valuation on K is ν_Y.

Example 2.7.1. If $Y = 0 \in X = \operatorname{Spec} \Bbbk[x] = \mathbb{A}^1$, then $\nu_Y(\sum a_n x^n) = \min\{n | a_n \neq 0\} \in \mathbb{Z}$.

Now, every $f \in K^*$ determines a divisor

$$(f) = \sum \nu_Y(f) \cdot Y.$$

This sum is finite by [Har77, Ch. 2, Lemma 6.1]. Divisors coming from rational functions in this way are called **principal divisors**. Two divisors D, D' are called **linearly equivalent** if $D - D'$ is a principal divisor. The **divisor class group** of X is the group $\operatorname{Cl} X$ of divisors modulo linear equivalence.

A Weil divisor D of X is called locally principal if X can be covered by open subsets U such that $D|_U$ is[12] principal for each U. A locally principal Weil divisor is called a **Cartier divisor** (see [Har77, Ch. 2, §6, pg. 140-142] for more details—the more general definition there does not require the niceness assumptions on X). If X is regular (i.e., all stalks of $\mathcal{O}_X$ are regular local rings—i.e., X is non-singular, cf. §3.4), or more generally, if X is locally factorial (stalks are UFD's), then every Weil divisor is a Cartier divisor. Principal Cartier divisors are the same as principal Weil divisors. Two Cartier divisors which differ by a principal divisor are called linearly equivalent, and the group of Cartier divisors modulo linear equivalence is denote $\operatorname{CaCl} X$.

Recall that an invertible sheaf is a rank 1 locally free $\mathcal{O}_X$-module. These sheaves are called "invertible" because they form a group under the operation of $\otimes$ with $\mathcal{O}_X$ as the identity [Har77, Ch. 2, Prop. 6.12]. The **Picard group** of X, denoted $\operatorname{Pic} X$, is the group of isomorphism classes of invertible sheaves, under the operation $\otimes$.

Let $D = \sum a_i Y_i$ be a Cartier divisor. Then D determines a sheaf $\mathcal{O}_X(D)$ which, on an open set U, is given by

$$[\mathcal{O}_X(D)](U) = \{f \in K | \nu_{Y_i \cap U}(f) \geq -a_i \text{ for all } i \text{ with } Y_i \cap U \neq \emptyset\}.$$

Since D is locally principal, we can cover X with open sets $\{U\}$ such that $D|_U$ is the principal divisor $(f|_U)$ for some $f \in K$, and then $[\mathcal{O}_X(D)](U) = f \cdot \mathcal{O}_X(U) \subset K$. Thus, $\mathcal{O}_X(D)$ is an invertible sheaf.

[11] By definition, the **codimension** of an irreducible closed subscheme $Y \subset X$ is the supremum of integers n such that there exists a chain $Y = Y_0 \subset Y_1 \ldots \subset Y_n$ of distinct closed irreducible subsets of X, beginning with Y. In nice cases (e.g., X an integral affine scheme of finite type over a field), the codimension of Y in X is the same as $\dim(X) - \dim(Y)$, as we expect (cf. [Har77, Ch. 2, §3, pg 86-87]). We give the precise definition of dim (dimension) in Remark 3.4.1.

[12] For $D = \sum a_i Y_i$, $D|_U$ means the divisor $\sum a_i (Y_i \cap U) \in \operatorname{Div}(U)$, where terms with $Y_i \cap U = \emptyset$ are disregarded (or treated as 0).

One can show that linearly equivalent Cartier divisors induce isomorphic invertible sheaves. We thus obtain a map $\mathrm{CaCl}\, X \to \mathrm{Pic}\, X$ which is in fact a group homomorphism, taking $+$ to $\otimes$. Furthermore, under our standing assumption that X is integral, the induced homomorphism $\mathrm{CaCl}\, X \to \mathrm{Pic}\, X$ is in fact an isomorphism [Har77, Ch. 2, Prop. 6.15].

Example 2.7.2. Recall the invertible sheaves $\mathcal{O}_{\mathbb{P}^r}(n)$ as in Example 2.5.1. If $n \in \mathbb{Z}_{\geq 0}$, then $\mathcal{O}_{\mathbb{P}^r}(n)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^r}(D)$ with $D = Z(f)$ for f a generic degree n homogeneous polynomial in $\Bbbk[x_0, \dots, x_r]$. Alternatively, for any $n \in \mathbb{Z}$, $\mathcal{O}_{\mathbb{P}^r}(n) \cong \mathcal{O}_{\mathbb{P}^r}(nH)$ for H the class of a hyperplane in $\mathbb{P}^r$.

2.8 Cohomology

2.8.1 Sheaf cohomology

Let $\mathfrak{A}$ be an Abelian category, i.e., a category such that each $\mathrm{Hom}(A, B)$ for $A, B \in \mathfrak{A}$ has the structure of an Abelian group; the composition law is linear; kernels, cokernels, and finite direct sums exist; and a handful of other desirable properties are satisfied. The standard example is the category $R - \mathfrak{Mod}$ of left (or right) modules over a ring R. In fact, every (small) Abelian category can be identified with a full subcategory of some such module category $R - \mathfrak{Mod}$ (cf. Mitchell's embedding theorem). The main examples we will care about is $\mathfrak{Mod}(X)$, the category of sheaves of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$. Similarly, $\mathfrak{Ab}(X)$ is the category of sheaves of Abelian groups on a topological space X (and $\mathfrak{Ab}$ is the category of Abelian groups).

Let I be an object of $\mathfrak{A}$. In general, the contravariant functor $\mathrm{Hom}(\cdot, I) : \mathfrak{A} \to \mathfrak{Ab}$ is left-exact, meaning that for any short exact sequence

$$0 \to A' \to A \to A'' \to 0,$$

the sequence

$$0 \to \mathrm{Hom}(A'', I) \to \mathrm{Hom}(A, I) \to \mathrm{Hom}(A', I)$$

is exact. If $\mathrm{Hom}(\cdot, I)$ is also right-exact (i.e., the last map above is surjective), then I is called **injective**.

An **injective resolution** of an object $A \in \mathfrak{A}$ is a complex $I^\bullet$, defined in degrees $i \geq 0$, together with a morphism $\epsilon A \to I^0$, such that I^i is injective in $\mathfrak{A}$ for each $i \geq 0$, and such that the sequence

$$0 \to A \xrightarrow{\epsilon} I^0 \to I^1 \to \dots$$

is exact. If every object of $\mathfrak{A}$ admits an injective resolution (equivalently, every object is isomorphic to a subobject of an injective object), then $\mathfrak{A}$ is said to have **enough injectives**.

One can show that $R - \mathfrak{Mod}$, $\mathfrak{Mod}(X)$, and $\mathfrak{Ab}(X)$ all have enough injectives [Har77, Ch. III, Prop. 2.1A, Prop, 2.2, cor. 2.3].

Let $\mathfrak{A}$ be an Abelian category with enough injectives, and let $F : \mathfrak{A} \to \mathfrak{B}$ be a covariant left exact functor to another Abelian category $\mathfrak{B}$. Given $A \in \mathfrak{A}$, let $I^\bullet$ be an injective resolution, and let $F(I^\bullet)$ be the complex

$$\dots \xrightarrow{d_{-2}} 0 \xrightarrow{d_{-1}} F(I^0) \xrightarrow{d_0} F(I^1) \xrightarrow{d_1} F(I^2) \xrightarrow{d_2} \dots.$$

Then the right derived functors R^iF, $i \geq 0$, of F are defined to be the i-th cohomology objects of the complex $F(I^\bullet)$. That is,

$$R^iF(A) := h^i(F(I^\bullet)) := \ker d^i / \operatorname{im} d^{i-1}$$

Example 2.8.1. In general, there is a natural isomorphism $R^0F \cong F$. Indeed, given an injective resolution $I^\bullet$ of an object A,

$$R^0F(A) = h^0(F(I^\bullet) = \ker(d^0)/\operatorname{im}(d^{-1}) = \ker(d^0).$$

By assumption, $0 \to A \to I^0 \to I^1$ is exact, and F is left exact, so $0 \to F(A) \to F(I^0) \xrightarrow{d^0} F(I^2)$ is exact. Hence, $F(A) \cong \ker(d^0) = R^0F(A)$.

Definition 2.8.2. For X any topological space, the i-th sheaf cohomology functors $H^i(X, \cdot) : \mathfrak{Ab}(X) \to \mathfrak{Ab}$ are defined by

$$H^i(X, \cdot) := R^i\Gamma(X, \cdot)$$

where $\Gamma(X, \cdot)$ is the global sections functor $\mathfrak{Ab}(X) \to \mathfrak{Ab}$. Similarly if we replace $\mathfrak{Ab}(X)$ with related categories like $\mathfrak{Mod}(X)$.

Example 2.8.3. By Example 2.8.1, $H^0(X, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$.

2.8.2 Čech Cohomology

Čech cohomology is often easier to compute or understand than sheaf cohomology, but the two turn out to often be equivalent (see below). We briefly review the definition of Čech cohomology here.

Let X be a topological space, and let $\mathfrak{U} = (U_i)_{i\in I}$ be an open cover for X, and denote $U_{i_0,\ldots,i_p} := U_{i_0} \cap \cdots \cap U_{i_p}$. Fix a well-ordering on the index set I. For any $\mathcal{F} \in \mathfrak{Ab}(X)$, one defines a complex of Abelian groups $C^\bullet(\mathfrak{U}, \mathcal{F})$ as follows:

$$C^p(\mathfrak{U}, \mathcal{F}) = \prod_{i_0<\ldots<i_p} \mathcal{F}(U_{i_0,\ldots,i_p}).$$

Given an element $\alpha \in C^p(\mathfrak{U}, \mathcal{F})$, i.e., an element $\alpha_{i_0,\ldots,i_p} \in \mathcal{F}(U_{i_0,\ldots,i_p})$ for each $(p+1)$-tuple $i_0 < \ldots < i_p$, the coboundary map $d : C^p \to C^{p+1}$ is determined by

$$(d\alpha)_{i_0,\ldots,i_{p+1}} := \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\ldots,\hat{i}_k,\ldots,i_{p+1}}|_{U_{i_0,\ldots,i_{p+1}}} \tag{2.6}$$

where $\hat{i}_k$ indicates that we skip i_k. One can check that $d^2 = 0$, so this yields a complex of Abelian groups.

Definition 2.8.4. The p-th Čech cohomology group of $\mathcal{F}$ on X, with respect to the open cover $\mathfrak{U}$, is

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) := h^p(C^\bullet(\mathfrak{U}, \mathcal{F})).$$

Example 2.8.5. An element of $C^0(\mathfrak{U}, \mathcal{F})$ is a choice of section $\alpha_i \in \mathcal{F}(U_i)$ for each i. Such a choice $(\alpha_i)_{i\in I}$ lies in the kernel of d if and only if $(\alpha_{i_0} - \alpha_{i_1})|_{U_{i_0,i_1}} = 0$ for all $i_0, i_1 \in I$, i.e., iff these sections α_i glue to form a global section $\Gamma(X, \mathcal{F})$. Thus, $\check{H}^0(\mathfrak{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

Theorem 2.8.6 ([Har77], Ch. III, Thm. 4.5)**.** *Suppose X is a Noetherian separated scheme, $\mathfrak{U}$ is an open affine cover of X (i.e., each $U_i \in \mathfrak{U}$ is affine), and $\mathcal{F}$ is quasi-coherent. Then there are natural isomorphisms*

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) \xrightarrow{\sim} H^p(X, \mathcal{F})$$

for all $p \geq 0$.

In general, for X any topological space and $\mathcal{F}$ a sheaf on X, one defines

$$\check{H}^p(X, \mathcal{F}) = \varinjlim_{U} \check{H}^p(\mathfrak{U}, \mathcal{F})$$

where on the right-hand side we take the direct limit over all open covers, viewed as a directed system via refinements (i.e., taking unions of covers).

For any Abelian group A (e.g., $A = \mathbb{Q}$), define $\check{H}^p(X, A) := \check{H}^p(X, \mathcal{F}_A)$ for $\mathcal{F}_A$ the constant sheaf on X determined by A (i.e., the sheaf whose stalks are all equal to A).

Theorem 2.8.7 (See Wikipedia or Math Overflow)**.** *If X is sufficiently nice (e.g., homotopy equivlaent to a CW-complex), then*

$$\check{H}^p(X, A) \cong H^p(X, A)$$

where the right-hand side indicates the singular cohomology of X with coefficients in A.

Chapter 3

Basic definitions and construction from fans and polytopes

My main reference is [Ful93]. Chapter 7 of [HKK+03] (available at https://www.claymath.org/library/monographs/cmim01c.pdf) also has a decent introduction from a different, more physics oriented point of view. Another good book (with which I am not very familiar) that is more recent and covers more is [CJS11].

3.1 Lattices, cones, and affine toric varieties

3.1.1 Definition and first examples

Definition 3.1.1. A **toric variety** is a (normal) algebraic variety X containing an algebraic torus $T \cong (\Bbbk^*)^r$ as a dense open subset (called the **big torus orbit**) such that the torus action extends to all of X. We'll always assume that X is normal (these are the cases which can be constructed from fans). The complement of the big torus orbit is called the **(toric) boundary**.

Examples 3.1.2.

1. Affine space $\mathbb{A}^r_\Bbbk = \Bbbk^r$ is a toric variety with the algebraic torus $(\Bbbk^*)^r$ acting in the natural way. We recall that the coordinate ring for $\mathbb{A}^r_\Bbbk$ is $\Bbbk[X_1, \ldots, X_r]$.

2. The algebraic torus $(\Bbbk^*)^r$ is itself a toric variety. Note that $(\Bbbk^*)^r$ could be obtained from $\mathbb{A}^r_\Bbbk$ by taking the complement of the coordinate hyperplanes $X_i = 0$, $i = 1, \ldots r$. This changes the coordinate ring by adjoining the inverses to each X_i. Thus, the coordinate ring for $(\Bbbk^*)^r$ is $\Bbbk[X_1^{\pm 1}, \ldots, X_r^{\pm 1}]$.

3. Projective space $\mathbb{P}^r_\Bbbk = \operatorname{Proj} \Bbbk[x_0, x_1, \ldots, x_r]$ is a toric variety with big torus orbit $(\Bbbk^*)^r = \operatorname{Spec} \Bbbk[X_1^{\pm 1}, \ldots, X_r^{\pm 1}]$ for $X_i = \frac{x_i}{x_0}$. I.e., if we think of $\mathbb{P}^r_\Bbbk$ as $\Bbbk^r$ plus a locus at infinity, then the algebraic torus is the natural one.

3.1.2 Lattices and the algebraic torus

Let $N \cong \mathbb{Z}^r$ be a finite rank lattice (called the **cocharacter lattice**) and $M := N^* := \mathrm{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^r$ the dual lattice (called the **character lattice**). We denote the dual paring by $\langle \cdot, \cdot \rangle : N \oplus M \to \mathbb{Z}$. For any lattice L and abelian group A (usually a field with its addition operation), let $L_A := L \otimes A$. In particular,

$$N_{\mathbb{R}} = N \otimes \mathbb{R}, \qquad M_{\mathbb{R}} = M \otimes \mathbb{R}.$$

We will also write $L_{\Bbbk^*}$ as T_L. I.e., T_N is the algebraic torus

$$T_N := N \otimes \Bbbk^* = \mathrm{Hom}(M, \Bbbk^*) \cong (\Bbbk^*)^r.$$

3.1.3 Convex polyhedral cones

A **convex polyhedral cone** in $N_{\mathbb{R}}$ is a set of the form

$$\sigma = \{a_1 n_1 + \ldots + a_r n_r | a_i \in \mathbb{R}_{\geq 0}\}$$

generated by a finite set of vectors $n_1, \ldots, n_s \in N_{\mathbb{R}}$. If each n_i is in N, we say σ is **rational**. The **dimension** of σ is dimension of the vector space $\mathbb{R} \cdot \sigma = \sigma + (-\sigma)$ spanned by σ.

Given a set $\sigma \subset N_{\mathbb{R}}$, the dual set $\sigma^\vee \subset M_{\mathbb{R}}$ is defined by

$$\sigma^\vee := \{m \in M_{\mathbb{R}} | \langle n, m \rangle \geq 0 \text{ for all } n \in \sigma\}.$$

A face of σ is a set of the form

$$\tau := \sigma \cap m^\perp = \{n \in \sigma | \langle n, m \rangle = 0\}$$

for some $m \in \sigma^\vee$. A codimension 1 face of σ is called a **facet**. We may write $\tau < \sigma$ or $\sigma > \tau$ to indicate that τ is a face of σ.

If σ is strongly convex (i.e., does not contain a line through the origin) and $n_1, \ldots, n_r$ is a minimal set of generators for σ, then the faces of σ are the cones generated by subsets of $n_1, \ldots, n_r$.

Examples 3.1.3.

1. If $m = 0 \in \sigma^\vee$, then $\sigma \cap m^\perp = \sigma$.

2. If m is in the relative interior of $\sigma^\vee$, then $\sigma \cap m^\perp$ is the minimal face of σ — this minimal face is $\{0\}$ iff σ is strongly convex.

3. More generally, if m is in the relative interior of face of $\sigma^\vee$ with codimension k in $M_{\mathbb{R}}$, then $\dim(\sigma \cap m^\perp)) = k$. We thus obtain a one-to-one order-reversing correspondence between faces of σ and faces of $\sigma^\vee$.

We list here some basic properties of convex polyhedral cones. See [Ful93, §1.2] for more details and additional properties (some of which I might add later).

1. If σ is a convex polyhedral cone, then $\sigma^\vee$ is a convex polyhedral cone, and $(\sigma^\vee)^\vee = \sigma$.

2. Any face is also a convex polyhedral cone.

3. Any intersection of faces is a face.

4. A face of a face is a face.

5. Any proper face is the intersection of the facets which contain it.

6. The relative boundary of a cone is the union of its proper faces.

7. If $m_1, \dots, m_s$ generate $\sigma^\vee$, then

$$\sigma = \{n \in N_\mathbb{R} | \langle n, m_i \rangle \geq 0, i = 1, \dots, s\}.$$

   Thus, a convex polyhedral cone can be equivalently defined as an intersection of closed half-spaces.

8. If $m \in \sigma^\vee$ and $\tau = \sigma \cap m^\perp$, then $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0} \cdot (-m)$.

Example 3.1.4. As an example of item (8) above, suppose $\sigma = \mathbb{R}_{\geq 0}\langle e_1, e_2 \rangle$, $\rho = \mathbb{R}_{\geq 0} e_2$, and $\tau = \{0\}$. Then $\rho = \sigma \cap (e_1^*)^\perp$ and $e_1^* \in \sigma^\vee$, and

$$\rho^\perp = \sigma + \mathbb{R}_{\geq 0}(-e_1^*) = \mathbb{R}e_1^* + \mathbb{R}_{\geq 0} e_2^*.$$

Similarly, $\tau = \sigma \cap m^\perp$ for any m in the interior of $\sigma^\vee$, and then

$$\tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0}(-m) = M_\mathbb{R}.$$

We may also need the following property of rational polyhedral cones.

Lemma 3.1.5 (Gordon's Lemma)**.**
If σ is a convex rational polyhedral cone, then $\sigma^\vee$ is also rational, and $S_\sigma := \sigma^\vee \cap M$ is a finitely generated monoid.

Lemma/Definition 3.1.6. *Let σ be a convex polyhedral cone. The following are equivalent:*

- $\sigma \cap (-\sigma) = \{0\}$*;*
- *σ contains no nonzero linear subspace;*
- *σ contains no line through the origin;*
- *there is an $m \in \sigma^\vee$ with $\sigma \cap m^\perp = \{0\}$;*
- *$\sigma^\vee$ spans $M_\mathbb{R}$.*

*A convex polyhedral cone σ satisfying these conditions is said to be **strongly convex**.*

For convenience, we may refer to strongly convex rational polyhedral cones as simply **toric** cones.

Some additional useful terminology regarding lattices: the **index** of an element $n \in N$, denoted $|n|$, is the largest positive integer such that $n = |n| \cdot n'$ for some $n' \in N$. An element of index 1 is called **primitive**.

3.1.4 Affine toric varieties from cones

For any monoid P and commutative ring R, define

$$R[P] := R[z^u | u \in P]/\langle z^u \cdot z^v = z^{u+v} \rangle,$$

where the addition in the exponent is the monoid operation.

Example 3.1.7. In particular, consider $\Bbbk[M]$. Choose a basis $\{e_1, \ldots, e_r\}$ for N with dual basis $\{e_1^*, \ldots, e_r^*\}$ for M, and let $X_i := z^{e_i^*} \in \Bbbk[M]$. Then

$$\Bbbk[M] = \Bbbk[X_1^{\pm 1}, \ldots, X_r^{\pm 1}].$$

Thus,

$$T_N = \operatorname{Spec} \Bbbk[M].$$

More generally, a toric cone σ in $N_{\mathbb{R}}$ determines an affine toric variety U_σ via

$$U_\sigma := \mathrm{TV}(\sigma) := \operatorname{Spec} \Bbbk[S_\sigma]$$

where we recall that $S_\sigma := \sigma^\vee \cap M$.

Alternatively, note that we may identify the geometric points of U_σ as the space of nonzero[1] semigroup[2] homomorphisms $\mathrm{Hom}_{\mathrm{sg}}(S_\sigma, \Bbbk) \setminus \{0\}$. Here, $\Bbbk$ is viewed a semigroup under multiplication, and a semigroup morphism taking $m \mapsto a$ corresponds to a $\Bbbk$-algebra homomorphism $\Bbbk[S_\sigma] \to \Bbbk$ taking $z^m \mapsto a$.

Remark 3.1.8 (The torus action). From the $\mathrm{Hom}_{\mathrm{sg}}$ perspective, the inclusion of the big torus orbit $T_N \subset U_\sigma$ corresponds to the inclusion $\mathrm{Hom}(M, \Bbbk^*) \subset \mathrm{Hom}_{\mathrm{sg}}(S_\sigma, \Bbbk) \setminus \{0\}$. This inclusion is defined by taking morphisms on M and restricting them to morphisms on S_σ, and the injectivity of this restriction morphism follows from the fact that $\sigma^\vee$ spans $M_{\mathbb{R}}$ (because σ is strongly convex).

The fact that the self-action of T_N extends to all of U_σ also follows from this semigroup-morphism perspective. Given $\lambda \in T_N = \mathrm{Hom}(M, \Bbbk^*)$ and $x \in \mathrm{Hom}_{\mathrm{sg}}(S_\sigma, \Bbbk) \setminus \{0\}$, the element $\lambda \cdot x \in \mathrm{Hom}_{\mathrm{sg}}(S_\sigma, \Bbbk) \setminus \{0\}$ is given by $m \mapsto (\lambda \cdot x)(m) := \lambda(m) \cdot x(m)$ for each $m \in S_\sigma$.

Example 3.1.9. Let $N = \mathbb{Z}^2$, $\sigma = \mathbb{R}_{\geq 0}\langle e_1, e_2 \rangle$, so $\sigma^\vee = \mathbb{R}_{\geq 0}\langle e_1^*, e_2^* \rangle$. Then $\Bbbk[\sigma^\vee \cap M] = \Bbbk[x, y]$ where $x := z^{e_1^*}$, $y := z^{e_2^*}$, and so $\operatorname{Spec} \Bbbk[S_\sigma] = \mathbb{A}^2$.

More generally, if $N = \mathbb{Z}^r$ and $\sigma = \mathbb{R}_{\geq 0}\langle e_1, \ldots, e_r \rangle$, then $\Bbbk[S_\sigma] = \Bbbk[x_1, \ldots, x_r]$, and so

$$\operatorname{Spec} \Bbbk[S_\sigma] = \mathbb{A}^r.$$

Example 3.1.10. Let $\rho = \mathbb{R}_{\geq 0}\langle e_2, \ldots, e_r \rangle$. Then

$$\rho^\vee = \mathbb{R}_{\geq 0}\langle \pm e_1^*, e_2^*, \ldots, e_r^* \rangle.$$

[1] [Ful93] doesn't appear to mention the requirement that the semigroup homomorphisms be nonzero, but I think this is important. Otherwise, we could have $m \mapsto 0$ for all $m \in S_\sigma$, and then the corresponding morphism $\Bbbk[S_\sigma] \to \Bbbk$ would be the 0-map, which is a ring homomorphism but not a $\Bbbk$-algebra homomorphism. This 0-morphism would correspond to the unit ideal of $\Bbbk[S_\sigma]$, and this is not a prime ideal, hence not an element of $\operatorname{Spec} \Bbbk[S_\sigma]$.

[2] Recall that a semigroup is a set with an associative binary operation i.e., a monoid without identity, or a group without identity or inverses.

So $\Bbbk[S_\rho] = \Bbbk[x_1^{\pm 1}, x_2, \ldots, x_r]$, and

$$\operatorname{Spec} \Bbbk[S_\rho] = \mathbb{A}^r \setminus \left(\bigcap_{i=2}^{r} \{x_1 = 0\} \right).$$

Example 3.1.11. Let $\rho = \mathbb{R}_{\geq 0}\langle e_1 \rangle$. Then

$$\rho^\vee = \mathbb{R}_{\geq 0}\langle e_1^*, \pm e_2^*, \ldots, \pm e_r^* \rangle.$$

So $\Bbbk[S_\rho] = \Bbbk[x_1, x_2^{\pm 1}, \ldots, x_r^{\pm 1}]$, and

$$\operatorname{Spec} \Bbbk[S_\rho] = \mathbb{A}^r \setminus \left(\bigcap_{i=2}^{r} \{x_i = 0\} \right).$$

Example 3.1.12. Let $\tau = \{0\}$. Then $\tau^\vee = M_\mathbb{R}$, so $\Bbbk[S_\tau] = \Bbbk[M]$, and $\operatorname{Spec} \Bbbk[S_\tau] = T_N$.

Note that the above examples may all be summarized and generalized as follows: if $\sigma = \mathbb{R}\langle e_1, \ldots, e_k \rangle$ for $e_1, \ldots, e_k$ part of a basis for $N \cong \mathbb{Z}^r$, then $\operatorname{TV}(\sigma) = \Bbbk^k \times (\Bbbk^*)^{r-k}$.

Remark 3.1.13. If $\tau \subset \sigma$, then $\sigma^\vee \subset \tau^\vee$, so there is an injection $\Bbbk[S_\sigma] \hookrightarrow \Bbbk[S_\tau]$, hence a morphism

$$\operatorname{TV}(\tau) \to \operatorname{TV}(\sigma).$$

In particular, suppose τ is a face of σ. Then there is some $m \in \sigma^\vee \cap M$ such that $\tau = \sigma \cap m^\perp$, and by Property 8 above, $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0}(-m)$. Hence,

$$\begin{aligned} \Bbbk[S_\tau] &= \Bbbk[S_\sigma][z^{-m}] \\ \operatorname{TV}(\tau) &= \operatorname{TV}(\sigma) \setminus Z(z^m). \end{aligned}$$

This is illustrated in the examples above.

In particular, note that every cone σ contains $\{0\}$ as a subcone. Thus, every affine toric variety $\operatorname{TV}(\sigma)$ contains the algebraic torus $\operatorname{TV}(\{0\}) = T_N$ as an open subvariety — indeed, $T_N = \operatorname{TV}(\sigma) \setminus Z(z^m)$ for m in the interior of $\sigma^\vee$.

3.2 Constructing an atlas for a toric variety from a fan

Definition 3.2.1. A fan Σ is a set[3] of toric cones in $N_\mathbb{R}$ such that each face of a cone in Σ is also a cone in Σ, and the intersection of any two cones is a face in each.

If $\rho \subset \sigma$, then $\sigma^\vee \subset \rho^\vee$, and $\operatorname{Spec} \Bbbk[\rho^\vee] \hookrightarrow \operatorname{Spec} \Bbbk[\sigma^\vee]$. Thus, if $\rho = \sigma_1 \cap \sigma_2$, then we can glue $\operatorname{Spec} \Bbbk[\sigma_1^\vee]$ to $\operatorname{Spec} \Bbbk[\sigma_2^\vee]$ along $\operatorname{Spec} \Bbbk[\rho^\vee]$. Performing this gluing for all intersecting pairs of cones in Σ yields a (separated) algebraic variety $\operatorname{TV}(\Sigma)$. Normal toric varieties are precisely the varieties which can be constructed from fans in this way.

Example 3.2.2. Every fan Σ contains the origin, and $\{0\}^\vee = M$, so every toric variety $\operatorname{TV}(\Sigma)$ contains the algebraic torus $\operatorname{Spec} \Bbbk[M] \cong T_N$. This is the torus T required in Definition 3.1.1.

[3] We will typically implicitly assume that all fans are finite—without this, the associated toric varieties are not of "finite type," hence are not technically considered to even be varieties. Still, much of what we will say applies to infinite fans as well, and these are occasionally of interest (e.g., the Tate curve can be understood in terms of a toric variety from an infinite fan).

Example 3.2.3. Let $N = \mathbb{Z}^r$, and let Σ be the fan consisting of the $r+1$ rays generated by $e_1, \dots, e_r$ and $-(\sum_{i=1}^r e_i)$, along with $\{0\}$ and the cones that these rays bound. Then $\mathrm{TV}(\Sigma) = \mathbb{P}^r$.

Let's check this for $r = 1$. In this case, Σ has two maximal cones, $\sigma_+ := \mathbb{R}_{\geq 0} e_1$ and $\sigma_- := \mathbb{R}_{\geq 0}(-e_1)$, plus the cone $\{0\}$. Then $\sigma_\pm^\vee = \mathbb{R}_{\geq 0}(\pm e_1)^*$, and $S_{\sigma_\pm} = \mathbb{Z}_{\geq 0}(\pm e_i)$. Thus, denoting $x := z^{e_1}$, $x^{-1} := z^{-e_1}$, we have

$$\Bbbk[S_{\sigma_+}] = \Bbbk[x], \qquad \Bbbk[S_{\sigma_-}] = \Bbbk[y]$$

and

$$\Bbbk[S_{\{0\}}] = \Bbbk[x,y]/\langle xy = 1\rangle = \Bbbk[x^{\pm 1}],$$

i.e., y is identified with x^{-1}. Thus, we have two copies of $\mathbb{A}^1_\Bbbk = \Bbbk$, glued along $\Bbbk^* = \mathbb{A}^1_\Bbbk \setminus \{0\}$ via the map taking a point a in one copy of $\mathbb{A}^1$ to the point a^{-1} in the other copy of $\mathbb{A}^1$. This indeed yields $\mathbb{P}^1_\Bbbk$.

As an exercise, you should check the case for $\mathbb{P}^2$. Hint: the charts associated to the three cones correspond to the three charts U_0, U_1, U_2 described in Example 2.4.7.

3.3 Cones correspond to torus orbits

Let σ be a toric cone in $N_\mathbb{R}$ and let τ be a face of σ. We would like to understand $U_\sigma \setminus U_\tau$. Recall from the definition of a face that there exist points $m_\tau \in \sigma^\vee$ such that $\tau = \sigma \cap m_\tau^\perp$, and then by Property 8, $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0}(-m_\tau)$. We may assume $m_\tau \in M$, so then

$$\Bbbk[S_\tau] = \Bbbk[S_\sigma]_{z^{m_\tau}}$$

hence

$$U_\tau = U_\sigma \setminus Z(z^{m_\tau}).$$

Note that m_σ can be any point in $\sigma^\perp$, and every other point in $\sigma^\vee \setminus \sigma^\perp$ is of the form $m_\sigma + m_\tau$ for some proper face τ of σ and some choices of $m_\sigma \in \sigma^\perp$ and m_τ as above. Thus,

$$\begin{aligned} O_\sigma :&= U_\sigma \setminus \bigcup_{\tau \subsetneq \sigma} U_\tau \\ &= U_\sigma \setminus \bigcup_{m \in \sigma^\vee \setminus \sigma^\perp} (U_\sigma \setminus Z(z^m)) \\ &= \bigcap_{m \in \sigma^\vee \setminus \sigma^\perp} Z(z^m) \\ &\cong \operatorname{Spec} \Bbbk[\sigma^\vee \cap M]/\langle z^m | m \in (\sigma^\vee \setminus \sigma^\perp) \cap M\rangle \\ &\cong \operatorname{Spec} \Bbbk[\sigma^\perp \cap M] \\ &\cong T_{\mathrm{Hom}(\sigma^\perp \cap M, \mathbb{Z})}. \end{aligned}$$

Given a fan Σ, these loci O_σ for $\sigma \in \Sigma$ are precisely the orbits for the action of the torus T_N on $\mathrm{TV}(\Sigma)$. To see this, first recall that we have understood the T_N action in terms of maps of semigroups; cf. Remark 3.1.8. From this semigroup perspective, when τ is a face of σ, we have

$$U_\sigma \setminus U_\tau = \mathrm{Hom}_{\mathrm{sg}}(\sigma^\vee \cap M, \Bbbk) \setminus \mathrm{Hom}_{\mathrm{sg}}(\tau^\vee \cap M, \Bbbk).$$

As before, we have $\tau = \sigma \cap m_\tau^\perp$ for various $m_\tau \in \sigma^\vee$, and then $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0}(-m_\tau)$. So we want to know which $p \in \mathrm{Hom}_{\mathrm{sg}}(\sigma^\vee \cap M, \Bbbk) \setminus \{0\}$ can *not* be extended to $-m_\tau$. The assumption that $p \neq 0$ implies that $p(0) = 1$, because $p(m) = p(0+m) = p(0) * p(m)$ for all m, so $p(m) \neq 0$ for some m implies $p(0) = 1$. But then we must have $p(-m_\tau) = \frac{1}{(}p(m_\tau))$, so p admits a (necessarily unique) extension to $-m_\tau$ if and only if $p(m_\tau) \neq 0$. Thus, $U_\sigma \setminus U_\tau$ consists of those p for which $p(m_\tau) = 0$. As before, we note that every element of $\sigma^\vee \setminus \sigma^\perp$ is of the form $m_\sigma + m_\tau$ for some $m_\sigma \in \sigma^\perp$ and some m_τ associated to some $\tau \subsetneq \sigma$, and then $p(m_\tau) = 0$ implies $p(m_\sigma + m_\tau) = p(m_\sigma)p(m_\tau) = 0$. So we have

$$\begin{aligned}
O_\sigma &= \mathrm{Hom}_{\mathrm{sg}}(S_\sigma, \Bbbk) \setminus \bigcup_{\tau \subsetneq \sigma} \mathrm{Hom}_{\mathrm{sg}}(S_\tau, \Bbbk) \\
&= \{p \in \mathrm{Hom}_{\mathrm{sg}}(S_\sigma, \Bbbk) \setminus \{0\} \text{ such that } p|_{\sigma^\vee \setminus \sigma^\perp} = 0\}. \qquad (3.1) \\
&\cong \mathrm{Hom}_{\mathrm{sg}}(\sigma^\perp \cap M, \Bbbk) \setminus \{0\} \\
&\cong \mathrm{Hom}(\sigma^\perp \cap M, \Bbbk^*)
\end{aligned}$$

By interpreting the torus action as in Remark 3.1.8, we see from (3.1) that O_σ is indeed an orbit for the T_N-action. Indeed. multiplication will clearly not change the property of $p|_{\sigma^\vee \setminus \sigma^\perp}$ equaling 0, so O_σ is closed. On the other hand, the action on $\mathrm{Hom}(\sigma^\perp \cap M, \Bbbk^*)$ is clearly transitive.

We will often be interested in the orbit closures $\overline{O_\sigma}$. Note that the correspondence between cones σ and orbit closures $\overline{O_\sigma}$ is inclusion-reversing. In particular,

$$\dim(\overline{O_\sigma}) = \dim(O_\sigma) = \mathrm{rank}(N) - \dim(\sigma).$$

The orbit closures $\overline{O_\sigma}$ are called the **toric strata** of $\mathrm{TV}(\Sigma)$. In particular, if $\rho \in \Sigma^{[1]}$ (the set of rays of Σ), then $D_\rho := \overline{O_\rho}$ is a divisor, called a **boundary divisor** of $\mathrm{TV}(\Sigma)$. The union $D := \bigcup_{\rho \in \Sigma^{[1]}} D_\rho$ is precisely the toric boundary $\mathrm{TV}(\Sigma) \setminus T_N$.

Example 3.3.1. As in Example 3.1.9, let $N = \mathbb{Z}^2$, and let $\sigma = \mathbb{R}_{\geq 0}\langle e_1, e_2 \rangle \subset N_\mathbb{R}$. For $i = 1, 2$, let ρ_i be the ray $\mathbb{R}_{\geq 0} e_i$. Then $U_\sigma = \mathbb{A}^2_\Bbbk$. The torus orbits are

$$\begin{aligned}
O_{(0,0)} &= T_N = (\Bbbk^*)^2 \\
O_{\rho_1} &= \{(0, y) | y \in \Bbbk^*\} \\
O_{\rho_2} &= \{(x, 0) | x \in \Bbbk^*\} \\
O_\sigma &= (0, 0).
\end{aligned}$$

In particular, the orbit closures $\overline{O_{\rho_1}}$ and $\overline{O_{\rho_2}}$ are the y- and x-axes, respectively.

Example 3.3.2. Consider the fan for $\mathbb{P}^2_\Bbbk$ as in Example 3.2.3. Label the rays $\rho_1 = \mathbb{R}_{\geq 0} e_1$, $\rho_2 = \mathbb{R}_{\geq 0} e_2$, and $\rho_3 = \mathbb{R}_{\geq 0}(-e_1 - e_2)$. Then D_{ρ_1} is the y-axis, D_{ρ_2} is the x-axis, and D_{ρ_3} is the axis at infinity (the "z-axis" in the homogeneous coordinates). If σ_{ij} denotes the 2-dimensional cone bounded by ρ_i and ρ_j, then $O_{\sigma_{ij}}$ is the point $D_{\rho_i} \cap D_{\rho_j}$.

3.3.1 Divisorial valuations

Let $\rho \in \sigma^{[1]}$ be a ray in Σ generated by a primitive vector $n \in N$. The associated Weil divisor $D_\rho := \overline{O_\rho}$ induces a discrete valuation $\mathrm{val}_n := \mathrm{val}_{D_\rho} := \nu_{D_\rho}$ on the function field $\Bbbk(M)$ of $\mathrm{TV}(\Sigma)$ as in §2.7. Elements of $\Bbbk(M)$ can be expressed as fractions $\frac{f}{g}$ for $f, g \in \Bbbk[M]$, $g \neq 0$, and $\mathrm{val}_n(\frac{f}{g}) = \mathrm{val}_n(f) - \mathrm{val}_n(g)$. We claim then that $\mathrm{val}_n(f)$ for $f \in \Bbbk[M]$ is given by

$$\mathrm{val}_n(\sum a_m z^m) = \min_{a_m \neq 0} \langle m, n \rangle. \tag{3.2}$$

Let us first consider an example.

Example 3.3.3. Let $N = \mathbb{Z}^r$ and $\rho = \mathbb{R}_{\geq 0} e_1$. Then $U_\rho = \mathrm{Spec}\, A_\rho$ for

$$\begin{aligned} A_\rho &= \Bbbk[z^m | \langle m, e_1 \rangle \geq 0] \subset \Bbbk[M] \\ &= \Bbbk[x_1, x_2^{\pm 1}, x_3^{\pm 1}, \ldots, x_r^{\pm 1}], \end{aligned}$$

and $D_\rho = Z(x_1)$. Then, by interpreting $\mathrm{val}_{e_1}(f)$ as the order of zero of f (or negative the order of pole of f) along D_ρ, we see that

$$\mathrm{val}_{e_1}(x_1^{k_1} \cdots x_r^{k_r}) = k_1,$$

and more generally,

$$\mathrm{val}_{e_1}(\sum a_{k_1,\ldots,k_r} x_1^{k_1} \cdots x_r^{k_r}) = \min_{a_{k_1,\ldots,k_r} \neq 0} k_1.$$

I.e., the valuation is indeed given as in (3.2) for $n = e_1$.

Since we can always choose a basis for N so that $\rho = \mathbb{R}_{\geq 0} e_1$, Example 3.3.3 actually implies (3.2) in general. We give an alternate derivation though that more closely follows the definition of ν_{D_n} as in §2.7, just for the sake of getting some practice with that definition.

In general, for ρ generated by primitive $n \in N$, $U_\rho = \mathrm{Spec}\, A_\rho$ for $A_\rho := \Bbbk[z^m | \langle m, n \rangle \geq 0]$, and the ideal associated to $D_\rho \subset U_\rho$ is $I_\rho := \langle z^m | \langle m, n \rangle \geq 1 \rangle$. For $X = \mathrm{TV}(\Sigma)$, the local ring $\mathcal{O}_{X, D_\rho}$ is $(A_\rho)_{I_\rho}$, i.e., the ring whose elements have the form $\frac{f}{g}$ for $f \in A_\rho$ and $g \in A_\rho \setminus I_\rho$. This is a discrete valuation ring with unique prime ideal $I'_\rho := (A_\rho)_{I_\rho} \cdot I_\rho$. Then for $f \in (A_\rho)_{I_\rho}$, $\mathrm{val}_n(f)$ is the maximal power of I'_ρ which contains f. For $f \in A_\rho \subset (A_\rho)_{I_\rho}$, this is given as in (3.2). Noting that every element of $\Bbbk[M]$ can be expressed as f/z^m for some $m \in M$ with $\langle m, n \rangle \geq 1$, we can extend (3.2) to all of $\Bbbk[M]$, as desired.

3.4 Singularities

Remark 3.4.1. Before defining singular/nonsingular points, I should introduce the notion of "dimension." Following [Har77, Ch I, §1, pg 5-7, or pg 86 for schemes], if Y is a topological space, then the **dimension** $\dim Y$ is defined to be the supremum of all integers n such that there exists a chain of distinct irreducible closed subsets $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ of Y. Algebraically, in a ring R, the **height** of a prime ideal $\mathfrak{p}$ is the supremum of all integers n such that there exists a chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n = \mathfrak{p}$. Then the (Krull) dimension of R is the supremum of the heights of all prime ideals. The dimension of an affine variety Y is then equal to the Krull dimension of its coordinate ring $A(Y)$ [Har77, Prop. 1.7]. In particular, it may be useful in Exercise 3.4.4 to know that $\dim \mathbb{A}^n = n$, $\dim Z(f) = n - 1$ for any non-constant $f \in \Bbbk[x_1, \ldots, x_n]$, and $\dim Y < n - 1$ for any proper subset $Y \subset \mathbb{A}^n$ which is not $Z(f)$ for some nonconstant f [Har77, Prop. 1.13]. Alternatively, feel free to use that $\dim U_\sigma = \mathrm{rank}(N)$ (e.g., because T_N is a $\mathrm{rank}(N)$-dimensional Zariski-dense open subset of U_σ, hence it's birational to U_σ, and birational maps of varieties preserve dimension).

Definition 3.4.2 ([Har77], Ch. I, §5, pg 31)**.** Let $Y \subset \mathbb{A}^n$ be an affine variety whose corresponding ideal $I(Y)$ is generated by $f_1, \ldots, f_t \in \Bbbk[x_1, \ldots, x_n]$. Then Y is said to be **nonsingular** at $P \in Y$ if the rank of the Jacobian matrix

$$\left(\frac{\partial f_i}{\partial x_j}(P)\right)_{i,j \in \{1,\ldots,t\}}$$

is $n - r$, where r is the dimension of Y.

If U is just a quasi-affine variety, then by definition, there exists an affine variety Y and a closed subset $V \subset Y$ such that $U \cong Y \setminus V$. Then U is said to be nonsingular at $P \in U$ if Y is nonsingular at P.[4]

In general, if Y is an algebraic variety, then each $P \in Y$ is contained in an open subvariety U of Y such that U is quasi-affine. Then Y is said to be nonsingular at P if U is nonsingular at P (the precise choice of U does not matter). Y is **nonsingular** if it is nonsingular at every point.

Example 3.4.3. Let $U = \mathbb{A}^k_{\Bbbk} \times (\Bbbk^*)^{n-k}$ for some $k \in \mathbb{Z}_{\geq 0}$, $r \in \mathbb{Z}_{\geq k}$. Let $P \in U$. Let $Y = \mathbb{A}^n \supset U$. Then Y is given by the ideal $\langle 0 \rangle \subset \Bbbk[x_1, \ldots, x_n]$. The rank of the Jacobian is 0, and this equals $n - \dim(Y)$. Thus, U is nonsingular.

Exercise 3.4.4. Let U_σ be the affine toric variety associated to a toric cone σ. We have seen that if σ is generated by k vectors from a basis for N, then U_σ is isomorphic to $\mathbb{A}^k \times (\Bbbk^*)^{r-k}$, so in particular, U_σ is nonsingular. The goal of this exercise is to prove the converse. For simplicity, we assume[5] that σ spans $N_{\mathbb{R}}$.

(a) (cf. [Ful93, Exercise on the middle of pg. 19]). Let $m_1, \ldots, m_t$ be a *minimal* set of generators for S_σ. Then

$$A_\sigma := \Bbbk[S_\sigma] = \Bbbk[z^{m_1}, \ldots, z^{m_t}] = \Bbbk[Y_1, \ldots, Y_t]/I$$

for some ideal I—here, view Y_i as the monomial z^{m_i} and I as the ideal coming from the relations in the monoid S_σ (I'm not asking you to show anything for this—just make sure you understand why it's true). Show that I is generated by polynomials of the form

$$Y_1^{a_1} \cdots Y_t^{a_t} - Y_1^{b_1} \cdots Y_t^{b_t} \tag{3.3}$$

for some non-negative integers $a_1, \ldots, a_t, b_1, \ldots, b_t$, such that[6]

$$\sum_{j=1}^{t} a_j \geq 2 \text{ and } \sum_{j=1}^{t} b_j \geq 2. \tag{3.4}$$

(b) Using (a), show that U_σ contains the origin, and show that $\frac{\partial f}{\partial x_i}$ equals 0 at the origin for each $i = 1, \ldots, t$ and each generator f for I as in (3.3).

(c) Apply parts (a) and (b) to show that if $m_1, \ldots, m_t$ are not part of a basis, then U_σ is singular at the origin.

Corollary 3.4.5. *A toric variety* $\mathrm{TV}(\Sigma)$ *is nonsingular if and only if each* $\sigma \in \Sigma$ *is* ***nonsingular****, i.e., is generated by part of a basis for* N*.*

[4]This is independent of the choice of compactification Y, as can be seen using the relationship to the regular local rings definition of nonsingular points given below.

[5]If σ does not span $N_{\mathbb{R}}$, we can split N as a direct sum $N_\sigma \oplus N''$ where σ is identified with a cone σ' which spans N_σ. Then $U_\sigma \cong U_{\sigma'} \times T_{N''}$ for $U_{\sigma'}$ singular, so U_σ is singular as well. Cf. [Ful93, pg. 29].

[6]In other words, show that any relation in S_σ corresponds to a generator for I of this form. When proving (3.4), you'll need the assumption that σ spans $N_{\mathbb{R}}$ and the assumption that the set of generators $m_1, \ldots, m_t$ is minimal.

Example 3.4.6. Consider $N = \mathbb{Z}^3$ and σ generated by e_1, e_2, e_3, and $e_1 + e_2 - e_3$. Then $\sigma^\vee$ is generated by $m_1 := e_1^*$, $m_2 := e_2^*$, $m_3 := e_1^* + e_3^*$, and $m_4 := e_2^* + e_3^*$. These generators satisfy the relation $m_1 + m_4 = m_2 + m_3$. Letting $X_i := z^{m_i}$ for $i = 1, 2, 3, 4$, we find

$$A_\sigma := \Bbbk[S_\sigma] = \Bbbk[X_1, X_2, X_3, X_4]/\langle X_1X_4 = X_2X_3\rangle.$$

That is, U_σ is the hypersurface in $\mathbb{A}^4_\Bbbk$ defined by $X_1X_4 = X_2X_3$. This is a "cone over a quadric surface" (the fibers of the cone being $X_1X_4 = X_2X_3 = t$ for varying $t \in \Bbbk$). The origin $0 \in U_\sigma$ is the standard example of a "conifold" singularity.

Note: Even when Σ contains singular cones, we can refine Σ by subdividing some cones until we get something nonsingular. The corresponding birational modification of $\mathrm{TV}(\Sigma)$ is a "resolution of singularities."

Cotangent spaces and regular local rings

Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $\Bbbk = A/\mathfrak{m}$. Then A is said to be a **regular** local ring if $\dim_\Bbbk \mathfrak{m}/\mathfrak{m}^2 = \dim A$. The $\Bbbk$-vector space $\mathfrak{m}/\mathfrak{m}^2$ is the algebro-geometric version of a cotangent space (the dual to the tangent space).

[Har77, Ch. I, Thm. 5.1] states that an affine variety Y is nonsingular at a point P if and only if the local ring $\mathcal{O}_{P,Y}$ (the stalk of $\mathcal{O}_Y$ at P) is a regular local ring. This viewpoint is used to generalize the notion of being nonsingular to abstract varieties over algebraically closed fields, cf. [Har77, Ch. II, §8, pg. 117].

The proof of the result from Exercise 3.4.4 in [Ful93, §2.1] is based on this regular local ring viewpoint. As before, we may assume that σ spans $N_\mathbb{R}$. In this case, U_σ contains the origin, and the maximal ideal associated to the origin is $\mathfrak{m} = \langle z^m | m \in S_\sigma\rangle$. A basis for $\mathfrak{m}/\mathfrak{m}^2$ is then given by those elements of $S_\sigma \setminus \{0\}$ which cannot be expressed as a sum of two elements of $S_\sigma \setminus \{0\}$; i.e., the dimension of the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ is the minimal number of elements needed to generated S_σ. Since $\dim U_\sigma = \dim(N_\mathbb{R}) = r$, being non-singular means that the number of generators for S_σ must be r, in which case $\sigma^\vee$, hence σ, are generated by bases.

Simplicial cones

A cone is called **simplicial** if it is generated by linearly independent vectors. A fan Σ is called **simplicial** if all of its cones are simplicial. As we shall see, Σ being simplicial implies that the singularities of $\mathrm{TV}(\Sigma)$ are not too bad—they can be understood as finite quotient singularities (i.e., $\mathrm{TV}(\Sigma)$ is smooth as an orbifold).

Example 3.4.7. $N = \mathbb{Z}^2$, σ generated by $(0, 1)$ and $(2, -1)$. Then $\sigma^\vee \cap M$ is generated by $(1, 0)$, $(1, 1)$, and $(1, 2)$ (using the standard inner product to identify N with M), and

$$\Bbbk[\sigma^\vee \cap M] = \Bbbk[x, xy, xy^2] \subset \Bbbk[x^{\pm 1}, y^{\pm 1}].$$

Letting $A = x$, $B = xy$, and $C = xy^2$, we can rewrite this as

$$\Bbbk[\sigma^\vee \cap M] \cong \Bbbk[A, B, C]/\langle AC - B^2\rangle.$$

This, in turn, is identified with the subring of $\Bbbk[U,V]$ generated by U^2, UV, and V^2, via $A \mapsto U^2$, $B \mapsto UV$, $C \mapsto V^2$; note that this is the subring of even functions, i.e., elements invariant under the $\mathbb{Z}/2\mathbb{Z}$-action $U \mapsto -U$ and $V \mapsto -V$:

$$\Bbbk[\sigma^\vee] \cong \Bbbk[U,V]^{\mathbb{Z}/2\mathbb{Z}}$$

where for R a ring and G a group acting on R, we write R^G for the subring of invariant elements.

Sidenote: By standard results from Geometric Invariant Theory (GIT),[7] if G is finite and $\operatorname{Spec} R$ is an affine algebraic variety, then $\operatorname{Spec} R^G$ is identified with the set of G-orbits in $\operatorname{Spec} R$, and the map $\operatorname{Spec} R \to \operatorname{Spec} R^G$ induced by the inclusion $R^G \hookrightarrow R$ is the quotient map.

Now, the action $U \mapsto -U$ and $V \mapsto -V$ of G on $\Bbbk[U,V]$ corresponds to the action $(u,v) \mapsto (-u,-v)$ on $\operatorname{Spec} \Bbbk[U,V] = \mathbb{A}^2_{\Bbbk}$. Thus, we have found that U_σ is isomorphic to the quotient of $\mathbb{A}^2$ by the action of negation:

$$U_\sigma \cong \mathbb{A}^2/(\mathbb{Z}/2\mathbb{Z}).$$

The image of the origin is a singular point, called the A_1-singularity.

Example 3.4.8. As a generalization of Example 3.4.7, let $N = \mathbb{Z}^2$ and consider σ generated by $(0,1)$ and $(n,-1)$ for $n \in \mathbb{Z}_{\geq 2}$. Then $\sigma^\vee \cap M$ is generated by $(1,0), (1,1), \ldots, (1,n)$, and so

$$A_\sigma = \Bbbk[\sigma^\vee \cap M] = \Bbbk[x, xy, \ldots, xy^{n-1}, xy^n] \subset \Bbbk[x^{\pm 1}, y^{\pm 1}].$$

We identify

$$A_\sigma \cong \Bbbk[U^n, U^{n-1}V, \ldots, UV^{n-1}, V^n] \subset \Bbbk[U,V]$$

via $x \mapsto U^m$ and $y \mapsto V/U$. I.e.,

$$A_\sigma \cong \Bbbk[U,V]^{\mathbb{Z}/n\mathbb{Z}}$$

where the generator of $\mathbb{Z}/n\mathbb{Z}$ acts via $(U,V) \mapsto (\zeta_n U, \zeta_n V)$ for ζ_n a primitive n-th root of unity. The induced action on $\mathbb{A}^2_{\Bbbk}$ is via $(u,v) \mapsto (\zeta_n u, \zeta_n v)$. We thus have

$$A_\sigma \cong \mathbb{A}^2_{\Bbbk}/(\mathbb{Z}/n\mathbb{Z}).$$

Proposition 3.4.9. *If Σ is simplicial, then $\mathrm{TV}(\Sigma)$ is an orbifold, i.e., it has only finite quotient singularities.*

More precisely, suppose σ is a simplicial toric cone generated by linearly independent primitive vectors $v_1, \ldots, v_k$ which span a rank-s sublattice N' of N. Let

$$N'_{\mathrm{sat}} := \{n \in N | kn \in N' \text{ for some } k \in \mathbb{Z}\}$$

*denote the **saturation** of N' in N, and let $G = N'_{\mathrm{sat}}/N' = (N/N')_{\mathrm{tor}}$. Then*

$$U_\sigma \cong (T_{N/N'_{\mathrm{sat}}}) \times (\mathbb{A}^s_{\Bbbk}/G).$$

[7] See https://math.ou.edu/~tmandel/GIT.pdf for some very concise notes I wrote summarizing the main ideas from GIT, following [Muk03]. Disclaimer: these are from when I was a grad student, and I remember being less that 100% confident in the correctness of the linearization section.

Proof. We work over $\Bbbk = \mathbb{C}$, but this is easily modified for other algebraically closed characteristic 0 fields $\Bbbk$. It suffices to prove the result for a single simplicial cone σ.

Let σ' denote σ viewed as a cone in $N'_{\text{sat}} \otimes \mathbb{R}$. Then $U_\sigma \cong U_{\sigma'} \times T_{N/N'_{\text{sat}}}$ (this is a special case of the product-of-fans construction that we'll cover later, but you should also be see this special case somewhat directly). So it suffices to prove the claim for σ', which is top-dimensional in $N'_{\text{sat}} \otimes \mathbb{R}$. For simplicity, we view this as allowing us to assume that σ is top-dimensional in $N_\mathbb{R}$.

Let $M' \supset M$ be the dual lattice to $N' \subset N$. Note that there is a canonical dual pairing

$$\langle \cdot, \cdot \rangle : M'/M \times N/N' \to \mathbb{Q}/\mathbb{Z}.$$

Let $\mu(\cdot, \cdot) : M'/M \times N/N' \to \mathbb{C}^*$ denote the map obtained by applying $\langle \cdot, \cdot \rangle$, followed by the inclusion $\mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}$, $q \mapsto \exp(2\pi i q)$. Then $G = N/N'$ acts on $\mathbb{C}[M]$ via

$$n \cdot z^{m'} = \mu(m', n) z^{m'} = \exp(2\pi i \langle m', n \rangle) z^{m'}.$$

We claim that $\mathbb{C}[M']^G = \mathbb{C}[M]$. Indeed, since the monomials $z^{m'}$ are a spanning set of eigenvectors for the action of any $n \in G$, and $\mathbb{C}[M']^G$ is the intersection overall all $n \in G$ of the eigenspaces with eigenvector 1, we just have to check that

$$\{m' \in M' | n \cdot z^{m'} = z^{m'}\} = M.$$

The left-hand side is clearly the same as the $m' \in M'$ with $\mu(m', n) = 1$ for all n, i.e., with $\langle m', n \rangle \in \mathbb{Z}$ for all n, and this is just

$$M' \cap \operatorname{Hom}(N, \mathbb{Z}) = \operatorname{Hom}(N', \mathbb{Z}) \cap \operatorname{Hom}(N, \mathbb{Z}) = \operatorname{Hom}(N, \mathbb{Z}) = M.$$

So indeed $\mathbb{C}[M']^G = \mathbb{C}[M]$.

It now follows that

$$\Bbbk[\sigma^\vee \cap M] = \Bbbk[(\sigma')^\vee \cap M']^G$$

hence

$$U_\sigma = U_{\sigma'}/G \cong \mathbb{A}^s_\Bbbk / G$$

where for the last equality we observed that $U_{\sigma'} \cong \mathbb{A}^s_\Bbbk$. □

3.5 Maps of fans

Consider two lattices N_1, N_2 with duals M_1, M_2, respectively. Let $\sigma_1 \subset N_{1,\mathbb{R}}$, $\sigma_2 \subset N_{2,\mathbb{R}}$, respectively. Let $\varphi : N_1 \to N_2$ be a linear map of lattices such that (abusing notation and extending φ to the lattices tensored with $\mathbb{R}$) we have $\varphi(\sigma_1) \subset \sigma_2$. Then dualizing φ yields a map

$$\varphi^\vee : \sigma_2^\vee \cap M_2 \to \sigma_1^\vee \cap M_1,$$

i.e., $\varphi^\vee : \Bbbk[S_{\sigma_2}] \to \Bbbk[S_{\sigma_1}]$. Thus, φ induces a map $\varphi_* : U_{\sigma_1} \to U_{\sigma_2}$.

We note that φ_* can also be defined in terms of the pullback by $\varphi^\vee$:

$$\begin{aligned} \varphi_* = (\varphi^\vee)^* : \operatorname{Hom}(S_{\sigma_1}, \Bbbk) \setminus \{0\} &\to \operatorname{Hom}(S_{\sigma_2}, \Bbbk) \setminus \{0\} \\ p &\mapsto p \circ \varphi^\vee. \end{aligned}$$

A map of fans $\varphi : \Sigma_1 \to \Sigma_2$ is a homomorphism $\varphi : N_1 \to N_2$, Σ_i a fan in N_i, such that for all $\sigma_1 \in \Sigma_1$, there exists some $\sigma_2 \in \Sigma_2$ containing $\varphi(\sigma_1)$. By the above discussion, this yields a map $\varphi_* : U_{\sigma_1} \to \mathrm{TV}(\Sigma_2)$ for each $\sigma_1 \in \Sigma_1$. These maps glue to yield a morphism $\varphi_* : \mathrm{TV}(\Sigma_1) \to \mathrm{TV}(\Sigma_2)$.

Example 3.5.1. We saw in Remark 3.1.13 that if τ is a face of σ, then there is an inclusion $U_\tau \hookrightarrow U_\sigma$. This inclusion of toric varieties is the map induced by the map of fans $\mathrm{Id} : N \to N$, $\tau \hookrightarrow \sigma$.

Example 3.5.2 (Products of fans)**.** We can take a product of two fans Σ_i in N_i, $i = 1, 2$. The cones in $\Sigma_1 \times \Sigma_2$ are the cones in $N_{1,\mathbb{R}} \times N_{2,\mathbb{R}}$ of the form $\{(x_1, x_2) | x_i \in \sigma_i \text{ for some } \sigma_i \in \Sigma_i\}$. Then

$$\mathrm{TV}(\Sigma_1 \times \Sigma_2) = \mathrm{TV}(\Sigma_1) \times \mathrm{TV}(\Sigma_2).$$

The projections $N_{1,\mathbb{R}} \times N_{2,\mathbb{R}} \to N_{i,\mathbb{R}}$, $i = 1, 2$, induce maps of fans $\Sigma_1 \times \Sigma_2 \to \Sigma_i$. The induced morphisms of toric varieties are the natural projections

$$\mathrm{TV}(\Sigma_1 \times \Sigma_2) \to \mathrm{TV}(\Sigma_i)$$

for $i = 1$ or 2. Similarly, the inclusions $N_{i,\mathbb{R}} \hookrightarrow N_{1,\mathbb{R}} \times N_{2,\mathbb{R}}$, $n_1 \mapsto (n_1, 0)$ or $n_2 \mapsto (0, n_2)$, induce maps of fans corresponding to the embeddings

$$\mathrm{TV}(\Sigma_i) \to \mathrm{TV}(\Sigma_1) \times \mathrm{TV}(\Sigma_2)$$

with $p \mapsto (p, \mathbf{1})$ or $(\mathbf{1}, p)$, respectively. Here, $\mathbf{1}$ denotes the identity in T_{N_1} or T_{N_2}.

The next lemma will be used in the example on fiber bundles.

Lemma 3.5.3. *Let N, N' be lattices, and let σ, σ' be toric cones in $N_\mathbb{R}, N'_\mathbb{R}$, respectively. Let $\varphi : N \to N'$ be a surjection of lattices such that each face of σ maps surjectively to a face of σ', and let $f : U_\sigma \to U_{\sigma'}$ be the corresponding map of afffine toric varieties. If τ is a face of σ with image τ' in σ', then φ maps $U_\sigma \setminus U_\tau$ to $U_{\sigma'} \setminus U_{\tau'}$.*

Proof. Let $m_{\tau'}$ be a point in $(\sigma')^\vee$ such that $\tau' = \sigma \cap m_{\tau'}^\perp$. As in Remark 3.1.13, $U_{\sigma'} \setminus U_{\tau'} = Z(z^{m_{\tau'}})$.

Let $\varphi^\vee : M' \to M$ be the dual to φ. We have $m_\tau := \varphi^\vee(m_{\tau'}) \in \sigma^\vee$ and

$$\begin{aligned} m_\tau^\perp \cap \sigma &= \varphi^\vee(m_{\tau'})^\perp \cap \sigma \\ &= \varphi^{-1}(m_{\tau'}^\perp) \cap \sigma \\ &= \varphi^{-1}(m_{\tau'}^\perp \cap \sigma') \cap \sigma \\ &= \varphi^{-1}(\tau') \cap \sigma \\ &= \tau. \end{aligned}$$

So

$$\begin{aligned} U_\sigma \setminus U_\tau &= Z(z^{m_\tau}) \\ &= Z(f^*(z^{m_{\tau'}})) \\ &= f^{-1}(Z(z^{m_{\tau'}})) \\ &= f^{-1}(U_{\sigma'} \setminus U_{\tau'}) \end{aligned}$$

as claimed. ∎

Proposition 3.5.4 (Fiber bundles: cf. the exercise at the bottom of pg 41 in [Ful93])**.** *Let*

$$0 \to N' \xrightarrow{\varphi_1} N \xrightarrow{\varphi_2} N'' \to 0$$

be an exact sequence of lattices, inducing maps of fans $\Sigma' \to \Sigma \to \Sigma''$ between fans in N', N, and N'', respectively. We thus obtain maps

$$\mathrm{TV}(\Sigma') \to \mathrm{TV}(\Sigma) \to \mathrm{TV}(\Sigma''). \tag{3.5}$$

Now suppose there exists a fan $\widetilde{\Sigma}''$ in N lifting Σ'' in the sense that the cones of $\widetilde{\Sigma}''$ are in bijection with the cones of Σ'', and for each $\sigma'' \in \Sigma''$, the restriction $\varphi_2|_{\widetilde{\sigma}''}$ to the corresponding cone $\widetilde{\sigma}'' \in \widetilde{\Sigma}''$ gives an isomorphism of integral[8] *cones from $\widetilde{\sigma}''$ to σ''. Suppose furthermore that the cones $\sigma \in \Sigma$ are exactly the cones of the form $\sigma = \sigma' + \widetilde{\sigma}''$ for σ' a cone of $\varphi_1(\Sigma')$ and $\widetilde{\sigma}'' \in \widetilde{\Sigma}''$. Then the sequence from (3.5) is a locally trivial fibration. That is, there is an open cover $\{U_i\}$ of $\mathrm{TV}(\Sigma'')$ such that for each i, $\varphi_2^{-1}(U_i) \cong \mathrm{TV}(\Sigma') \times U_i$.*

Proof. In fact, we can take this open cover to be $\{U_{\sigma''} | \sigma'' \in \Sigma''\}$. Indeed, for each σ'', let $\Sigma_{\sigma''} \subset \Sigma$ be those cones of the form $\sigma' + \widetilde{\sigma}''$ for this fixed choice of σ''. Then $\mathrm{TV}(\Sigma_{\sigma''}) \subset \mathrm{TV}(\Sigma)$ is precisely $\varphi_2^{-1}(U_{\sigma''})$—to show this, we just have to show that the other points of $\mathrm{TV}(\Sigma)$ don't map to $U_{\sigma''}$, and this follows from Lemma 3.5.3.

Let $s : N'' \to N$ be a linear section of φ_2 whose restriction to σ'' is inverse to $\varphi_2|_{\widetilde{\sigma}''}$. This yields a decomposition $N = N' \times N''$. We wish to show that $\sigma = \sigma' + \widetilde{\sigma}''$ in $\widetilde{\Sigma}''$ is, under this decomposition, equal to $\sigma' \times \widetilde{\sigma}''$. This is straightforward after noting that each point of σ' is of the form $(n', 0)$ for $n' \in N'_{\mathbb{R}}$, and each point of $\widetilde{\sigma}''$ is of the form $(0, n'')$ for $n'' \in N''_{\mathbb{R}}$. □

Example 3.5.5. As a special case of Proposition 3.5.4, Figure 3.5 shows a map of fans yielding a $\mathbb{P}^1$-bundle $\mathbb{F}_a$ over $\mathbb{P}^1$. These ruled surfaces $\mathbb{F}_a$ are called the Hirzebruch surfaces. One can show that $\mathbb{F}_a \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1})$, i.e., $\mathbb{F}_a$ is the projectivization of the line bundle $\mathcal{O}_{\mathbb{P}^1}(a)$ on $\mathbb{P}^1$ (cf. Example 2.5.2).

3.6 The quotient construction

We now construct toric varieties as quotients, generalizing the standard quotient construction of $\mathbb{P}^n$. This approach (or at least the cool fancy interpretation of it that we'll get to later) is due to [Cox95], and it is the primary approach in [HKK$^+$03].

Let $\Sigma(1)$ denote the rays of Σ, generated by $v_1, \ldots, v_s$. Let $\widetilde{N} := \mathbb{Z}^{\Sigma(1)} = \mathbb{Z}^s$ denote the lattice freely generated by these rays, with generators denoted $\widetilde{v}_1, \ldots, \widetilde{v}_s$. We define a fan $\widetilde{\Sigma}$ in $\mathbb{Z}^{\Sigma(1)}$ as follows: If $\sigma = \langle v_{i_1}, \ldots, v_{i_k} \rangle$ is a cone in Σ, then $\widetilde{\Sigma}$ contains a cone $\widetilde{\sigma}$ generated by the corresponding $\widetilde{v}_{i_1}, \ldots, \widetilde{v}_{i_k}$. We consider $\mathrm{TV}(\widetilde{\Sigma})$.

Alternatively, let $Z \subset \operatorname{Spec} \Bbbk[\widetilde{v}_1^*, \ldots, \widetilde{v}_s^*]$ be the union of all sets of the form

$$Z(\{z^{\widetilde{v}_{i_j}^*}, j = 1, \ldots, \ell | \text{ such that the } v_{i_j}\text{'s do not generate a cone in } \Sigma\}).$$

Then $\mathrm{TV}(\widetilde{\Sigma})$ is $\operatorname{Spec} \Bbbk[\widetilde{v}_1^*, \ldots, \widetilde{v}_m^*] \setminus Z = \mathbb{A}^s \setminus Z$.

[8] By an isomorphism of *integral* cones, we mean that, in addition to being a linear map taking $\widetilde{\sigma}''$ bijectively to σ'', we also have $\varphi_2(\widetilde{\sigma}'' \cap N) = \sigma'' \cap N''$.

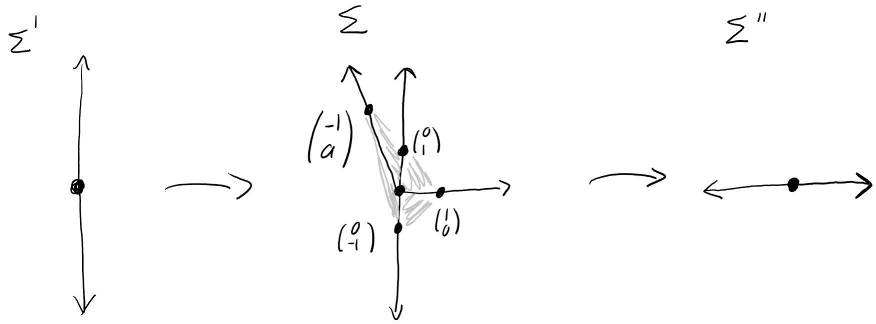

Figure 3.5.1: The map of fans identifying the Hirzebruch surface $\mathbb{F}_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1})$ as a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Here, the fans Σ' and Σ'' lie in one-dimensional lattices—the orientation with which Σ' s drawn is meant to intuitively show what the first map is, i.e., the inclusion $\mathbb{Z} \to (0, \mathbb{Z}) \subset \mathbb{Z} \oplus \mathbb{Z}$. The shading in Σ is meant to indicate that the two-dimensional chambers are also part of the fan.

We have a map $\pi : \mathbb{Z}^{\Sigma(1)} \to N$ induced by $\widetilde{v_i} \mapsto v_i$. By construction, this is a map of fans, so we get a map $\pi_* : \mathrm{TV}(\widetilde{\Sigma}) \to \mathrm{TV}(\Sigma)$. Let us assume that $\Sigma(1)$ spans $N_{\mathbb{R}}$ — this will imply that π_* is surjective.[9] We would like to view this as a quotient of $\mathrm{TV}(\widetilde{\Sigma})$ by the action of some group G—this way we get a construction of $\mathrm{TV}(\Sigma)$ that does not depend on our prior construciton involving dual cones and gluing. We treat the nonsingular case first.

The kernel K of π is the lattice of relations among the v_i's. The short exact sequence

$$0 \to K \to \mathbb{Z}^{\Sigma(1)} \to N \to 0 \tag{3.6}$$

induces (in the nonsingular cases) a short exact sequence of the corresponding toric varieties which realizes $\mathrm{TV}(\Sigma)$ as the quotient of $\mathrm{TV}(\widetilde{\Sigma})$ by the action of the torus $G := T_K$:

$$\mathrm{TV}(\Sigma) \cong \mathrm{TV}(\widetilde{\Sigma})/T_K$$

Indeed, in these nonsingular cases, the sequence (3.6) is a special case of the fiber bundle construction in Proposition 3.5.4 — here the fan in K is just the origin $\{0\}$. Explicitely, the action of $K \otimes \Bbbk^*$ on $\mathrm{TV}(\widetilde{\sigma})$ is given by

$$\left(\sum_{i=1}^{s} a_i v_i\right) \otimes \lambda : (x_1, \ldots, x_s) \mapsto (\lambda^{a_1} x_1, \ldots, \lambda^{a_n} x_s).$$

Example 3.6.1. Let $N = \mathbb{Z}^s$. Denote $v_i = e_i$ for $i = 1, \ldots, s$, and let $v_{s+1} = -e_1 - \ldots - e_s$. Let Σ be the fan consisting of all cones spanned by proper subsets of $\{v_1, \ldots, v_{s+1}\}$. I've claimed before that $\mathrm{TV}(\Sigma) \cong \mathbb{P}^s$. Let's check this using the quotient construction.

We have $\widetilde{N} = \mathbb{Z}^{s+1}$. The locus Z equals $Z(z^{\widetilde{v}_1^*}, \ldots, z^{\widetilde{v}_{s+1}^*}) = (0, \ldots, 0) \in \mathbb{A}^{s+1}$. The lattice K is one-dimensional, generated by the relation $v_1 + \ldots + v_s + v_{s+1} = 0$. So $T_K \cong \Bbbk^*$ with action on $\mathrm{TV}(\widetilde{\Sigma}) = \mathbb{A}^{k+1} \setminus \{0\}$ given by $\lambda \cdot (x_1, \ldots, x_{s+1}) = (\lambda x_1, \ldots, \lambda x_{s+1})$. So the quotient

$$\mathrm{TV}(\Sigma) = (\mathbb{A}^{s+1} \setminus \{0\})/\Bbbk^*$$

[9] If $\Sigma(1)$ does not span $N_{\mathbb{R}}$, we could pick some additional primitive elements of N until we do get a spanning set, and then enlarge $\widetilde{N}$ to also include generators associated to these elements. The fan $\widetilde{\Sigma}$ is then defined in the same way (so the cones will have nothing going in these extra directions), and we can thus obtain a quotient construction in general.

is exactly the usual quotient construction of $\mathbb{P}^s_{\Bbbk}$.

We now extend to the possibly singular cases. Dualizing the map $\widetilde{N} \to N$ and identifying $\widetilde{M} = \mathrm{Hom}(\widetilde{N}, \mathbb{Z})$ with $\mathbb{Z}^{\Sigma(1)}$, we obtain a short exact sequence

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to A \to 0 \tag{3.7}$$

(for A defined as the cokernel of the map $M \to \mathbb{Z}^{\Sigma(1)}$).[10] If Σ is nonsingular, then A is torsion free and equals $K^* := \mathrm{Hom}(K, \mathbb{Z})$ for $K := \ker(\widetilde{N} \to N)$. But in general, $A \cong K^* \times A_{\mathrm{tor}}$. Applying $\mathrm{Hom}(\cdot, \Bbbk^*)$ to (3.7) yields

$$0 \to \mathrm{Hom}(A, \Bbbk^*) \to (\Bbbk^*)^{\Sigma(1)} \to T_N \to 0.$$

We take G to be the first term of the sequence, i.e.,

$$G := \mathrm{Hom}(A, \Bbbk^*) \cong T_K \times \mathrm{Hom}(A_{\mathrm{tor}}, \Bbbk^*) \cong T_K \times \mathrm{Hom}(A_{\mathrm{tor}}, \mathbb{Q}/\mathbb{Z}).$$

The inclusion of G into $(\Bbbk^*)^{\Sigma(1)}$ induces an action of G on $\mathrm{TV}(\widetilde{\Sigma}) = \mathbb{A}^{\Sigma(1)} \setminus Z$.

Theorem 3.6.2 ([Cox95], Thm. 2.1)**.** $\mathrm{TV}(\Sigma)$ *is naturally isomorphic to the* ***categorical quotient***[11] $\mathrm{TV}(\widetilde{\Sigma}) /\!\!/ G$ *(for each* $\sigma \in \Sigma$*, there is an isomorphism between* $\Bbbk[S_\sigma]$ *and* $\Bbbk[S_\sigma]^G$*, and these are compatible with gluing). Furthermore, this is a* ***geometric quotient*** *(a categorical quotient where fibers are orbits and the topology on the quotient space is the quotient topology) if and only if* Σ *is simplicial.*

Example 3.6.3. Proposition 3.4.9 can be interpreted as a special case of Theorem 3.6.2. Assuming that $\dim(\sigma) = \mathrm{rank}(N)$, the lattice N' there is the same as $\widetilde{N} = \mathbb{Z}^{\Sigma(1)}$ here. There we interpret G as N/N'. Here we interpret G as $\mathrm{Hom}(M'/M, \mathbb{Q}/\mathbb{Z})$.

Example 3.6.4. Weighted projective space is a generalization of projective space in which the $\Bbbk^*$-action is modified. More precisely, for any positive integers $d_0, \ldots, d_r$, there is a corresponding weighted projective space

$$\mathbb{P}(d_0, \ldots, d_r) = (\Bbbk^{n+1} \setminus \{0\})/\Bbbk^*$$

where $\Bbbk^*$ acts via

$$\lambda.(x_0, \ldots, x_r) = (\lambda^{d_0} x_0, \ldots, \lambda^{d_n} x_r).$$

So the usual projective space $\mathbb{P}^r$ is the case $d_0 = \ldots = d_r = 1$. More generally, suppose $\gcd(d_1, \ldots, d_r) = 1$.[12] Take $v_i = e_i$ for $i = 1, \ldots, r$ and take $v_0 = -\frac{1}{d_0} \sum_{i=1}^r d_i e_i \in N_{\mathbb{Q}}$. Replace N with the lattice N' in $N_{\mathbb{Q}}$ generated over $\mathbb{Z}$ by $v_0, v_1, \ldots, v_r$. Note that the v_i's are all primitive in N' and that $\sum_{i=0}^r d_i v_i = 0$. Take Σ to be the fan in N' whose cones are generated by proper subsets of $\{v_0, \ldots, v_r\}$. The quotient constructions above quickly imply that $\mathrm{TV}(\Sigma) = \mathbb{P}(d_0, d_1, \ldots, d_r)$.

[10] As we'll see in §4.3, A here can be identified with the divisor class group $\mathrm{Cl}(X) = A_{n-1}(X)$ for $X = \mathrm{TV}(\Sigma)$.

[11] The categorical quotient is the universal morphism which is equivariant with respect to the G-action. GIT quotients are known to be categorical quotients. Again, see my very concise GIT primer `https://math.ou.edu/~tmandel/GIT.pdf`, or for details, see [Muk03].

[12] This can in fact be assumed without loss of generality—see the property of being "well-formed" at `https://en.wikipedia.org/wiki/Weighted_projective_space`.

3.7 Toric varieties from polytopes

In addition to [Ful93], I will use the exposition in [GS11] to help me here.

Let E be a real vector space. A **convex polyhedron** K in E is an intersection of finitely many closed affine half-spaces; i.e., a set of the form

$$K = \{v \in E | \langle u_1, v\rangle \geq -a_1, \ldots, \langle u_r, v\rangle \geq -a_r\} \tag{3.8}$$

for some $u_1, \ldots, u_r \in E^*$ and some $a_1, \ldots, a_r \in \mathbb{R}$. If E is given as $L \otimes \mathbb{R}$ for some lattice L, then then K is said to be **rational** if we can take $u_i \in L^*$ and $a_i \in \mathbb{Z}$ for each i. For K given as in (3.8), the **faces** of K are the subsets of K where a subset of the inequalities $\langle u_i, v\rangle \geq -a_i$ are actually equalities. Equivalently, a face is a subset of K given by $\langle u, v\rangle = r$ for some $u \in E^*$ and $r \in \mathbb{R}$ satisfying $\langle u, v\rangle \geq r$ for all $v \in K$. A codimension-1 face is called a **facet**.

A convex polyhedron K is bounded if and only if it is a convex hull of a finite set [Ful93, Execrise on pg 25]. In this case, K is called a **convex polytope**. The polytope K is rational if and only if it is a convex hull of points in $L_{\mathbb{Q}}$, and K is called **integral** if it is a convex hull of points in L.

Suppose K is a rational convex polyhedron in $M_{\mathbb{R}}$. Consider the **cone** over K defined by

$$CK := \overline{\mathbb{R}_{\geq 0}(K \times \{1\})} \subset M_{\mathbb{R}} \times \mathbb{R}.$$

Taking the closure here is important for the unbounded cases since it adds the **asymptotic cone**

$$K_\infty = \lim_{a \to 0} aK \subset M_{\mathbb{R}} \times \{0\}.$$

Of course, if K is bounded, then $K_\infty = \{0\}$ and $\Bbbk[K_\infty \cap M] = \Bbbk$.

The ring $S_K = \Bbbk[CK \cap (M \oplus \mathbb{Z})]$ is a graded $\Bbbk[K_\infty \cap M]$-algebra with $\deg(z^{(m,r)}) = r \in \mathbb{Z}$ and $(S_K)_0 = \Bbbk[K_\infty, \cap M]$. It turns out that $X = \operatorname{Proj} S_K$ is a toric variety! Note that X is projective over $\operatorname{Spec} \Bbbk[K_\infty \cap M]$, and $[\mathcal{O}_X(1)](X)$ is naturally identified with $\Bbbk\langle z^m | m \in K \cap M\rangle$.

Indeed, for each face T of K, let us choose a rational point m_T in the relative interior of T, and fix a point $\widetilde{m}_T = (km_T, k) \in CT \cap (M \oplus \mathbb{Z})$ for some $k \in \mathbb{Z}_{\geq 1}$. Then $\operatorname{Proj} S_K$ is covered by the distinguished open subsets $D_+(z^{\widetilde{m}_T}) = \operatorname{Spec} \Bbbk[CK \cap (M \oplus \mathbb{Z})]_{(z^{\widetilde{m}_T})}$. Let

$$\begin{aligned} C_T K := \mathbb{R}_{\geq 0}(K - T) : &= \mathbb{R}_{\geq 0}\{k - t \in M_{\mathbb{R}} | k \in K, t \in T\} \\ &= (CK - CT) \cap (M_{\mathbb{R}} \times \{0\}), \end{aligned}$$

or equivalently,

$$\begin{aligned} C_T K &= \mathbb{R}_{\geq 0}(K - m_T) \\ &= (CK - Cm_T) \cap (M_{\mathbb{R}} \times \{0\}). \end{aligned}$$

Noting that localizing by $z^{\widetilde{m}}$ corresponds to subtracting Cm_T from CK, and taking the degree-0 subring corresponds to intersecting the monoid with $M \oplus \{0\}$, we see that

$$\Bbbk[CK \cap (M \oplus \mathbb{Z})]_{(z^{\widetilde{m}})} = \Bbbk[(CK - Cm_T) \cap \{0\}] = \Bbbk[C_T K \cap M].$$

Thus, $\operatorname{Proj} S_K$ is covered by sets of the form $\Bbbk[C_T K \cap M]$. One can show (cf. [Ful93, Proposition on pg 26]) that the cones $(C_T K)^\vee \subset N_{\mathbb{R}}$ form a fan Σ_K in $N_{\mathbb{R}}$, called the **normal fan** of K. Hence, $\operatorname{Proj} S_K \cong \operatorname{TV}(\Sigma_K)$.

In fact, if K is bounded (i.e., a polytope) then [Ful93, Proposition on pg 26] gives another way to find this fan Σ_K. Define the **polar set** K° of the polytope K by

$$K^\circ = \{n \in N_\mathbb{R} | \langle n, m\rangle \geq -1 \text{ for all } m \in K\}. \tag{3.9}$$

There is a one-to-one order-reversing correspondence between proper faces of K and proper faces of K°, and K being rational implies that K° is rational [Ful93, Prop. on pg 24]. Furthermore [Ful93, Proposition on pg 26], if K contains the origin in its interior (this can always be made true because re-scaling and translating K do not affect $\operatorname{Proj} S_K$), then the fan Σ_K consists precisely of the cones over the faces of K°.

Note that the Proj construction comes with extra information: the choice of K determines the graded ring S_K, hence the graded S_K-module $S_K(1)$, hence an associated sheaf of modules $\mathcal{O}_{\operatorname{Proj} S_K}(K) := \widetilde{S_K(1)}$. That is, while the projective toric variety $X = \operatorname{Proj} S_K$ is unchanged, the induced embedding of X into projective spaces *does* change.

Example 3.7.1. Let $N = \mathbb{Z}^2$, and let $K \subset M_\mathbb{R}$ be the convex hull of the points $(1,0)$, $(0,1)$, and $(0,0)$. Letting $x = (1,0,1)$, $y = (0,1,1)$, and $z = (0,0,1)$, we identify S_K with $\Bbbk[x,y,z]$ with each variable having degree 1, Thus, $\operatorname{Proj} S_K = \mathbb{P}^2$.

For $T = (0,0) \subset K$, we have $C_T K = \mathbb{R}_{\geq 0}\langle (1,0),(0,1)\rangle$, and so the dual cone is the first quadrant in $N_\mathbb{R}$. The cones associated to other faces are similarly computed, and one finds that Σ_K is the usual fan for $\mathbb{P}^2$ as in Example 3.2.3.

Suppose we replace K with the convex hull of $(-1,-1)$, $(2,-1)$, and $(-1,2)$; that is, we re-scale by a factor of 3 and translate by $(-1,-1)$. Then $\operatorname{Proj} S_K$ is again isomorphic to $\mathbb{P}^2$, but the re-scaling changes the line bundle induced by K from $\mathcal{O}_{\mathbb{P}^2}(1)$ to $\mathcal{O}_{\mathbb{P}^2}(3)$.[13] For this new K, the polar polytope is the convex hull of $(1,0)$, $(0,1)$, and $(-1,-1)$. We immediately see that the fan obtained as the cone over K° is the fan for $\mathbb{P}^2$.

3.7.1 Polar polytopes and line bundles

Consider a divisor $D = \sum_{i=1}^s a_i D_{\rho_i}$ be a divisor supported on the boundary of $\operatorname{TV}(\Sigma)$ for Σ a fan with rays $\rho_1, \ldots, \rho_s$. Let $n_i' \in N$ be the primitive generator for ρ_i, and let $n_i = \frac{1}{a_i} n_i' \in N_\mathbb{Q}$. Let $K^\circ \subset N_\mathbb{R}$ be the convex hull of $\{n_i | i = 1, \ldots, s\}$, the polar set (3.9) of some $K \subset M_\mathbb{R}$. So by (3.2), $\operatorname{val}_{D_{\rho_i}}(\sum c_m z^m) = \min_{c_m \neq 0}\langle m, n_i'\rangle = a_i \min_{c_m \neq 0}\langle m, n_i\rangle$. Thus,

$$\left\{f \in \Bbbk[M] | \operatorname{val}_{D_{\rho_i}}(f) \geq -a_i\right\} = \left\{ \sum_{m | \langle m, n_i\rangle \geq -1} c_m z^m \in \Bbbk[M] \right\}. \tag{3.10}$$

Consider $X = \operatorname{Proj} S_K = \operatorname{TV}(\Sigma_K)$. Recall that

$$\Gamma(X, \mathcal{O}_X(D)) = \{f \in \Bbbk[M] | \operatorname{val}_{D_{\rho_i}} \geq -a_i \text{ for } i = 1, \ldots, s\}.$$

So by (3.10) and the definition (3.9) of the polar set $(K^\circ)^\circ = K$, we see that

$$\Gamma(X, \mathcal{O}_X(D)) = \left\{ \sum_{m \in K \cap M} c_m z^m \right\}.$$

[13] This construction actually yields the degree-3 Veronese embedding of $\mathbb{P}^2$ into $\mathbb{P}^9$.

Example 3.7.2. Consider the polytope K from Example 3.7.1 after re-scaling by a factor of 3 and translating so that the origin is in the interior. Recall that the vertices of K° were $(1,0),(0,1)$, and $(-1,-1)$, so the corresponding divisors are the three lines $D_{\rho_1}, D_{\rho_2}, D_{\rho_3}$ in $\mathrm{TV}(\Sigma_K) = \mathbb{P}^2$ (the x-, y, and z-axes in $\mathbb{P}^2$, where by the z-axis we mean the axis at infinity). So the corresponding line bundle is

$$\mathcal{O}_{\mathbb{P}^2}(D_{\rho_1} + D_{\rho_2} + D_{\rho_3}) \cong \mathcal{O}_{\mathbb{P}^2}(3H) \cong \mathcal{O}_{\mathbb{P}^2}(3)$$

where H denotes the class of a hyperplane (i.e., a line) in $\mathbb{P}^2$.

3.8 Characters and cocharacters

A (multiplicative) character of a group G is a group homomorphism from G to the multiplicative group of a field—for us, $G = T_N$ and the characters are homomorphisms to $\Bbbk^*$. The lattice M can be identified with the characters of $T_N = N \otimes \Bbbk^* \cong \mathrm{Hom}(M, \Bbbk^*)$. Given $m \in M$, the associated map $N \otimes \Bbbk^* \to \Bbbk^*$ is $n \otimes \lambda \mapsto \lambda^{\langle m,n\rangle}$. Equivalently, m corresponds to the evaluation map $\mathrm{Hom}(M, \Bbbk^*) \to \Bbbk^*$, $f \mapsto f(m)$. One therefore often refers to M as the **character lattice**.

Dually, N can be viewed as the **cocharacter lattice**, with each element n corresponding to a 1-paramater families (or cocharacters) $\rho_n \in \mathrm{Hom}(\Bbbk^*, T_N)$:

$$\rho_n : \Bbbk^* \to N \otimes \Bbbk^* \qquad \lambda \mapsto n \otimes \lambda$$

or equivalently,

$$\rho_n : \Bbbk^* \to \mathrm{Hom}(M, \Bbbk^*) \qquad \lambda \mapsto (m \mapsto \lambda^{\langle m,n\rangle}).$$

Note that if $\langle m,n\rangle > 0$, then $\lim_{\lambda\to 0}[\rho_n(\lambda)](m) = 0^{\langle m,n\rangle} = 0$. If $\langle m,n\rangle = 0$, then $\lim_{\lambda\to 0}[\rho_n(\lambda)](m) = \lim_{\lambda\to 0} \lambda^0 = 1$. On the other hand, if $\langle m,n\rangle < 0$, then the limit $\lim_{\lambda\to 0}[\rho_n(\lambda)](m)$ does not exist in $\Bbbk$. It follows that $\lim_{\lambda\to 0} \rho_n(\lambda) \in U_\sigma$ if and only if $n \in \sigma$ (because then $\langle m,n\rangle \geq 0$ for all $m \in \sigma^\vee$, hence the limit lies in $\mathrm{Hom}_{\mathrm{sg}}(\sigma^\vee \cap M, \Bbbk) \setminus \{0\} = U_\sigma$).

We thus obtain the following:

Lemma 3.8.1. $\lim_{\lambda\to 0} \rho_n(\lambda) \in \mathrm{TV}(\Sigma)$ *if and only if n is contained in a cone of Σ.*

Chapter 4

Properties of toric varieties

4.1 Compactness and properness

Given a fan Σ in $N_{\mathbb{R}}$, the **support** of Σ is the set

$$|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma \subset N_{\mathbb{R}}.$$

Proposition 4.1.1. *A toric variety* $\mathrm{TV}(\Sigma)$ *is* ***compact*** *if* $|\Sigma| = N_{\mathbb{R}}$.

More generally, a map of complex varieties $\varphi : X \to Y$ is called **proper**[1] if the inverse images of compact subsets are compact. In particular, a complex variety X is compact if and only if $X \to \operatorname{Spec} \mathbb{C}$ is proper (often one calls such X "complete," cf. [Har77, Ch. 2, §4. Def. on pg 105 and Exercise 4.5]). Proposition 4.1.1 is thus a special case of the following:

Proposition 4.1.2. *Let* $\varphi : N_1 \to N_2$ *be a morphism of lattices which induces a map of fans from* Σ_1 *to* Σ_2*, thus induces a morphism* $\varphi_* : \mathrm{TV}(\Sigma_1) \to \mathrm{TV}(\Sigma_2)$*. Then* φ_* *is proper if and only if* $\varphi^{-1}(|\Sigma_2|) = |\Sigma_1|$.

Proof sketch. $\implies$: We always have $\varphi^{-1}(|\Sigma_2|) \supset |\Sigma_1|$ by the definition of a map of fans. So if $\varphi^{-1}(|\Sigma_2|) \neq |\Sigma_1|$, that means there is some $n \in N_1$ which is not in $|\Sigma_1|$, but which does map to some cone $\sigma \in \Sigma_2$. Consider the associated 1-parameter subgroups ρ_n and $\varphi_*(\rho_n) = \rho_{\varphi(n)}$ as in §3.8. By Lemma 3.8.1, $\lim_{\lambda \to 0} \rho_{\varphi(n)}(\lambda)$ exists in $\mathrm{TV}(\Sigma_2)$, but ρ_n has no converging subsequence as $\lambda \to 0$, contradicting properness.

$\Leftarrow$: The converse uses the Valuative Criterion of Properness [Har77, Ch. 2, Thm 4.7]. This criterion is very often used to prove properness in algebraic geometry, but I don't intend to cover it in this class. You can find the proof in [Ful93, Prop. on pg 39, §2.4].

In case you're already familiar with the valuative criterion of properness, I'll also sketch the proof here. Let R be a discrete valuation ring with field of fractions K, and suppose we have a maps

[1] The definition of properness for more general morphisms of schemes is much more technical [Har77, Ch. 2, §4, Def. on pg 100], but this won't be necessary for us.

$\alpha_* : \operatorname{Spec} K \to \mathrm{TV}(\Sigma_1)$ and $f : \operatorname{Spec} R \to \mathrm{TV}(\Sigma_2)$ such that the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{\alpha_*} & \mathrm{TV}(\Sigma_1) \\ \downarrow & & \downarrow \varphi_* \\ \operatorname{Spec} R & \xrightarrow{f} & \mathrm{TV}(\Sigma_2) \end{array}.$$

We want to show that there exists a (unique) lift $\widetilde{f} : \operatorname{Spec} R \to \mathrm{TV}(\Sigma_1)$ which can be inserted above to yield a still-commuting diagram. We can assume that α_* has image in T_{N_1}, hence is induced by a non-vanishing map $\Bbbk[M_1] \to K$, or equivalently, by a homomorphism $\alpha : M_1 \to K^*$. Then $\widetilde{f}|_{T_{N_1}}$ necessarily satisfies $i \circ \widetilde{f}^\sharp(z^m) = \alpha(m)$ for i the inclusion $R \hookrightarrow K$, and then f must satisfy $i \circ f^\sharp = \alpha \circ \varphi^* : M_2 \to K^*$. Let ν denote the discrete valuation on K associated to R. Then $\operatorname{Spec} R$ mapping to some U_{σ_2} for $\sigma_2 \in \Sigma_2$ means that $\operatorname{im} f^\sharp|_{U_{\sigma_2}} \subset i(R)$, hence $\nu(f^\sharp(z^m)) \in \mathbb{Z}_{\geq 0}$ for all $m \in \sigma_2^\vee \cap M_2$. I.e., $\nu \circ \alpha \circ \varphi^*$ can be identified with an element of $\sigma_2 \cap N_2$. The condition $\varphi^{-1}(|\Sigma_2|) = |\Sigma_1|$ implies that there exists a cone $\sigma_1 \in \Sigma_1$ such that $\varphi(\sigma_1) \ni \nu \circ \alpha \circ \varphi^*$, i.e., such that $\sigma_1 \ni \nu \circ \alpha$. So then $\nu \circ \alpha(m) \geq 0$ for all $m \in \sigma_1^\vee \cap M_1$; i.e., $\nu \geq 0$ on $\Bbbk[\sigma_1^\vee \cap M_1]$. So the map $\alpha^* : \Bbbk[\sigma_1^\vee \cap M_1] \to K$ has image contained in $i(R)$, hence yields to a map $\widetilde{f} : \operatorname{Spec} R \to U_{\sigma_1} \subset \mathrm{TV}(\Sigma_1)$ extending α_* and lifting f, as desired. □

Remark 4.1.3. Proposition 4.1.2 assumes that Σ is finite, i.e., includes only finitely many cones. If Σ is infinite, then $\mathrm{TV}(\Sigma)$ is never compact even if $|\Sigma| = N_{\mathbb{R}}$. We want to show that there exists a map $\operatorname{Spec} R \to \mathrm{TV}(\Sigma_1)$ which can be inserted above to yield a diagram which is still commutative. We can assume that the image of α lies in T_{N_1}, so α is induced by a map $\alpha : M_1 \to K^*$.

Example 4.1.4. The fibration $(\varphi_2)_* : \mathrm{TV}(\Sigma) \to \mathrm{TV}(\Sigma'')$ in Proposition 3.5.4 is proper.

4.1.1 Blowups and refinements of Σ

Let $N = \mathbb{Z}^r$ and consider a cone σ generated by a basis $\{v_1, \ldots, v_r\}$ for N. Let $v_0 = \sum_{i=1}^r v_i$. Let Σ be the fan consisting of all cones generated by subsets of $\{v_0, v_1, \ldots, v_r\}$ which do not include all of $\{v_1, \ldots, v_r\}$. I.e., Σ is a refinement of the cone σ. The identity map on N gives a map of fans $\Sigma \to \sigma$ (where by σ here we actually mean the fan consisting of the faces of σ), hence a map $p : \mathrm{TV}(\Sigma) \to U_\sigma$. This map is birational[2] and proper (by Proposition 4.1.2). In fact, we claim that $p : \mathrm{TV}(\sigma) \to U_\sigma$ is the blowup of $U_\sigma \cong \mathbb{A}^r$ at the origin.

To see this, let $\mathfrak{m} = \langle x_1, \ldots, x_r \rangle$ denote the maximal ideal associated to the origin, using the notation $x_i := z^{e_i^*}$. Then the graded algebra $S = \bigoplus_{d \geq 0} \mathfrak{m}^d$ can be identified with

$$\Bbbk[x_1, \ldots, x_r][tx_1, \ldots, tx_r] \subset \Bbbk[t, x_1, \ldots, x_r]$$

where $\deg(t) = 1$ and $\deg(x_i) = 0$ for each i. Here, the degree of a homogeneous element (i.e., the power of t) indicates power of $\mathfrak{m}$ that the element is contained in. So by definition (cf. §2.6), $\operatorname{Proj} S$ is the blowup of $\mathbb{A}^r$ at the origin, and the map to $\mathbb{A}^r$ is the map induced by $\Bbbk[x_1, \ldots, x_r] \xrightarrow{\sim} S_0$.

[2]In general, for $\varphi : N \xrightarrow{\sim} N'$ an isomorphism of lattices inducing a map of fans, the induced map φ_* between toric varieties is always birational since it restricts to an isomorphism of Zariski open subsets $T_N \xrightarrow{\sim} T_{N'}$.

On the other hand, the fan Σ is the normal fan associated to the polytope $\{m \in M_{\mathbb{R}} | \langle m, v_i \rangle \geq -1 \text{ for } i = 0, 1, \ldots, r\}$, or after some translation, the polytope[3]

$$K = \{m \in M_{\mathbb{R}} | \langle m, v_i \rangle \geq 0 \text{ for } i = 1, \ldots, r; \langle m, v_0 \rangle \geq 1\}.$$

Taking the cone over K, the monoid $CK \cap (M \oplus \mathbb{Z})$ is generated by the elements $(v_i^*, 0)$, $i = 1, \ldots, r$, together with the elements $(v_i^*, 1)$, $i = 1, \ldots, r$. Letting $x_i := z^{(v_i^*, 0)}$ and $tx_i = z^{(v_i^*, 1)}$, we thus identify $\Bbbk[CK \cap (M \oplus \mathbb{Z})]$ with the homogeneous ring S. Thus,

$$\mathrm{TV}(\Sigma) = \mathrm{Proj}\, \Bbbk[CK \cap (M \oplus \mathbb{Z})] = \mathrm{Proj}\, S$$

with the map to U_σ being induced by the inclusion of the degree 0 part. Thus, $\mathrm{TV}(\Sigma) \to U_\sigma$ is the blowup at the origin, as claimed.

Exercise 4.1.5. Let $N = \mathbb{Z}^2$, and let $\sigma = \mathbb{R}_{\geq 0}\langle (1, 0), (1, 3)\rangle$.

(a) What is $A_\sigma := \Bbbk[\sigma^\vee \cap M]$? Write your answer as a subring of $\Bbbk[x^{\pm 1}, y^{\pm 1}]$ (with the usual identifications).

(b) What is the maximal ideal $\mathfrak{m}$ associated to the origin in $U_\sigma := \mathrm{Spec}\, A_\sigma$? (By the origin, I mean the torus orbit O_σ).

(c) Suppose we refine σ by inserting a single ray $\mathbb{R}_{\geq 0}(1, 1)$ to obtain a fan Σ. Describe a convex rational polyhedron $K \subset M_{\mathbb{R}}$ whose normal fan is Σ; choose K so that it's vertices are $(0, 1)$ and a point on the x-axis.

(d) For K as above, identify $\mathrm{Proj}\, \Bbbk[CK \cap (M \oplus \mathbb{Z})]$ with the blowup of U_σ at some ideal $I \subset \mathfrak{m}$ by identifying the corresponding graded rings.

4.1.2 Resolution of singularities

As noted in §4.1.1, if Σ' is a refinement of the fan Σ, then the identity map on N induces a proper birational map $\mathrm{TV}(\Sigma') \to \mathrm{TV}(\Sigma)$. In general (cf. [Har77, Ch. 5, Rmk. 3.8.1]) the problem of **resolution of singularities** is, given a variety V, to find a proper birational morphism $f : V' \to V$ with V' nonsingular. For toric varieties, it turns out that it is always possible to find a resolution of singularities via refinement of the fan. This is described for toric surfaces via an explicit algorithm (related to the Euclidean algorithm and certain continued fractions, called Hirzebruch-Jung continued fractions) at the start of [Ful93, §2.6], and the general statement is [Ful93, Prop. on pg 48].

I will not review the general constructions here, but there are a couple special cases that I think are worth mentioning.

Example 4.1.6 (A_n-singularities)**.** Let $N = \mathbb{Z}^2$ and let $\sigma := \mathbb{R}_{\geq 0}\langle (1, 0), (1, n + 1)\rangle$. Then σ is a singular cone, but it can be refined into $n+1$ non-singular cones by inserting the rays $\rho_k = \mathbb{R}_{\geq 0}(1, k)$, $k = 1, \ldots, n$. This gives a (minimal) resolution of singularities for U_σ with n exceptional divisors D_{ρ_k}, each having self-intersection number -2. Note that the dual graph to the exceptional locus is the A_n-Dynkin diagram. The singular point in U_σ is therefore called an A_n-singularity (see `https://en.wikipedia.org/wiki/Du_Val_singularity`).

[3]More generally, refining a cone of Σ_K corresponds to "slicing" through the corresponding face of the associated normal polytope K.

Example 4.1.7 (The conifold singularity and the Atiyah flop)**.** Recall the conifold singularity from Example 3.4.6. Let σ be the singular cone introduced there. There are three natural ways to resolve σ:

- Add the cone $\mathbb{R}_{\geq 0}\langle e_1, e_2\rangle$ to get a fan Σ_1.
- Add the cone $\mathbb{R}_{\geq 0}\langle e_3, e_1 + e_2 - e_3\rangle$ to get a fan Σ_2.
- Add the ray generated by $v = e_1 + e_2$, and then add the four 2-dimensional cones generated by v and one of the original boundary rays. Call the resulting fan Σ_3.

Note that Σ_3 is a common refinement of Σ_1 and Σ_2 — the exceptional locus of $\mathrm{TV}(\Sigma_3) \to U_\sigma$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and this can be partially blown down to get just $\mathbb{P}^1$ in two different ways corresponding to Σ_1 and Σ_2. We thus have a commutative diagram of proper birational maps

$$\begin{array}{ccc} \mathrm{TV}(\Sigma_3) & \longrightarrow & \mathrm{TV}(\Sigma_2) \\ \downarrow & & \downarrow \\ \mathrm{TV}(\Sigma_1) & \longrightarrow & U_\sigma \end{array} .$$

The induced birational map $\mathrm{TV}(\Sigma_1) \dashrightarrow \mathrm{TV}(\Sigma_2)$ is called the Atiyah flop. Flops are a fundamental operation in the minimal model program.

4.2 T-invariant divisors

Let Σ be a fan in $N_{\mathbb{R}}$. Recall from §3.3 that each ray $\rho \in \Sigma^{[1]} \subset \Sigma$ corresponds to a prime Weil divisor $D_\rho = \overline{O}_\rho \subset \mathrm{TV}(\Sigma)$.

A Weil divisor in a toric variety is said to be T-invariant if it is mapped to itself by the action of the torus T_N. In particular, each boundary divisor D_ρ is T-invariant. More generally, the T-invariant Weil divisors are precisely the divisors of the form

$$\sum_{\rho \in \Sigma^{[1]}} a_\rho D_\rho.$$

for $a_\rho \in \mathbb{Z}$. We denote the set of T-invariant Weil divisors by $\mathrm{Weil}_T(\mathrm{TV}(\Sigma))$. Let $A_{n-1}(\mathrm{TV}(\Sigma))$ denote the group of T-invariant Weil divisors modulo linear equivalence.

We next wish to describe the T-invariant Cartier divisors, i.e., the group of locally principal T-invariant Weil divisors. These are the Weil divisors which are locally cut out by T-invariant rational functions; i.e., by monomials z^m.

Let us begin with the affine case. Let σ be a cone with rays $\rho_1, \ldots, \rho_k$ generated by primitive vectors $n_1, \ldots, n_k \in N$, respectively. An element $m \in M$ corresponds to a monomial z^m on U_σ, and the corresponding principal divisor is

$$(z^m) = \sum_{i=1}^{k} \langle n_i, m\rangle D_{\rho_i}. \tag{4.1}$$

One can show that every T-invariant Cartier divisor on U_σ has this form.[4] Furthermore, for $m' \in M$, note that $(z^m) = (z^{m'})$ if and only if $\langle n_i, m\rangle = \langle n_i, m'\rangle$ for each $i = 1, \ldots, k$; i.e., if and only if $m - m' \in \sigma^\perp$. Thus, the T-Cartier divisors of U_σ correspond to points in $M/(\sigma^\perp \cap M)$.

For notational convenience, if σ is any toric cone in $N_\mathbb{R}$, we write $M(\sigma) := \sigma^\perp \cap M$.

Now for more general Σ, a T-cartier divisor on $\mathrm{TV}(\Sigma)$ will be a T-Cartier divisor on each U_σ which is compatible with restrictions; i.e., a choice of $m_\sigma \in M/M(\sigma)$ for each σ such that if $\sigma' \subset \sigma$ (so $\sigma^\perp \subset (\sigma')^\perp$) then $m_{\sigma'}$ is the image of m_σ under the projection $M(\sigma) \to M(\sigma')$. In other words, the group $\mathrm{Div}_T \mathrm{TV}(\Sigma)$ of T-Cartier divisors on $\mathrm{TV}(\Sigma)$ is

$$\begin{aligned}\mathrm{Div}_T \mathrm{TV}(\Sigma) &= \varprojlim M/M(\sigma) \\ &= \ker\left(\bigoplus_i M/M(\sigma_i) \to \bigoplus_{i<j} M/M(\sigma_i \cap \sigma_j)\right)\end{aligned}$$

where σ_i are the maximal cones of Σ.

We say that a function on a subset of $N_\mathbb{R}$ is **integral** if it takes integer values on N. Note that the elements $m_i \in M/M(\sigma_i)$ can be viewed as the integeral linear functions on the cones σ_i. Lying in the kernel above exactly means that these linear functions glue to give a piecewise-linear function on $|\Sigma|$ which only bends along non-maximal cones of Σ; we call such functions **Σ-piecewise linear**. We thus obtain the following:

Proposition 4.2.1. $\mathrm{Div}_T \mathrm{TV}(\Sigma)$ *can be identified with the set of integral Σ-piecewise linear functions on $|\Sigma|$. More precisely, an integral Σ-piecewise linear function ψ_D on $|\Sigma|$ is associated to the T-Cartier divisor D given by*

$$D = \sum_{\rho_i \in \Sigma^{[1]}} -\psi_D(n_i) D_i. \tag{4.2}$$

Here, the sum is over rays ρ_i in Σ, n_i is a primitive generator for ρ_i, and $D_i := D_{\rho_i}$ is the corresponding divisor.

The sign convention in (4.2) is perhaps different from what (4.1) might suggest, but this is the convention used in [Ful93]. Leaving the minus sign out here would result in an awkward sign elsewhere.

Example 4.2.2. One can deduce from Proposition 4.2.1 that Σ is simplicial if and only if every Weil divisor D is $\mathbb{Q}$-Cartier (i.e., some integer multiple of D is a Cartier divisor).

4.3 Line bundles on toric varieties

Let us denote the divisor class group $\mathrm{Cl}(X)$ (i.e., Weil divisors modulo principal divisors) by $A_{n-1}(X)$. Recall from §2.7 that the group $\mathrm{CaCl}\, X$ of Cartier divisors modulo linear equivalence is isomorphic to $\mathrm{Pic}\, X$ (the group of invetible sheaves up to isomorphism. Since Cartier divisors are a subgroup of Weil divisors, we have an embedding $\mathrm{Pic}(X) \hookrightarrow A_{n-1}(X)$ which we know is an isomorphism if X is

[4] See the argument at the start of [Ful93, pg. 61]. Briefly, being T-invariant implies that the fractional ideal I corresponding to the divisor is M-graded, hence has a basis of monomials. Let $x_\sigma \subset U_\sigma$ be the intersection of all the boundary divisors. Being locally principal implies that the localization of I at $\mathfrak{m}_{x_\sigma}$ is principal, hence generated by some z^m. From this it can be deduced that I must be generated by a single monomial.

nonsingular (since then all Weil divisors are Cartier). Recall that the T-invariant Weil divisors are $\mathrm{Weil}_T(X) = \bigoplus_{i=1}^d \mathbb{Z} \cdot D_i$. The following proposition says that $\mathrm{Pic}(X)$ and $A_{n-1}(X)$ can in fact be constructed using only the T-invariant divisors.

Proposition 4.3.1. *Let Σ be a fan not contained in any proper subspace of $N_{\mathbb{R}}$. Let $X = \mathrm{TV}(\Sigma)$. Let $\rho_1, \ldots, \rho_d$ be the rays of Σ, and let $D_i = D_{\rho_i}$ be the corresponding boundary divisors. Then there is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \mathrm{Div}_T X & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & 0\,. \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & \bigoplus_{i=1}^d \mathbb{Z} \cdot D_i & \longrightarrow & A_{n-1}(X) & \longrightarrow & 0 \end{array} \tag{4.3}$$

Proof. Let D be a prime divisor with generic point in T_N. I.e., $D \cap T_N$ is a hypersurface in $T_N = \mathrm{Spec}\,\Bbbk[M]$, hence $D = (f)$ for some $f \in \Bbbk[M]$. So then we have a linear equivalence $D \sim \sum_i \eta_{D_i}(f) D_i \in \mathrm{Weil}_T(X)$. Hence, every divisor is linear equivalent to a divisor supported in the toric boundary, i.e., to a T-invariant Weil divisor. The surjectivity claims follow.

The maps from M are given by $m \mapsto -(z^m) = \sum_i -\langle m, n_i \rangle D_i$. The injectivity of these maps from M follows from assumption that Σ is not contained in any proper subspace of $N_{\mathbb{R}}$, hence the vectors n_i span $N_{\mathbb{R}}$. □

Remark 4.3.2. Note that the short exact sequence (3.7) from the quotient construction of toric varieties is the bottom short exact sequence in (4.3). In particular, the Abelian group A from (3.7) is $A_{n-1}(X)$.

Now let $D = \sum_i a_i D_i \in \mathrm{Div}_T(X)$, so the associate integeral Σ-piecewise linear function ψ_D is determined by $\psi_D(n_i) = -a_i$. Let P_D be the rational convex polyhedron in $M_{\mathbb{R}}$ defined by

$$\begin{aligned} P_D :&= \{m \in M_{\mathbb{R}} | \langle m, n_i \rangle \geq -a_i \forall i\} \\ &= \{m \in M_{\mathbb{R}} | m \geq \psi_D \text{ on } |\Sigma|\}. \end{aligned}$$

Remark 4.3.3. Let $Q_D \subset N_{\mathbb{R}}$ be the convex hull of

$$\left\{ \frac{1}{a_i} n_i | i = 1, \ldots, d \text{ such that } a_i \neq 0 \right\} \cup \{\rho_i | i = 1, \ldots, d \text{ such that } a_i = 0\} \cup \{0\}.$$

Recall the definition of a polar set (3.9). Note that P_D is exactly the polar set of Q_D:

$$P_D = Q_D^\circ.$$

Furthermore, since we included the origin before taking the convex hull, one can show that

$$Q_D = P_D^\circ.$$

Furthermore, assuming that D is effective (i.e., each $a_i \geq 0$), one can show that ψ_D is convex (in the sense that will be defined below) if and only if $Q_D = \{n \in N_{\mathbb{R}} | \psi_D(n) \geq -1\}$. Indeed, convexity of ψ_D exactly means that the element $m \in P_D$ which gives the minimal possible value at $n \in |\Sigma|$ (i.e., the element which is relevant to determining whether or not n is in P_D°) is precisely the m which is equal to ψ_D at n. (WARNING: I've corrected this remark multiple times. Hopefully it's correct now, but I'm not ready to promise it is).

Lemma 4.3.4. $\Gamma(X, \mathcal{O}(D)) = \bigoplus_{m \in P_D \cap M} \Bbbk \cdot z^m$.

Proof. Recall that

$$\eta_{D_i}(\sum c_m z^m) = \min_{c_m \neq 0} \langle m, n_i \rangle$$

and

$$\Gamma(X, \mathcal{O}(D)) = \{f \in K(X) | \eta_{D_i}(f) \geq -a_i \text{ for all } i \text{ and for all } c_m \neq 0\}.$$

Since $D \cap T_N = \emptyset$, the rational functions f here cannot have any poles in T_N, so we can restrict to $f \in \Bbbk[M]$. It now follows that $\Gamma(X, \mathcal{O}(D))$ is spanned by those z^m such that $\langle m, n_i \rangle \geq -a_i$ for all i; i.e., by those z^m with $m \in P_D \cap M$. $\square$

In fact, we can say more. For each $\sigma \in \Sigma$, let

$$\begin{aligned} P_D(\sigma) : &= \{m \in M_\mathbb{R} | \langle m, n_i \rangle \geq -a_i \text{ for all } n_i \in \sigma\} \\ &= \{m \in M_\mathbb{R} | m_\sigma \geq \psi_D|_\sigma\}. \end{aligned}$$

Then we have

$$\begin{aligned} \Gamma(U_\sigma, \mathcal{O}(D)) &= \{f \in \Bbbk[M] | \eta_{D_i}(f) \geq -a_i \text{ for all } \rho_i \in \sigma\} \\ &= \bigoplus_{\substack{m \in M \\ \langle m, n_i \rangle \geq -a_i \ \forall \ n_i \in \sigma}} \Bbbk \cdot z^m \\ &= \bigoplus_{m \in P_D(\sigma) \cap M} \Bbbk \cdot z^m. \end{aligned}$$

Assume that σ is rank(N)-dimensional, and let $m_\sigma = \psi_D|_\sigma$. I.e., $m_\sigma(n_i) = -a_i$ for all $n_i \in \sigma$. Then

$$P_D(\sigma) = m_\sigma + \sigma^\vee.$$

Hence,

$$\Gamma(U_\sigma, \mathcal{O}(D)) = z^{m_\sigma} \cdot \mathcal{O}_X(U_\sigma).$$

Note that z^{m_σ} extends to a global section of $\mathcal{O}(D)$ if and only if $m \in P_D$; i.e., if and only if $m_\sigma \geq \psi_D$ on all of $|\Sigma|$. So, assuming that all maximal cones of Σ are rank(N)-dimensional, we see that $\mathcal{O}(D)$ is generated by global sections if and only if, for each maximal cone σ, $m_\sigma := \psi_D|_\sigma$ is $\geq \psi_D$ on all $|\Sigma|$; i.e., if and only if, for each pair of maximal cones $\sigma_1, \sigma_2 \in \Sigma$,

$$m_{\sigma_1}|_{\sigma_2} \geq m_{\sigma_2}|_{\sigma_2} \tag{4.4}$$

I.e, the graph of ψ_D lies under the graph of each m_σ. An integral piecewise-linear function satisfying this is called **convex**. If the inequality in (4.4) is a strict inequality whenever $\sigma_1 \neq \sigma_2$, then the function is called **strictly convex**.

In summary:

Proposition 4.3.5. *Assume that all maximal cones of* Σ *are* rank(N)*-dimensional. Let* $D \in \mathrm{Div}_T(\mathrm{TV}(\Sigma))$. *Then* $\mathcal{O}(D)$ *is generated by global sections if and only if* ψ_D *is convex.*

Recall from §2.5.2 that $\mathcal{O}(D)$ being generated by global sections means that the associated map $\varphi_D := \varphi_{\mathcal{O}}(D) : \mathrm{TV}(\Sigma) \dashrightarrow \mathbb{P}(\Gamma(X, \mathcal{O}(D))$ is defined on all of $\mathrm{TV}(\Sigma)$. Writing $P_D \cap M = \{m_1, \dots, m_r\}$, this map φ_D can be written as

$$\begin{aligned} \varphi_D : \mathrm{TV}(\Sigma) &\to \mathbb{P}^{r-1} = \operatorname{Proj} \Bbbk[T_1, \dots, T_r] \\ z^{m_i} &\leftarrow\!\shortmid T_i. \end{aligned}$$

Now recall from Defintion 2.5.5 that $\mathcal{O}(D)$ is called very ample if this morphism φ_D is an embedding.

Proposition 4.3.6. *Suppose $|\Sigma| = N_{\mathbb{R}}$. Then D is very ample if and only if ψ_D is strictly convex and for all maximal cones $\sigma \in \Sigma$, the semigroup $S_\sigma = \sigma^\vee \cap M$ is generated by $\{m - m_\sigma | m \in P_D \cap M\}$.*

Proof. First, assuming that ψ_D is convex, note that the condition on the semigroups S_σ actually implies that ψ_D is *strictly* convex. Indeed, if $m_{\sigma_1} = m_{\sigma_2}$, then the sets $\{m - m_{\sigma_i} | m \in P_D \cap M\}$ for $i = 1, 2$ are the same, so $S_{\sigma_1} = S_{\sigma_2}$, hence $\sigma_1 = \sigma_2$. We may therefore replace the strict convexity in the statement of the proposition with convexity.

Now, whether we assume that $\mathcal{O}(D)$ is very ample or that φ_D is convex, it follows that $\mathcal{O}(D)$ is generated by global sections, hence that each m_σ is contained in $P_D \cap M$. Let i_σ denote the index such that $m_{i_\sigma} = m_\sigma$.

Consider the distinguished open subset where z^{m_σ} is nonzero:

$$\mathbb{A}_\sigma := D_+(z^{m_\sigma}) = \operatorname{Spec} \Bbbk\left[\frac{T_1}{T_{i_\sigma}}, \dots, \frac{T_r}{T_{i_\sigma}}\right] \cong \Bbbk^{r-1}.$$

Then $\varphi_D^{-1}(\mathbb{A}_\sigma)$ consists of the points in $\mathrm{TV}(\Sigma)$ where $z^{m_j - m_\sigma}$ is finite for all j. This always includes all of T_N (since all monomials are regular on T_N). More generally, for each cone $\tau \in \Sigma$, $\varphi_D^{-1}(\mathbb{A}_\sigma)$ it includes U_τ if and only if $m_j - m_\sigma \in \tau^\vee$; i.e., if and only if

$$m_j|_\tau \geq m_\sigma|_\tau \qquad \text{for all } j = 1, \dots, r. \tag{4.5}$$

Convexity of ψ_D implies that (4.5) holds if and only if $\tau \subset \sigma$; i.e., if and only if $U_\tau \subset U_\sigma$, and so ψ_D strictly convex implies that $\varphi_D^{-1}(\mathbb{A}_\sigma) = U_\sigma$ for all maximal σ. Conversely, (4.5) holding for all σ and $\tau = \sigma$ is the definition of convexity of ψ_D.

We now have morphisms $U_\sigma \to \mathbb{A}^{r-1}_{\Bbbk}$ for all maximal cones σ via $\frac{T_i}{T_{i_\sigma}} \mapsto z^{m_i - m_\sigma}$. This is an embedding if and only if the $z^{m_i - m_\sigma}$, $i = 1, \dots, r$, generate $\Bbbk[S_\sigma] = \Bbbk[\sigma^\vee \cap M]$ over $\Bbbk$; i.e., if and only if $\{m - m_\sigma | m \in P_D \cap M\}$ generates S_σ, as desired. ∎

Now recall from Definition 2.5.5 that D is called ample if and only if kD is very ample for some positive integer k.

Proposition 4.3.7. *Suppose $|\Sigma| = N_{\mathbb{R}}$. Then a T-Cartier divisor D is ample if and only if ψ_D is strictly convex.*

Proof. $\Longrightarrow$: If D is ample, then there is some positive integer k such that kD is very ample, hence ψ_{kD} is strictly convex by Proposition 4.3.6.

$\Longleftarrow$: The semigroup $S_\sigma = \sigma^\vee \cap M$ is finitely generated. Let $m \in S_\sigma$ be a generator. We wish to show that $m' := m + km_\sigma \in P_{kD} \cap M$ for some positive integer k, because then $m = m' - km_\sigma$ for $m' \in P_{kD} \cap M$, and so the condition on S_σ from Proposition 4.3.6 is satisfied for kD. By strict convexity, if $n_i \notin \sigma$, then $\langle m_\sigma, n_i \rangle > -a_i$, hence $\langle m + km_\sigma, n_i \rangle \geq -ka_i$ for sufficiently large k. We also have $\langle m + km_\sigma, n_i \rangle \geq -ka_i$ for $n_i \in \sigma$ because $m \in \sigma^\vee$ and $\langle m_\sigma, n_i \rangle = -a_i$. So we can indeed find a positive integer k such that $m + km_\sigma \in P_{kD} \cap M$, as desired. □

Definition 4.3.8. Let Σ be a fan in $N_\mathbb{R}$, and let ψ be an integral Σ-piecewise linear function on $|\Sigma|$. Let σ_1, σ_2 be two r-dimensional cones of Σ (for $r = \operatorname{rank} N$) whose intersection $\tau \in \Sigma$ has dimension $r-1$. Let u_τ be the unique primitive element of M which vanishes along τ and is positive on σ_2. Using the notation $m_{\sigma_i} = \psi|_{\sigma_i}$, we define the **bending parameter** $b_\tau \in \mathbb{Z}$ of ψ along τ by[5]

$$b_\tau u_\tau = m_{\sigma_2} - m_{\sigma_1}. \tag{4.6}$$

Morally, b_τ is a discrete version of the second derivative of ψ along τ.

Exercise 4.3.9. Let Σ be a fan in $N_\mathbb{R}$.

(a) For $\sigma_1, \sigma_2 \in \Sigma$ and $\tau = \sigma_1 \cap \sigma_2$ as in Definition 4.3.8, show that ψ is convex on $\sigma_1 \cup \sigma_2$ if and only if $b_\tau \leq 0$. Show ψ is strictly convex on $\sigma_1 \cup \sigma_2$ if and only if $b_\tau < 0$.

(b) Now suppose[6] $|\Sigma| = N_\mathbb{R}$. Prove that ψ is convex (resp., strictly convex) if and only if each bending parameter b_τ is non-positive (resp., negative). **Hint:** let $\gamma : [0,1] \to N_\mathbb{R}$. $\gamma(t) = x + tn$ be a line segment in $|\Sigma| \setminus \bigcup_{\dim(\sigma) < r-1} \sigma$ with x in the interior of a maximal cone σ and $n \in N_\mathbb{R}$, and prove that $m_\sigma - \psi$ is non-decreasing on γ (and increases whenever γ crosses a wall with negative bending parameter). Note that for $t \ll 1$, $(m_\sigma - \psi)[\gamma(t)] = 0$, and when crossing a wall, $(m_\sigma - d\psi_{\gamma(t)})(v)$ increases (or at least doesn't decrease); here, $d\psi_{\gamma(t)}$ denotes the element of M which locally equals ψ near $\gamma(t)$.

(c) Edit: I've decided to move part (c) to the new Exercise 4.4.5 at the end of §4.4.

Example 4.3.10. We demonstrate these concepts in an example (illustrated in Figure 4.3.1). Let's consider the invertible sheaf $\mathcal{O}_X(2)$ on $X = \mathbb{P}^2$, viewed as a toric variety with fan Σ as in Example 3.2.3; i.e., $N = \mathbb{Z}^2$ and the rays are $\rho_1 = \mathbb{R}_{\geq 0}(1,0)$, $\rho_2 = \mathbb{R}_{\geq 0}(0,1)$, and $\rho_3 = \mathbb{R}_{\geq 0}(-1,-1)$. Let us understand $\mathcal{O}_X(2)$ using toric geometry. Note that $\mathcal{O}_X(2) \cong \mathcal{O}_X(D)$ where D is a conic or a pair of lines. Let's choose $D = D_2 + D_3$ (the y-axis plus the axis at infinity). Then ψ_D take the value 0 at $(1,0)$, -1 at $(0,1)$ and -1 at $(-1,-1)$, and is linear on cones of Σ. You may be able to picture ψ_D and see intuitively that it is strictly convex. Indeed, all the bending parameters are -2, as can be checked directly or seen using intersection numbers as in Exercise 4.4.5 (the intersection numbers $[D_i].[D_j]$ equal 1 for each $i,j = 1,2,3$). This reaffirms our knowledge that $\mathcal{O}_X(2)$ is ample.

Now let's find the polytope P_D. Using the standard coordinates x, y on $M_\mathbb{R} \cong \mathbb{R}^2$ (and identifying this with $N_\mathbb{R} \cong \mathbb{R}^2$ using the standard inner product), this polytope is defined by

$$x \geq 0, \quad y \geq -1, \quad x + y \leq 1.$$

There are 6 lattice points in here:

$$P_D \cap M = \{(0,1), (0,0), (1,0), (0,-1), (1,-1), (2,-1)\}$$

[5] Note that b_τ is well-defined independent of the labelling of σ_1, σ_2, because reversing the labelling would change the sign of both u_τ and the right-hand side of (4.6).

[6] Or more generally, just assume that $|\Sigma| \setminus \bigcup_{\dim(\sigma) < r-1} \sigma$ is path-connected.

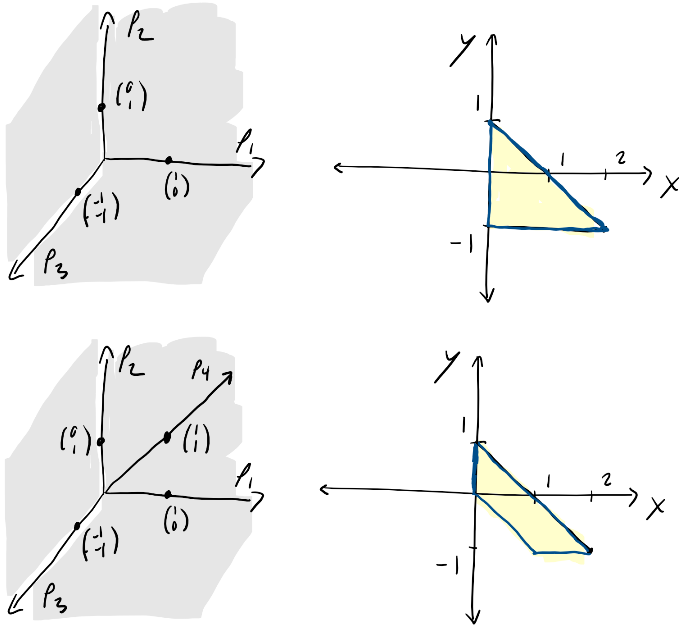

Figure 4.3.1: Top left: The fan for $X = \mathbb{P}^2$. Top right: The polytope P_D for $D = D_2 + D_3 \in \mathrm{Div}(\mathbb{P}^2)$. Bottom left: The fan for X' equal to $\mathbb{P}^2$ blown up at a point. Bottom right: The polytope $P_{\widetilde{D}}$ for $\widetilde{D} = D_2 + D_3 \in \mathrm{Div}(X')$.

corresponding to a $\Bbbk$-basis $\{y, 1, x, y^{-1}, xy^{-1}, x^2y^{-1}\}$ for the global sections $\Gamma(X, \mathcal{O}_X(D_2 + D_3))$. We obtain an embedding $\varphi_D : X \to \mathbb{P}^5 = \mathrm{Proj}\, \Bbbk[t_0, \ldots, t_5]$ via mapping the 6 homogeneous coordinates t_i to the 6 basis elements above.

Now suppose we blow up the point $O_{\sigma_{12}}$ for $\sigma_{12} = \mathbb{R}_{\geq 0}\langle(1,0),(0,1)\rangle$. This corresponds to refining σ_{12} by adding a new ray $\mathbb{R}_{\geq 0}(1,1)$. Let X' be the corresponding toric variety. The proper transform $\widetilde{D}$ of D from before is still $D_2 + D_3$ (now viewed as being in $\mathrm{Div}(X')$). The associated piecewise-linear function $\psi_{\widetilde{D}}$ takes the same values as before on ρ_i, $i = 1, 2, 3$, but now $\psi_{\widetilde{D}}(1,1) = 0$. This $\psi_{\widetilde{D}}$ is still strictly convex: the bending parameters on ρ_2 and ρ_3 are still -2, while the bending parameters on ρ_1 and ρ_4 are each -1. The blowup affects the polytope $P_{\widetilde{D}}$ by "slicing off" the bottom left corner, adding the new condition $x + y \geq 0$ (this is true more generally; refining the fan corresponds to chopping off lower-dimensional faces of the polytope). This removes a single lattice point $(0, -1)$ from $P_D \cap M$, so now $\varphi_{\widetilde{D}} : X' \to \mathbb{P}^4 = \mathrm{Proj}\, \Bbbk[t_0, \ldots, t_4]$ with the t_i's mapping to the 5 basis elements $\{y, 1, x, xy^{-1}, x^2y^{-1}\}$ for the global sections $\Gamma(X', \mathcal{O}_{X'}(\widetilde{D}))$.

4.4 Chow groups

Here I'll follow [Ful93, §5.1]. See [Har77, Appendix A] for more details on intersection theory. Or if you really want to learn intersection theory in great generality and detail, the canonical reference is [Ful98]. As in [Ful93], lets us start using the notation $V(\sigma)$ for the orbit closure $\overline{O_\sigma}$.

On a variety X, the Chow group $A_k(X)$ is defined to be the free Abelian group generated by

k-dimensional irreducible closed subvarieties of X (called **cycles**), modulo the subgroup generated by cycles of the form (f) for f a nonzero rational function on a $(k+1)$-dimensional subvariety Y of X and (f) the corresponding principal divisor from Y embedded into X (i.e., modulo **rational equivalence**).

In particular, for $r = \operatorname{rank}(N)$ and $X = \operatorname{TV}(\Sigma)$, we saw in Proposition 4.3.1 that $A_{r-1}(X)$ is generated by the T_N-invariant Weil divisors $\operatorname{Weil}_T(X) = \bigoplus_{\rho_i \in \Sigma[1]} \mathbb{Z} \cdot D_{\rho_i}$. The following generalization is true as well:

Proposition 4.4.1. *The Chow group $A_k(X)$ of a toric variety $X = \operatorname{TV}(\Sigma)$ is generated by the classes of the orbit closures $V(\sigma)$ for the $(r-k)$-dimensional cones $\sigma \in \Sigma$.*

To prove Proposition 4.4.1, we will use the following two general facts regarding Chow groups:

Lemma 4.4.2. *$A_k(\mathbb{A}^r)$ is equal to $\mathbb{Z} \cdot [\mathbb{A}^r]$ if $k = r$ and 0 if $k \neq r$.*

Lemma 4.4.3. *Let X be a variety, Y a closed subvariety of X, and $U = X \setminus Y$. Then the following sequence is exact for each k:*

$$A_k(Y) \to A_k(X) \to A_k(U) \to 0.$$

The first map here is via inclusion, and the second is via restriction (i.e., intersection with U)

Proof of Prop 4.4.1. Let $r = \dim X$. Define

$$X_i := \bigcup_{\substack{\sigma \in \Sigma \\ \dim(\sigma) \geq r-i}} V(\sigma) \subset X.$$

This gives a filtration $X = X_r \supset X_{r-1} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset$ by closed subschemes. Note that

$$X_i \setminus X_{i-1} = \bigcup_{\substack{\sigma \in \Sigma \\ \dim(\sigma) = r-i}} O_\sigma.$$

So by Lemma 4.4.3, we have the exact sequence

$$A_k(X_{i-1}) \to A_k(X_i) \to \bigoplus_{\dim \sigma = r-i} A_k(O_\sigma) \to 0. \tag{4.7}$$

To compute $A_k(O_\sigma)$ for $\dim(\sigma) = r - i$, we can view O_σ as an open subset of $\mathbb{A}^i$. Then by Lemma 4.4.3 again, we have

$$A_k(\mathbb{A}^i \setminus O_\sigma) \to A_k(\mathbb{A}^i) \to A_k(O_\sigma) \to 0.$$

By Lemma 4.4.2, it follows that $A_k(O_\sigma) = \mathbb{Z} \cdot [O_\sigma]$ if $k = i$ and $A_k(O_\sigma) = 0$ for $k \neq i$.

We now proceed by induction on i. Suppose that $A_k(X_{i-1})$ is generated by classes $V(\sigma)$ for $\dim \sigma = r - k$ (for $i = 0$ this is trivial since $X_{-1} = \emptyset$ and $A_k(\emptyset) = 0$). If $k \neq i$, then each $A_k(V(\sigma))$ summand in (4.7) is 0, hence $A_k(X_{i-1})$ surjects onto $A_k(X_i)$ and so $A_k(X_i)$ is also generated by the classes $V(\sigma)$ for $\dim \sigma = r - k$. On the other hand, if $k = i$, then since $\dim X_{i-1} = i - 1$, we have $A_k(X_{i-1}) = 0$. So then (4.7) implies that $A_k(X_i) \cong \bigoplus_{\dim \sigma = r-i} \mathbb{Z} \cdot [O_\sigma]$. The isomorphism here is via restriction, and restriction maps $[V(\sigma)]$ to $[O_\sigma]$, so we have $A_k(X_i) = \bigoplus_{\dim \sigma = r-i} \mathbb{Z} \cdot [V(\sigma)]$. In particular, $A_k(X_i)$ is generated by the classes $[V(\sigma)]$ with $\dim \sigma = r - i = r - k$, as desired. □

On a non-singular quasi-projective variety X over an algebraically closed field, one denotes $A^k(X) := A_{r-k}(X)$ and then defines an intersection product

$$A^p(X) \otimes A^q(X) \to A^{p+q}(X)$$

which makes $A^*(X) := \bigoplus_{p=0}^{\dim X} A^p(X)$ into a commutative graded ring with identity $[X]$.

Two subvarieties Y and Z are said to meet **properly**[7] if every irreducible component of $Y \cap Z$ has codimension $\operatorname{codim}(Y) + \operatorname{codim}(Z)$. In this case, the intersection product $[Y].[Z]$ is a $\mathbb{Z}$-linear combination of the irreducible components of $Y \cap Z$. If D is a (Cartier) divisor meeting a subvariety V properly, then the intersection product $[D].[V]$ is given by $[D|_V]$, i.e., by the (inclusion in X of the) Cartier divisor in V given by restricting the local defining equations of D to V. If D' is linearly equivalent to D, then $[D'].[V]$ is rationally equivalent to $[D].[V]$.

Now suppose $X = \mathrm{TV}(\Sigma)$ is a nonsingular toric variety. If $\sigma_1, \sigma_2 \in \Sigma$, then

$$V(\sigma_1) \cap V(\sigma_2) = \begin{cases} V(\gamma) & \text{if } \sigma_1 \text{ and } \sigma_2 \text{ span a cone } \gamma; \\ \emptyset & \text{otherwise.} \end{cases}$$

The intersection of $V(\sigma_1)$ and $V(\sigma_2)$ is proper exactly when $\dim(\gamma) = \dim(\sigma_1) + \dim(\sigma_2)$ (or trivially if the intersection is empty), and in this case, the intersection product is

$$[V(\sigma_1)].[V(\sigma_2)] = [V(\gamma)].$$

So by induction, if γ is the cone bounded by rays $\rho_1, \ldots, \rho_k$, then

$$[V(\gamma)] = [D_{\rho_1}]. \ldots .[D_{\rho_k}].$$

All intersections product computations are thus reduced to the case of divisors.

If ρ is a ray and σ a cone such that D_ρ and $V(\sigma)$ do not intersect properly, then ρ is a face of σ. In this case, $[D_\rho].[V(\sigma)]$ can be computed by replacing D_ρ with a linearly equivalent divisor whose summands *do* intersect $V(\sigma)$ properly. Indeed, by the non-singularity assumption, the generators of σ form part of a basis, so we can find some m such that $\langle m, n_\rho \rangle = 1$ (for n_ρ the primitive generator of ρ) while $m \in (\rho')^\perp$ for each of the other rays ρ' in σ. Then

$$\sum_{\substack{\tau \in \Sigma^{[1]} \\ \tau \neq \rho}} -\langle m, n_\tau \rangle D_\tau$$

is linearly equivalent to D_ρ, but now each component intersects $V(\sigma)$ properly.

Remark 4.4.4. If Σ is only simplicial, one can still define an intersection product making

$$A^*(X)_{\mathbb{Q}} = \bigoplus_p A^p(X) \otimes \mathbb{Q}$$

into a commutative graded $\mathbb{Q}$-algebra with identity $[X]$ (this is true for all smooth orbifolds). The intersections can again be reduced to intersections of divisors, though now these must be computed with multiplicities (see the discussion on [Ful93, pg 97-98 and pg 100]). Alternatively, one can refine Σ to make it non-singular and then take advantage of properties of the intersection product under pullback and pushforward.

[7] Chow's moving lemma states that for Y, Z, cycles in a quasiprojective variety X, there exist a cycle Z' rationally equivalent to Z such that Y and Z intersect properly. So all intersection products on such X can be reduced to the case of proper intersections.

Exercise 4.4.5. (a) By part (b) of Exercise 4.3.9, a line bundle $\mathcal{O}(D)$ is generated by global sections (resp., ample) if and only if ψ_D has non-positive (resp., negative) bending parameters. Note that each codimension-one cone τ corresponds to a curve $C_\tau \cong \mathbb{P}^1 \subset \mathrm{TV}(\Sigma)$. Assuming that Σ is nonsingular, prove that the bending parameter b_τ of ψ_D along τ is equal to the negative **intersection number**[8] $-[D].[C_\tau] \in \mathbb{Z}$.[9]

(b) Let Σ be a complete non-singular fan and consider $X = \mathrm{TV}(\Sigma)$. Let K be the lattice from (3.6); i.e., K is the lattice of relations between the primitive generators n_i of the rays $\rho_i \in \Sigma^{[1]}$ (i.e., elements of K correspond to linear combinations $\sum_i a_i n_i$ which equal 0). Let $N^1(X)$ denote the lattice of curves in X considered up to numerical equivalence.[10] Prove that $N_1(X) \cong K$ via

$$[C] \mapsto \sum_{\rho_i \in \Sigma^{[1]}} [C].[D_{\rho_i}] n_i. \tag{4.8}$$

Hint: Recall from Proposition 4.4.1 that $N^1(X)$ is generated by $[C_\tau]$ for $\dim \tau = r - 1$, and check that each $[C_\tau]$ gives a relation via (4.8). On the other hand, if we fix σ_0 generated by a basis $e_1, \ldots, e_r$, then for any other σ and path γ from σ_0 to σ crossing codimension-one cones $\tau_1, \ldots, \tau_k$ and maximal cones $\sigma_0, \sigma_1, \ldots, \sigma_k = \sigma$, the relations associated to the classes $[C_{\tau_i}]$ can be used inductively express the generators of each σ_i in terms of the basis $e_1, \ldots, e_r$; thus, these relations coming from the classes $[C_\tau]$ span all of K.

4.5 Moment maps

Notation: For A, B semigroups and $0 \in B$ an element satisfying $0 \cdot b = 0$ for all $b \in B$, let us denote $\mathrm{Hom}'_{\mathrm{sg}}(A, B) := \mathrm{Hom}_{\mathrm{sg}}(A, B) \setminus \{0\}$ (i.e., the nonzero semigroup homomorphisms).

Recall that for each toric cone σ in $N_\mathbb{R}$, the corresponding affine toric variety U_σ can be viewed as $\mathrm{Hom}'_{\mathrm{sg}}(S_\sigma, \Bbbk)$ for $S_\sigma = \sigma^\vee \cap M$. This construction is compatible with inclusions of faces of cones $\tau \subset \sigma \rightsquigarrow U_\tau \subset U_\sigma$. Then for a fan Σ, these inclusions are used to glue the affine varieties U_σ for $\sigma \in \Sigma$ to construct $\mathrm{TV}(\Sigma)$.

More generally, we can replace the field $\Bbbk$ with other semigroups. In particular, take $\Bbbk = \mathbb{C}$ and consider $\mathbb{R}_{\geq 0} \subset \mathbb{C}$, viewed as a semigroup under multiplication. We consider

$$(U_\sigma)_{\geq 0} := \mathrm{Hom}'_{\mathrm{sg}}(S_\sigma, \mathbb{R}_{\geq 0}) \subset U_\sigma = \mathrm{Hom}'_{\mathrm{sg}}(\sigma^\vee \cap M, \mathbb{C}).$$

These spaces glue to yield $\mathrm{TV}(\Sigma)_{\geq 0} \subset \mathrm{TV}(\Sigma)$. If σ is nonsingular of dimension k and $r = \operatorname{rank} N$, then $U_\sigma \cong (\mathbb{R}_{\geq 0})^k \times \mathbb{R}^{r-k}$.

The retraction $\mathbb{C} \to \mathbb{R}_{\geq 0}$, $z \mapsto |z|$ induces a retraction $U_\sigma \to (U_\sigma)_{\geq 0}$ for each σ. Furthermore, these retractions are compatible with the gluing, thus yielding a retraction

$$\mathrm{TV}(\Sigma)_{\geq 0} \subset \mathrm{TV}(\Sigma) \to \mathrm{TV}(\Sigma)_{\geq 0}.$$

[8] To define the intersection number, note that for X connected, $A^{\dim X}(X) \cong \mathbb{Z}$ with the class of a point mapping to $1 \in \mathbb{Z}$. The the "intersection number" of a collection of classes in $A^*(X)$ is obtained by taking the intersection product and then projecting to $A^{\dim X}(X) \cong \mathbb{Z}$.

[9] Since the classes $[C_\tau]$ generate the cone of effective curve classes $\mathrm{NE}(\mathrm{TV}(\Sigma))$ (I learned this from [HKK+03, Thm. 7.4.4] available here, which references [Rei83, Prop. 1.6]), this gives an example of Kleiman's criterion for ampleness.

[10] Two curve classes $[C_1], [C_2] \in A_1(X)$ are said to be numerically equivalent if $[C_1].[D] = [C_2].[D]$ for all divisor classes $[D] \in A^1(X)$. Then $N_1(X) = A_1(X)/\sim$ where $\sim$ denotes numerical equivalence. For simplicial toric varieties, it turns out that the intersection pairing is a perfect pairing [Ful93, Exercise on pg 104-105] so in fact $N_1(X) = A_1(X)$. But in general, numerically equivalent cycles need not be rationally equivalent.

Example 4.5.1. For $\sigma = \{0\}$, $U_\sigma = N \otimes \mathbb{C}^*$, $(U_\sigma)_{\geq 0} = N \otimes \mathbb{R}_{>0} \xrightarrow{\sim} N \otimes \mathbb{R}$ (the last isomorphism being induced by the log map), and the retraction is given by

$$\sum n \otimes \lambda \mapsto \sum n \otimes |\lambda| \in N \otimes \mathbb{R}_{>0} \mapsto n \otimes \log|\lambda| \in N \otimes \mathbb{R}.$$

Identifying U_σ with $(\mathbb{C}^*)^r$, this retraction to $N \otimes \mathbb{R}$ can be expressed as $(x_1, \ldots, x_r) \mapsto (\log|x_1|, \ldots, \log|x_r|)$.

Example 4.5.2. Let $X = \mathbb{P}^r_{\mathbb{C}} = \mathrm{TV}(\Sigma)$ for Σ as in Example 3.2.3. Let σ_i be the unique maximal cone of σ not containing the ray ρ_i. In homogeneous coordinates $(x_0 : \ldots, x_r)$ for $\mathbb{P}^r$, D_{ρ_i} can be identified with the locus $x_i = 0$, and U_{σ_i} with the locus $x_i \neq 0$ (which via the scaling action can be identified with $x_i = 1$). Then $(U_{\sigma_i})_{\geq 0}$ corresponds to points $(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_r)$ with each $x_i \in \mathbb{R}_{\geq 0}$. Gluing yields

$$\begin{aligned}\mathbb{P}^r_{\geq} := \mathrm{TV}(\Sigma)_{\geq} &= (\mathbb{R}^{r+1}_{\geq 0} \setminus \{0\})/\mathbb{R}_{>0} \\ &= \{(x_0, \ldots, x_r) \in \mathbb{R}^{r+1}_{\geq 0} : x_0 + \ldots + x_r = 1\}.\end{aligned}$$

I.e., $\mathbb{P}^r_{\geq}$ is the standard r-simplex. The retraction $\mathbb{P}^r \to \mathbb{P}^r_{\geq}$ is given by

$$(x_0 : \ldots : x_r) \mapsto \frac{1}{\sum_{i=0}^r |x_i|}(x_0, \ldots, x_r). \tag{4.9}$$

The fiber over a point $(a_0, \ldots, a_r)$ is the compact torus $(S^1)^k$ where $k+1$ is the number of nonzero coordinates $\#\{i | a_i \neq 0\}$; i.e., the fiber over a point in the interior of a k-dimensional face is $(S^1)^k$. See Figure 4.5.2

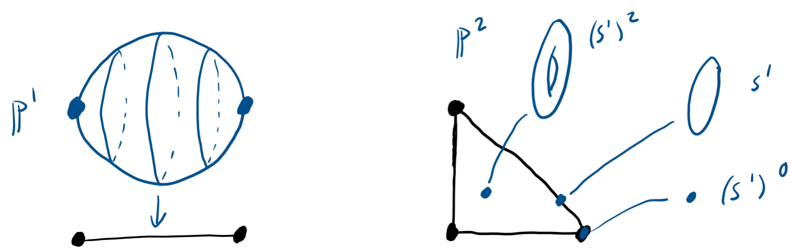

Figure 4.5.2: Left: The retraction of $\mathbb{P}^1$ to $\mathbb{P}^1_{\geq} \cong [0,1]$. Generic fibers are circles, while fibers over the boundary are points. Right: $\mathbb{P}^2$ retracts onto a closed triangle. Fibers over interior points are $(S^1)^2$. Fibers over points in the relative interiors of the edges are S^1. Fibers over vertices are points. Each orbit closure $V(\sigma)$ fibers over a face F with $\dim F = \operatorname{codim} \sigma$. E.g., the preimages of the edges are the boundary divisors, and the preimages of the vertices are O_σ for σ maximal. The big torus orbit fibers over the interior. Note that the boundary divisors are copies of $\mathbb{P}^1$, so their retractions to the edges are just cases of the example on the left where $\mathbb{P}^1$ retracts to a closed interval.

More generally, the algebraic torus $T_N \cong (\mathbb{C}^*)^r$ contains the compact torus $S_N \cong (S^1)^r$ via

$$S_N := \mathrm{Hom}(M, S^1) \subset \mathrm{Hom}(M, \mathbb{C}^*) = T_N$$

(equivalently, identifying T_N with $(\mathbb{C}^*)^r$, S_N corresponds to the points $(x_1, \ldots, x_r)$ with $|x_i| = 1$ for each i). In fact, we can write identify

$$\begin{aligned} T_N &= S_N \times \operatorname{Hom}(M, \mathbb{R}_{>0}) \\ &= S_N \times \operatorname{Hom}(M, \mathbb{R}) \\ &= S_N \times N_{\mathbb{R}} \end{aligned}$$

where the second equality is via the identification $\log : \mathbb{R}_{>0} \to \mathbb{R}$. One can deduce the following (cf. [Ful93, Prop. on pg 79]):

Proposition 4.5.3. *The retraction* $\mathrm{TV}(\Sigma) \to \mathrm{TV}(\Sigma)_{\geq 0}$ *identifies* $\mathrm{TV}(\Sigma)_{\geq 0}$ *with the quotient space of* $\mathrm{TV}(\Sigma)$ *under the* S_N*-action.*

The explicit contraction of (4.9) in fact extends much more generally. Let $K \subset M_{\mathbb{R}}$ be a convex integral polytope. As in §3.7, we consider the associated projective variety $X := \operatorname{Proj} S_K$ for $S_K = \Bbbk[CK \cap (M \oplus \mathbb{Z})]$. Consider the map

$$\begin{aligned} \mu : X &\to M_{\mathbb{R}} \\ x &\mapsto \frac{1}{\sum_{m \in K \cap M} |z^m(x)|} \sum_{m \in K \cap M} |z^m(x)| m. \end{aligned}$$

The map μ is S_N-invariant because $|z^m(t)| = 1$ for all $t \in S_N$ and all $m \in M$, so μ factors through $X/S_N = X_{\geq}$. In fact:

Proposition 4.5.4. *The map* μ *induces a homeomorphism from* $X_{\geq}$ *to* K*.*

Furthermore, the preimage μ^{-1} of the interior of a face $F \subset K$ is the corresponding torus orbit O_σ for $\sigma^\vee = C_F K$ as in §3.7. I'll just refer to [Ful93, §4.2] for the proof of Proposition 4.5.4 (this is a bit involved).

Proposition 4.5.4 remains true if we modify μ by using any subset of points $m \in K \cap M$ which includes the vertices of K (i.e., only the convex hull matters). It follows that Proposition 4.5.4 also remains true if we modify μ as follows:

$$\mu_K : x \mapsto \frac{1}{\sum_{m \in K \cap M} |z^m(x)|^2} \sum_{m \in K \cap M} |z^m(x)|^2 m \tag{4.10}$$

By the exercises on [Ful93, pg. 83], μ_K agrees with the usual construction of moment maps (in this cases the moment map associated to the S_N-action on X)

Furthermore, let $r = |K \cap M|$, and recall that points in $K \cap M$ correspond to global sections of an ample basepoint-free line bundle, so we have an associated morphism $\varphi : X \hookrightarrow \mathbb{P}^{r-1} = \operatorname{Proj} \mathbb{C}[x_1, \ldots, x_r]$ given by $x_i \mapsto z^{m_i}$. Let μ_Δ be the moment map given for $\mathbb{P}^r$ as in (4.10) for Δ equal to the convex hull of $\{e_1^*, e_2^*, \ldots, e_r^*\} \subset M'_{\mathbb{R}}$ for $M' := \mathbb{Z}^r$. We have a map

$$\pi : M' \to M, \qquad \sum_i c_i e_i^* \mapsto \sum_i c_i m_i$$

inducing a map $\pi_* : T_N = \operatorname{Hom}(M, \mathbb{C}^*) \to \operatorname{Hom}(M', \mathbb{C}^*) =: T$. The map φ respects the actions of T_N and T; i.e., for $t \in T_n$ and $x \in X$, we have $\varphi(t.x) = \pi_*(t).\varphi(x)$. [Ful93, Ex. on pg 83] shows that $\mu_K = \pi_{\mathbb{R}} \circ \mu_\Delta \circ \varphi$. I.e., the moment map μ_K for X factors through the map φ to projective space.

In fact, [Rud14, §1] shows that μ_K gives a special Lagrangian[11] torus fibration of the big torus orbit of X over the interior of K (Lagrangian with respect to the Fubini-Study form induced by φ and special with respect to the holomorphic volume form $d\log X_1 \wedge \cdots \wedge d\log X_r$ on T_N for $X_i := z^{e_i^*}$). Such fibrations are very important in SYZ mirror symmetry (cf. loc. cit.).

4.6 Differentials and the tangent bundle

For X a nonsingular variety. Let Ω_X be cotangent sheaf on X (the sheaf of differentials over $\Bbbk$), and let $\mathcal{T}_X$ be the tangent sheaf of X (the sheaf of $\Bbbk$-derivations). The **canonical sheaf** ω_X of X is defined to be the top exterior power of the cotangent sheaf:

$$\omega_X := \Lambda^{\dim X}\Omega_X.$$

One says that a divisor D in X is a **canonical divisor** if $\omega_X \cong \mathcal{O}(D)$; one often denotes a canonical divisor by K_X. Similarly, one says that D is an **anticanonical divisor** if $\omega_X^\vee = \Lambda^{\dim X}\mathcal{T}_X \cong \mathcal{O}(D)$; i.e., if $\omega_X \cong \mathcal{O}(-D)$; i.e., $D \sim -K_X$.

Proposition 4.6.1. *Let X be a nonsingular toric variety $\mathrm{TV}(\Sigma)$ with toric boundary $D_1 \cup \ldots \cup D_d$. Then $-\sum_{i=1}^d D_i$ is a canonical divisor.*

Equivalently, one might say that the toric boundary $D = \sum_i D_i$ is an anticanonical divisor.

Proof. Let $e_1, \ldots, e_r$ be a basis of N, and let $X_i := z^{e_i^*}$. Then

$$\omega := d\log X_1 \wedge \cdots \wedge d\log X_r = \frac{dX_1}{X_1} \wedge \cdots \wedge \frac{dX_r}{X_r}$$

is a rational section of ω_X, finite and nonvanishing on all of T_N. Note that the choice of basis for N does not affect ω except up to sign.[12]

Now let $\sigma \in \Sigma$. Since we assumed Σ is nonsingular, σ is generated by part of a basis, say $e_1, \ldots, e_k$, so $U_\sigma = \operatorname{Spec} \Bbbk[X_1, \ldots, X_k, X_{k+1}^{\pm 1}, \ldots, X_r^{\pm 1}]$. Then $\omega = \pm \frac{1}{X_1 \cdots X_r} dX_1 \wedge \cdots dX_r$ has simple poles along the boundary divisors $\{X_1 = 0\}, \ldots, \{X_k = 0\}$, so (ω) and $-\sum D_i$ have the same restriction to U_σ, as desired.[13] ☐

Remark 4.6.2. In fact, canonical divisors can be defined more generally than just the nonsingular cases. In particular, for any normal scheme X of finite type over a field, X is regular outside some locus Z of codimension at least 2. We can therefore define a canonical divisor K_U on $U := X \setminus Z$, and this extends to a canonical divisor K_X. If K_X is Cartier, we can define an associated sheaf $\omega_X := \mathcal{O}_X(K_X)$. With this definition of canonical divisors, Proposition 4.6.1 extends to arbitrary toric varieties. If X is Gorenstein, then this ω_X will satisfy many of the same properties as in the

[11] Here, Lagrangian means that each fiber L satisfies $\dim_{\mathbb{R}} L = \frac{1}{2}\dim_{\mathbb{R}} X$ and $\omega|_L = 0$ (for ω a fixed symplectic form on X). Special means that we additionally have $\operatorname{Im}\Omega|_L = 0$ for Ω a nowhere-vanishing holomorphic volume form; such Ω exists for (log) Calabi-Yau varieties, and for the toric case, $\Omega = \frac{1}{(2\pi i)^r} d\log X_1 \wedge \cdots d\log X_r$ is the unique-up-to-scaling such volume form on the big torus orbit (cf. §4.6).

[12] For example, if $r = 2$ and we replace e_1^* with $e_1^* + e_2^*$, then ω becomes $d\log z^{e_1^*+e_2^*} \wedge d\log z^{e_2^*} = d\log(X_1X_2) \wedge d\log X_2 = (d\log X_1 + d\log X_2) \wedge d\log X_2 = d\log X_1 \wedge d\log X_2$, which is the same as before.

[13] I don't think I've said this, but given an invertible sheaf $\mathcal{L}$, you can find a divisor D satisfying $\mathcal{L} \cong \mathcal{O}(D)$ by taking the divisor of zeroes and poles for a global rational section of $\mathcal{L}$.

nonsingular cases (in particular, Serre duality applies). If X is normal as above, then X is Gorenstein if and only if K_X is Cartier and X is Cohen-Macaulay. Toric varieties are always Cohen-Macaulay (because $\Bbbk[P]$ for P a finitely generated monoid is Cohen-Macaulay) and normal (at least under our definition), so for $\mathrm{TV}(\Sigma)$, Gorenstein is equivalent to the boundary divisor $\sum D_i$ being Cartier.

A nonsingular compact complex variety X is said to be **Calabi-Yau** if the canonical sheaf ω_X is trivial, or equivalently, if X admits a nowhere-vanishing holomorphic volume form Ω. Examples include Abelian varieties (i.e., the arbitrary-dimension generalization of elliptic curves) and hypersurfaces in $\mathbb{P}^r$ of degree $r+1$ (e.g., elliptic curves, K3-surfaces, the qunitic three-fold, etc.).

More generally,[14] consider a pair (X, D) with X a compact variety and D a union of distinct prime divisors. The pair (X, D) is called **log Calabi-Yau** if D is anti-canonical (i.e., there is a nonvanishing holomorphic volume form Ω on $U := X \setminus D$ with simple poles along D).[15] Proposition 4.6.1 can be interpreted as saying that the pair (X, D) is log Calabi-Yau (for X a complete toric variety and D its toric boundary).

We let $\Omega^1_X(\log D)$ denote the sheaf of "log differentials" on X: for a point $x \in D_1 \cap \ldots \cap D_k$ with x not in any other boundary divisors, and $e_1, \ldots, e_r$ a basis such that for $X_i = z^{e_i^*}$, the components D_i can locally be viewed as the coordinate hyperplanes of $\operatorname{Spec} \Bbbk[X_1, \ldots, X_k, X_{k+1}^{\pm 1}, \ldots, X_r^{\pm 1}]$, we say that

$$d \log X_1, \ldots, d \log X_k, dX_{k+1}, \ldots, dX_r \tag{4.11}$$

gives a $\Bbbk$-basis for the space of log differentials at x (and an $\mathcal{O}_{X,x}$-module basis for the stalk of $\Omega^1_X(\log D)$ at x). That is, we allow log differentials to have simple poles along the boundary divisors.

Proposition 4.6.3. *1. The sheaf $\Omega^1_X(\log D)$ is trivial (i.e., it's isomorphic to a free $\mathcal{O}_X$-module).*

2. There is an exact sequence of sheaves

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \to \bigoplus_{i=1}^d \mathcal{O}_{D_i} \to 0$$

where $\mathcal{O}_{D_i}$ is the sheaf of functions on D_i extended by 0 to X (i.e., given on an open subset U of X by the structure sheaf of D_i at $U \cap D_i$).

Proof. For (1), we claim that

$$\begin{aligned} h : M \otimes_{\mathbb{Z}} \mathcal{O}_X &\to \Omega^1_X(\log D) \\ m \otimes f &\mapsto f \cdot d \log z^m \end{aligned}$$

gives an isomorphism of sheaves. We can check this on sufficiently small affine open subsets. Let $\sigma \in \Sigma$ be the minimal cone satisfying $x \in U_\sigma$. I.e., for $x \in D_1, \ldots D_k$ as above, σ will have k rays, one corresponding to each of these D_i's. Then $\mathcal{O}_X(U_\sigma) = \Bbbk[X_1, \ldots, X_k, X_{k+1}^{\pm 1}, \ldots, X_r^{\pm 1}]$, so $h[(M \otimes \mathcal{O}_X)(U_\sigma)]$ is indeed generated over $\mathcal{O}_X(U_\sigma)$ by the elements in (4.12). In (2), the mapping to $\bigoplus_{i=1}^d \mathcal{O}_{D_i}$ is the residue mapping $\omega = \sum_i f_i d \log X_i \mapsto \bigoplus_i f_i|_{D_i}$. The residue is 0 iff each f_i is divisible by X_i; equivalently, iff ω is a section of Ω^1_X. This proves the exactness. □

[14] X Calabi-Yau means that $(X, \emptyset)$ is log Calabi-Yau.

[15] Many slight modifications of this definition are commonly used. In particular, D is often required to have normal crossings.

Dual to $\Omega^1_X(\log D)$, we have the sheaf of log derivations $\mathcal{T}_X(\log D)$: for x as above, the log derivations at x have

$$X_1\partial_{X_1}, \ldots, X_k\partial_{X_k}, \partial_{X_{k+1}}, \ldots, \partial_{X_r} \tag{4.12}$$

as a basis for the space of log derivations at x. As with the isomorphism h in the proof of Proposition 4.6.3, we have an isomorphism $N \otimes \mathcal{O}_X \to \mathcal{T}_X(\log D)$ given by $n \otimes f \mapsto f\partial_n$ where $\partial_n(z^m) := \langle m, n\rangle z^m$ (one easily checks that ∂_n is a derivation). Thus, $\mathcal{T}_X(\log D)$ is also trivial.

Remark 4.6.4. In Gromov-Witten theory, one "counts" algebraic curves in a variety X by integrating certain cycles over a moduli space $\mathcal{M}_{g,n}(X, \beta)$ of genus g stable maps of n-marked curves to X with some specified homology class $\beta \in H_2(X)$. This moduli space is a Deligne-Mumford stack, often very singular and not even of the expected dimension, necessitating the construction of a "virtual fundamental class" over which we can integrate (meaning that the counts we get are just "virtual," not actually naive enumerations of curves). However, we do get a smooth Deligne-Mumford stack when $g = 0$ if X is "convex" meaning that for all genus-0 stables maps $f : C \to X$, $f^*\mathcal{T}_X$ is generated by global sections (i.e., $H^1(C, f^*\mathcal{T}_X) = 0$). This is satisfied in a handful of very nice examples, like projective space and homogeneous spaces G/P for P a parabolic subgroup of G (e.g., Grassmannians and flag varieties). I am often interested in a modification of this construction called "log Gromov-Witten theory" for a pair (X, D). For the analogous convexity condition here, one just replaces $\mathcal{T}_X$ with the log tangent bundle $\mathcal{T}_X(\log D)$. It follows from the above that all toric varieties are log convex, so their log Gromov-Witten invariants are naive enumerations of curves like we would want.

4.6.1 Toric Fano varieties

A complete variety X is called **Fano** if the anticanonical divisor $-K_X$ is ample. Examples include projective space and any hypersurface in $\mathbb{P}^r$ of degree $\leq r$ (recall that the degree $(r+1)$-hypersurfaces are Calabi-Yau; the higher-degree hypersurfaces are said to be of "general type").

Let $X = \mathrm{TV}(\Sigma)$ be a complete toric variety with toric boundary $D = \sum_{\rho_i \in \Sigma^{[1]}} D_i$, $n_i \in N$ a primitive generator for ρ_i. Let $Q \subset N_{\mathbb{R}}$ be the convex hull of the elements n_i, so Σ consists of the cones over faces of Q (or possibly refinements of these cones). Then D is a Cartier divisor generated by global sections if and only if there exists a convex integral Σ-piecewise linear function ψ on $N_{\mathbb{R}}$ satisfying $\psi(n_i) = -1$ for each i. Such ψ will exist if and only if the polar polytope Q° is integral—indeed, if F is a maximal face of Q, and $\sigma_F \in \Sigma$ is a maximal cone of Σ contained in $\mathbb{R}_{\geq 0}F$, then $\psi|_{\sigma_F} = m_{\sigma_F}|_{\sigma_F}$ for some $m_{\sigma_F} \in M$, and this m_{σ_F} will be the vertex of Q° associated to F (convexity means that these m_{σ_F}'s will all be vertices). Finally, ampleness of D is equivalent to the additional condition that ψ is *strictly* convex, and this is equivalent to saying that these elements m_{σ_F} are distinct; i.e., that the cones of Σ really are cones over the faces F of Σ rather than refinements of these cones.

A convex integral polytope Q containing the origin in its interior is called **reflexive** if the polar polytope Q° is also an integral polytope. Recall from Remark 4.6.2 that a toric variety is Gorenstein if and only if D is Cartier. The above discussion yields the following:

Proposition 4.6.5. *There is a bijective correspondence (up to isomorphism) between Gorenstein toric Fano varieties and reflexive polytopes.*

Reflexive polytopes can be quickly recognized visually using the following:

Proposition 4.6.6. *A convex integral polytope is reflexive if and only if the origin is the only lattice point in the interior of Q.*

To see this, let F be a facet of Q, let AF be the affine linear subspace of $N_{\mathbb{R}}$ obtained by extending F, and let LF be the corresponding linear space (through the origin) parallel to F. Then note that the existence of an element $m \in M$ which equals -1 on F is equivalent to there being no lattice points of N strictly between AF and LF.

Examples 4.6.7.

- The only Fano curve is $\mathbb{P}^1$. This of course is toric. The corresponding reflexive polytope is the closed interval $[-1, 1]$.
- The nonsingular Fano surfaces are known as Del Pezzo surfaces. There are 10 of these, one for each degree 1 through 9, except two of degree 8 (the "degree" is the self-intersection number $K_X.K_X$). With the exception of $\mathbb{P}^1 \times \mathbb{P}^1$ (which has degree 8), the Del Pezzo surfaces of degree $9 - k$ for $0 \leq k \leq 6$ are obtained by blowing up k generic points in $\mathbb{P}^2$. These can easily be realized as toric varieties for $k = 0, 1, 2, 3$ (and of course also for $\mathbb{P}^1 \times \mathbb{P}^1$).

  Note: There are many more two-dimensional reflexive polytopes associated to singular Gorenstein toric Fano surfaces, including the polar duals to the reflexive polytopes for the non-singular cases. According to slides I found here, there are a total of 16 two-dimensional reflexive polytopes.

Remark 4.6.8. The duality between reflexive polytopes Q and Q° has geometric significance as well. There are interesting dualities between the associated toric varieties which are studied in mirror symmetry. It looks like a decent brief exposition of this is given here.

4.7 Euler characteristics and Betti numbers

The Euler characteristic is the number of maximal cones. This is because $\mathrm{TV}(\Sigma)$ is stratified by the orbits $O_\sigma \cong (S^1)^{\operatorname{codim} \sigma} \times N(\sigma)_{\mathbb{R}}$, and these each contribute Euler characteristic 0 except when $\operatorname{codim} \sigma = 0$, in which case O_σ is a point and its Euler characteristic is 1. The Betti numbers can be computed similarly, though some fancy stuff goes into justifying this decomposition into strata. See [Ful93, §4.5]

4.8 Cohomology of toric line bundles

Here we follow [Ful93, §3.5]. Let Σ be a fan with rays ρ_i generated by $n_i \in N$, and let $D = \sum_i a_i D_i$ be a T-Cartier divisor on $\mathrm{TV}(\Sigma)$. Recall that for such D, there is a corresponding integral Σ-piecewise linear function $\psi = \psi_D$ determined by $\psi_D(n_i) = -a_i$. We defined a polytope

$$\begin{aligned} P_D := &\{m \in M_{\mathbb{R}} | \langle m, n_i \rangle \geq -a_i \text{ for all } i\} \\ = &\{m \in M_{\mathbb{R}} | m \geq \psi_D \text{ on } |\Sigma|\}. \end{aligned}$$

I.e., P_D was defined so that $m \in P_D \cap M$ if and only if $z^m \in H^0(\mathrm{TV}(\Sigma), \mathcal{O}_X(D))$—as a reminder, this follows because

$$H^0(\mathrm{TV}(\Sigma), \mathcal{O}_X(D)) = \{z^m | \operatorname{val}_{D_i}(z^m) \geq -a_i \text{ for all } i\}$$

and $\mathrm{val}_{D_i}(z^m) = \langle m, n_i \rangle$.

In fact, we saw that $H^0(X, \mathcal{O}_X(D)) = \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot z^m$, so we can decompose $H^0(X, \mathcal{O}_X(D))$ as

$$H^0(X, \mathcal{O}_X(D)) = \bigoplus_{m \in M} H^0(X, \mathcal{O}_X(D))_m$$

where

$$H^0(X, \mathcal{O}_X(D))_m = \begin{cases} \mathbb{C} \cdot z^m & \text{if } m \in P_D \cap M \\ 0 & \text{otherwise.} \end{cases}$$

The condition $m \in P_D \cap M$ can be equivalently described as follows: for each $m \in M$, let

$$Z(m) = \{n \in |\Sigma| | \langle m, n \geq \psi(v)\}.$$

Then $m \in P_D$ if and only if $Z(m) = |\Sigma|$, or equivalently, if and only if

$$H^0(|\Sigma| \setminus Z(m)) = 0.$$

Here H^* means the ordinary (singular) or sheaf cohomology of the topological space with complex coefficients (cf. Theorem 2.8.7).

Equivalently, let $H^*_{Z(m)}(|\Sigma|) = H^*(|\Sigma|, |\Sigma| \setminus Z(m))$ denote the local/relative cohomology group of the pair $(|\Sigma|, |\Sigma| \setminus Z(m))$—recall that the relative homology $H_*(X, Y)$ is defined via the chain complex $C_n(X, Y) := C_n(X)/C_n(Y)$, and relative cohomology is defined by dualizing this complex. We have the short exact sequence

$$0 \to C_*(Y) \to C_*(X) \to C_*(X)/C_*(Y) \to 0,$$

and taking the associated long exact sequence in cohomology yields

$$0 \to H^0(X, Y) \to H^0(X) \to H^0(Y) \to H^1(X, Y) \to H^1(X) \to H^1(Y) \to \dots \tag{4.13}$$

(take all coefficients to be in $\mathbb{C}$). Thus, we can identify

$$H^0_{Z(m)}(|\Sigma|) = H^0(|\Sigma|, |\Sigma| \setminus Z(m)) = \ker(H^0(|\Sigma|) \to H^0(|\Sigma| \setminus Z(m)).$$

This kernel nonzero if and only if $Z(m)$ is all of $|\Sigma|$, i.e., if and only if $m \in P_D \cap M$, and in this case the kernel is all of $H^0(|\Sigma|) = \mathbb{C} \cdot [X] \cong \mathbb{C} \cdot z^m$. Thus,

$$H^0(X, \mathcal{O}(D)) = \bigoplus_{m \in M} H^0(X, \mathcal{O}(D))_m \quad , \quad H^0(X, \mathcal{O}(D))_m = H^0_{Z(m)}(|\Sigma|). \tag{4.14}$$

More generally, letting

$$H^p_{Z(m)}(|\Sigma|) := H^p(|\Sigma|, |\Sigma| \setminus Z(m); \mathbb{C}),$$

we will see the following:

Proposition 4.8.1. *For all $p \geq 0$, there are canonical isomorphisms*

$$H^p(X, \mathcal{O}(D)) = \bigoplus_{m \in M} H^p(X, \mathcal{O}(D))_m \quad , \quad H^p(X, \mathcal{O}(D))_m = H^p_{Z(m)}(|\Sigma|).$$

Remark 4.8.2. Since $|\Sigma|$ is always contractible, $H^p(|\Sigma|) = 0$ for all $p \geq 1$. So (4.13) implies

$$H^p(|\Sigma|, |\Sigma| \setminus Z(m)) \cong H^{p-1}(|\Sigma| \setminus Z(m))$$

for all $p \geq 2$. In the $p = 1$ case we have

$$H^1(|\Sigma|, |\Sigma| \setminus Z(m)) \cong H^0(|\Sigma| \setminus Z(m))/\text{Image}((H^0(|\Sigma|)).$$

Example 4.8.3. If $X = \text{TV}(|\Sigma|)$ is affine, then $|\Sigma|$ is a strongly convex cone and ψ_D is linear on $|\Sigma|$. So for each $m \in M$, $Z(m)$ and $|\Sigma| \setminus Z(m)$ are complementary subcones of $|\Sigma|$. Using Remark 4.8.2, we see that $H^p(X, \mathcal{O}(D)) = H^p_{Z(m)}(|\Sigma|) = 0$ for all $p \geq 1$ (as is always the case with affine varieties X).

Corollary 4.8.4. *If $|\Sigma|$ is convex and $\mathcal{O}(D)$ is generated by global sections, then $H^p(X, \mathcal{O}(D)) = 0$ for all $p > 0$.*

Proof. Recall from Proposition 4.3.5 that $\mathcal{O}(D)$ being generated by global sections is equivalent to ψ_D being convex. Note that

$$\begin{aligned} |\Sigma| \setminus Z(m) &= \{n \in |\Sigma| | \langle m, n\rangle < \psi_D(n)\} \\ &= \{n \in |\Sigma| | 0 < [\psi_D - m](n)\}. \end{aligned}$$

Since ψ_D is convex and m is linear, $\psi_D - m$ is convex, and then since $|\Sigma|$ is convex, the set $|\Sigma| \setminus Z(m)$ above is a convex cone in $|\Sigma|$. Thus, the sets $|\Sigma|$ and $|\Sigma| \setminus Z(m)$ are contractible. So $H^p(|\Sigma| \setminus Z(m)) = 0$ for all $p > 0$, and $H^0(|\Sigma| \setminus Z(m))/\text{Image}((H^0(|\Sigma|)) = 0$, so by Remark 4.8.2, $H^p(|\Sigma|, |\Sigma| \setminus Z(m)) = 0$ for all $p \geq 1$ as claimed. ☐

Proof of Proposition 4.8.1. We consider the Čech cohomology computed using the affine open cover $\mathfrak{U} = \{U_\sigma\}_{\sigma \in \Sigma}$ with some fixed well-ordering on the cones of Σ. That is,

$$C^p(\mathfrak{U}, \mathcal{F}) = \bigoplus_{\sigma_0 < \ldots < \sigma_p} H^0(U_{\sigma_0} \cap \ldots \cap U_{\sigma_p}), \mathcal{O}(D)).$$

Let $\sigma_{0,\ldots,p} = \sigma_0 \cap \ldots \cap \sigma_p$, and note that $U_{\sigma_{0,\ldots,p}} = U_{\sigma_0} \cap \ldots \cap U_{\sigma_p}$. By (4.14), we have

$$C^p = \bigoplus_{m \in M} \bigoplus_{\sigma_0 < \ldots < \sigma_p} H^0_{Z(m) \cap \sigma_{0,\ldots,p}}(\sigma_{0,\ldots,p}).$$

Since the boundary maps (2.6) are given by taking linear combinations of elements, they preserve the M-grading, so the cohomology will also be M-graded. As before, we know that $H^i_{Z(m) \cap \sigma_{0,\ldots,p}}(\sigma_{0,\ldots,p}) = 0$ for each $i > 0$ because $(\psi_D - m)|_{\sigma_{0,\ldots,p}}$ is linear, and so $(|\Sigma| \setminus Z(m)) \cap \sigma_{0,\ldots,p}$ is a convex subcone of $\sigma_{0,\ldots,p}$, hence we can simultaneously contract $\sigma_{0,\ldots,p}$ and $Z(m) \cap \sigma_{0,\ldots,p}$. The proposition now follows from the following Lemma

Lemma 4.8.5. *Let Z be a closed subspace of a space Y that is a union of a finite number of closed subspaces Y_j, and let $\mathcal{F}$ be a sheaf on Y such that $H^i_{Z \cap Y'}(Y', \mathcal{F}) = 0$ for all $i > 0$ and all $Y' = Y_{j_0} \cap \ldots \cap Y_{j_p}$. Then*

$$H^i_Z(Y, \mathcal{F}) = H^i(C^*(\{Y_j\}, \mathcal{F})),$$

where

$$C^p(\{Y_j\}, \mathcal{F}) = \bigoplus_{j_0, \ldots, j_p} \Gamma_{Z \cap Y_{j_0} \cap \ldots \cap Y_{j_p}}(Y_{j_0} \cap \ldots \cap Y_{j_p}, \mathcal{F})$$

In our setting, Z is the set $Z(m)$, the Y_{j_k}'s are the σ_k's, and $\mathcal{F}$ is the constant sheaf $\overline{\mathbb{C}}$ (i.e., the sheaf whose stalks are $\mathbb{C}$). Note that this is essentially a relative cohomology analog of Theorem 2.8.7. □

Example 4.8.6. Consider $X = \mathbb{P}^n$ realized as $\mathrm{TV}(\Sigma)$, where the rays of Σ are generated by $e_1, \ldots, e_r, e_0 := -e_1 - \ldots - e_r$. Consider $\mathcal{O}(kD_0) \cong \mathcal{O}(k)$, so $\psi_{kD_0}(e_i) = -k\delta_{0i}$. Then ψ_{kD_0} is convex ($\mathcal{O}(k)$ is generated by global sections) iff $k \geq 0$. So in these cases we have $H^i(X, \mathcal{O}(kD_0)) = 0$ for all $i \geq 1$, and we have $H^0(X, \mathcal{O}(kD_0)) = \bigoplus_{m \in P_{mD_0}} \mathbb{C} \cdot z^m$ where P_{kD_0} consists of those $m = (a_1, \ldots, a_r)$ with each $a_i \geq 0$ and $\langle m, e_0 \rangle = -\sum_{i=1}^r a_i \geq -k$, i.e., $\sum_i a_i \leq k$.

Now suppose $k < 0$. Then $P_D = \{0\}$ and $H^0(X, \mathcal{O}_{kD_0}) = \mathbb{C} \cdot 1$. Furthermore, ψ_{kD_0} is strictly concave (negative a strictly convex function), so each $Z(m) = \{n \in |\Sigma| | \langle m, n \rangle \geq \psi_{kD_0}(n)\}$ is a strongly convex cone. The complement $N_{\mathbb{R}} \setminus Z(m)$ is thus contractible (it is a non-convex cone) except in the case where $Z(m) = \{0\}$. In this case, $N_{\mathbb{R}} \setminus Z(m)$ can be retracted to a sphere S^{r-1}, and so for $p \geq 2$ (with some slight modification for the $p = 1$ cases), we have

$$H^p(|\Sigma| \setminus Z(m)) \cong H^{p-1}(S^{r-1}) = \begin{cases} \mathbb{C} & \text{for } p = r \\ 0 & \text{for all other } p \geq 1. \end{cases}$$

For $m = (a_1, \ldots, a_r)$, we have $Z(m) = \{0\}$ iff each $a_i < 0$ and $a_1 + \ldots + a_r > k$; i.e., for m in the interior of $-P_{-kD_0}$. So we have $H^p(X, \mathcal{O}(kD_0)) = 0$ for $0 < p \neq r$, and we have $H^r(X, \mathcal{O}(kD_0)) = \bigoplus_m \mathbb{C} \cdot z^m$ where the sum is over m in the interior of $-P_{-kD_0}$.

Higher direct image sheaves

Given a morphism of schemes $f : Y \to X$ and a sheaf $\mathcal{F}$ on Y, recall that $f_*\mathcal{F}$ is the sheaf on X given by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$. The functor f_* is called the direct image functor. It is left exact but generally not right exact. The right derived functors $R^i f_*$ are called the higher direct image functors, and $R^i f_*\mathcal{F}$ is the **higher direct image sheaf**. One can show that $R^i f_*\mathcal{F}$ is the sheaf on X given by $R^i f_*\mathcal{F}(U) = H^i(f^{-1}(U), \mathcal{F})$.

A scheme X is said to have **rational singularities** if it is normal, of finite-type over a field of characteristic 0, and there exists a resolution of singularities $f : Y \to X$ whose higher direct images sheaves are 0 for all $i > 0$.

Proposition 4.8.7. *Let Σ' be a refinement of Σ, giving a proper birational map $f : X' = \mathrm{TV}(\Sigma') \to X = \mathrm{TV}(\Sigma)$. Then*

$$f_*(\mathcal{O}_{X'}) = \mathcal{O}_X \quad \textit{and} \quad R^i f_*(\mathcal{O}_{X'}) = 0 \quad \textit{for all} i > 0.$$

In particular, since we can take f to be a resolution of singularities, every toric variety X has rational singularities.

Proof. Since the claims are local, we can assume that Σ is a single cone σ plus its faces, so $X = U_\sigma$. The first claim is now that $\Gamma(X', \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$, and this is clear since both rings are $A_\sigma = \mathbb{C}[\sigma^\vee \cap M]$ (and in fact the equality of these rings is a general fact for X normal and f birational).

The second claim just says that $H^i(X, \mathcal{O}_{X'}) = 0$ for all $i > 0$. This follows from Corollary 4.8.4 because $\mathcal{O}_{X'}$ is generated by global sections and $|\Sigma'| = \sigma$ is convex. □

Chapter 5

Cluster Varieties

My intention here is to introduce cluster varieties using the perspective from which I came to the subject; that is, following the path of Gross-Hacking-Keel. I will therefore begin by introducing log Calabi-Yau surfaces following [GHK15b]. I do not plan to go into great depth here (in particular, I will not discuss scattering diagrams or theta functions), but we will at least see that cluster varieties are natural higher-dimensional generalizations of log Calabi-Yau surfaces and are closely related to toric geometry [FG09, GHK15a]. (This section is still in-progress).

5.1 Log Calabi-Yau Surfaces

Here we largely follow introductory parts of [GHK15b].

By a **log Calabi-Yau surface** (with maximal boundary), we mean a pair (Y, D) where Y is a smooth rational projective surface over an algebraically closed field $\Bbbk$ of characteristic 0, and $D = D_1 + \ldots + D_n$ is a singular nodal curve anticanonical curve. By anticanonical, we mean $D \in |-K_Y|$, i.e., $K_Y = \mathcal{O}(-D)$ where K_Y is the canonical bundle on Y. The assumption that D is singular precludes the case where $D = \emptyset$ and Y is compact Calabi-Yau (e.g., Y an Abelian surface or a $K3$-surface). It also precludes cases like $Y = \mathbb{P}^2$ and $D =$(a smooth cubic). This is why the parenthetical phrase "with maximal boundary" is sometimes used to describe such surfaces.

The divisor D is necessarily an irreducible nodal curve or a cycle of $n \geq 2$ rational curves. The pair (Y, D) may also be called a Looijenga pair. The complement $Y \setminus U$ may be called a Looijenga interior, or just the interior of the surface.

Examples 5.1.1.

1. The most basic examples are where Y is a toric variety and D is the toric boundary. These are the cases where the interior $U = Y \setminus D$ is just an algebraic torus $(\Bbbk^*)^2$.

2. For $Y = \mathbb{P}^2$, D may be a nodal cubic, a line plus a conic, or a triangle of lines (this last possibility being the toric case).

3. The cubic surface can famously be obtained by blowing up 6 points in $\mathbb{P}^2$ (see [Har77, Ch. V]). Suppose $\overline{Y} = \mathbb{P}^2$ and $\overline{D} = D_1 + D_2 + D_3$ is a triangle of lines. Then blowing up two points on each component of D, not including any nodal points of D, results in a cubic surface Y, and the proper transform D of $\overline{D}$ makes (Y, D) a Looijenga pair. The complement $U = Y \setminus D$ is an affine cubic surface.

For (Y, D) a Looijenga pair, a **toric blowup** of (Y, D) is a birational morphism $\pi : \widetilde{Y} \to Y$ such that if $\widetilde{D}$ is the reduced scheme structure on $\pi^{-1}(D)$, then $(\widetilde{Y}, \widetilde{D})$ is a Looijenga pair. In the toric setting, these are the blowups that come from refining your fan by adding new rays.

A **toric model** of (Y, D) is a birational morphism $p : (Y, D) \to (\overline{Y}, \overline{D})$ such that$(\overline{Y}, \overline{D})$ is a smooth toric surface with its toric boundary and $D \to \overline{D}$ is an isomorphism. Note that $p : Y \to \overline{Y}$ consists of **non-toric blowups**—i.e., p blows up only non-nodal points of the boundary (and the new boundary comes from taking the proper transform of the old), whereas toric blowups as above only blow up nodal points of the boundary (possibly repeatedly, and the new boundary is the reduced inverse image of the old).

Proposition 5.1.2 ([GHK15b], Prop. 1.3)**.** *Every Looijenga pair has a toric blowup which admits a toric model.*

Thus, every Looijenga interior U can be realized as $Y \setminus D$ for some Looijenga pair (Y, D) which is obtained by taking repeated non-toric blowups of a toric variety $(\overline{Y}, \overline{D})$. By a toric model of U, I will mean a toric model for a Looijenga pair (Y, D) whose interior is U, and I will view two toric models $\pi_i : (Y_i, D_i) \to (\overline{Y}_i, \overline{D}_i)$ of U as being equivalent of there exists a toric model $\pi : (\widetilde{Y}, \widetilde{D}) \to (\overline{Y}, \overline{D})$ factoring through both π_1 and π_2. I.e., I really only care about the non-toric blowdowns, not the boundary part.

Looijenga interiors typically admit many non-equivalent toric models. Indeed, given one toric model, another toric model can be produced via **mutation**. This is illustrated in Figure 5.1.1 and explained in the caption.

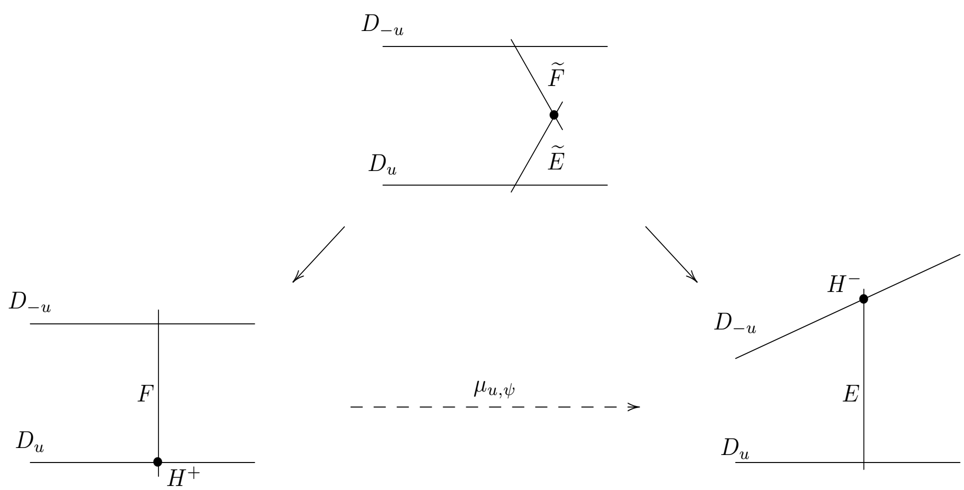

Figure 5.1.1: Consider a non-toric blowup of a point H^+ in a toric boundary divisor D_u associated to a direction $u \in N$. Let Σ be fan in $N_{\mathbb{R}}$ consisting of the two rays $\pm\mathbb{R}_{\geq 0}u$, so $\mathrm{TV}(\Sigma) \cong \mathbb{P}^1 \times \Bbbk^*$ with $D_u = 0 \times \Bbbk^*$ and $D_{-u} = \infty \times \Bbbk^*$. Consider the projection to D_u with fibers $\mathbb{P}^1$, and let F be the $\mathbb{P}^1$-fiber over H^+. Let $\widetilde{F}$ be the proper transform of F. Since $F^2 = 0$, $\widetilde{F}^2 = -1$ (in general, blowing up a smooth point on a curve in a surface like this reduces the self-intersection number by 1), and so $\widetilde{F}$ can then be blown down (smooth -1 curves in surfaces can always be blown down to smooth points). The result is a new toric variety, and so the blowdown of $\widetilde{F}$ yields a new toric model. The birational map between the two toric varieties (or rather, between their big torus orbits) is called a **mutation**.

Recall that a birational map is an isomorphism between Zariski open subsets. By identifying these Zariski open subsets, birational maps can thus be used to glue schemes together to produce new schemes. Note that gluing the two big torus orbits from Figure 5.1.1 yields the interior of the blown up space, except for the point $\widetilde{F} \cap \widetilde{E}$. Since this point has codimension-two in the scheme, it has no impact on the space of global regular functions, or more generally, on global sections of line bundles (this is Hartogs's lemma in the holomorphic setting, and holds in the algebraic setting as a consequence of Serre's condition S2). So for many purposes, these codimension-two issues can be ignored. We may thus view U as consisting of a number of algebraic tori glued together via mutations. Cluster varieties are a generalization of this viewpoint.

5.2 Cluster varieties

Here I will introduce the notion of cluster varieties following their definition in [FG09] and their interpretation in [GHK15a].

A **seed** is a tuple of data $\mathbf{s} = (N, I, E = \{e_i\}_{i\in I}, F, \omega)$ where N is a lattice of finite-rank r, E is a basis for N (or at least for a sublattice of N) indexed by the finite index-set I, and F is a subset of I called the frozen indices (and the e_i with $i \in F$ may be called frozen vectors). Let N_{uf} denote the span of $\{e_i\}_{i\in I\setminus F}$. Then ω is a $\mathbb{Z}$-valued[1] bilinear form on N which is skew-symmetrizable, by which we mean that there exist positive rational numbers $\{d_j\}_{j\in I\setminus F}$ and a skew-symmetric form $\{\cdot,\cdot\}$ on N_{uf} such that $\omega(e_i, e_j) = d_j\{e_i, e_j\}$ for each $i, j \in I_{\mathrm{uf}}$. For simplicity, the reader may just assume that ω is skew-symmetric without sacrificing too much generality. We may denote N by $N_{\mathbf{s}}$, etc., to indicate the associated seed.

Remark 5.2.1. If we assume that ω is skew-symmetric and E is a basis for N, then seeds naturally correspond to quivers without loops or oriented two-cycles. The vertices of the quiver correspond to the indices $i \in I$. The number of arrows from i to j is $\omega(e_i, e_j)$, where arrows actually go from j to i if $\omega(e_i, e_j)$ is negative. Vertices associated to frozen indices are sometimes illustrated as boxes rather than points to indicate that they are frozen.

Let $M := N^* = \mathrm{Hom}(N, \mathbb{Z})$. We have two maps $\omega_1, \omega_2 : N \to M$ given by $n \mapsto \omega(n, \cdot)$ and $n \mapsto \omega(\cdot, n)$, respectively. Let $K_i := \ker p_i$, and denote the inclusion $\kappa_i : K_i \hookrightarrow N$. Let $\langle\cdot,\cdot\rangle : N \oplus M \to \mathbb{Z}$ denote the dual pairing between N and M.

Given a lattice L with dual lattice L^*, let $T_L := L \otimes \Bbbk^* = \mathrm{Spec}\, \Bbbk[L^*]$. A choice of $u \in L$ and $\psi \in L^*$ satisfying $\psi(u) = 0$ determines a birational map $\mu_{u,\psi} : T_L \dashrightarrow T_L$ defined by

$$\mu_{u,\psi}^{\sharp}(z^{\varphi}) := z^{\varphi}(1 + z^{\psi})^{-\varphi(u)} \qquad \text{for } \varphi \in L^*.$$

This birational map $\mu_{u,\psi}$ is called a **mutation**.

The mutation $\mu_{u,\psi}$ is interpreted geometrically as follows (cf. Figure 5.1.1). Let $u' \in L$ denote the primitive vector in the direction u, and let $|u|$ denote the index of u, so $u = |u|u'$. Let Σ denote the fan in $L\otimes\mathbb{Q}$ with rays generated by u and $-u$. The map $L \to L/\mathbb{Z}u'$ induces a $\mathbb{P}^1$ fibration of the toric variety $\mathrm{TV}(\Sigma)$ over $T_{L/\mathbb{Z}u'}$. The mutation $\mu_{u,\psi}$ is the birational map $T_L \dashrightarrow T_L$ given geometrically by including T_L into $\mathrm{TV}(\Sigma)$, blowing up the locus $H^+ := D_u \cap \overline{V}((1 + z^{\psi})^{|u|})$ (left arrow of Figure

[1] More generally, one may allow ω to be $\mathbb{Q}$-valued but require $\omega(e_i, n) \in \mathbb{Z}$ for all $i \in I \setminus F$. I think that for our purposes it makes sense to require $\omega(e_i, n) \in \mathbb{Z}$ for all $i \in I$ but then to not require E to span N.

5.1.1), contracting the proper transform $\widetilde{F}$ of the fibers F which hit H^+ down to a hypertorus H^- in D_{-u} (right arrow of Figure 5.1.1), and then taking the complement of the proper transforms of the boundary divisors. In the figure, $\widetilde{E}$ denotes the exceptional divisor, with E being its image after the contraction of $\widetilde{F}$. The result of gluing the two copies of T_L via $\mu_{u,\psi}$ is given by the top picture minus D_u, D_{-u}, and $\widetilde{E}\cap\widetilde{F}$.

Now, let $\mathbf{s}$ be a seed (N, I, E, F, ω) as above. Given $j \in I \setminus F$, we define a new seed $\mu_j(\mathbf{s}) = (N, I, E', F, \omega)$ where $E' = \{e'_i \in N\}_{i\in I}$ is defined via

$$e'_i := \mu_j(e_i) := \begin{cases} e_i + \max(\omega(e_i, e_j), 0)e_j & \text{if } i \neq j \\ -e_i & \text{if } i = j. \end{cases}$$

Now for each $i \in I \setminus F$, we may apply the mutations

$$\mu^{\mathcal{A}} := \mu_{e_j,\omega_1(e_j)} : T_{M_{\mathbf{s}}} \dashrightarrow T_{M_{\mu_j(\mathbf{s})}} \tag{5.1}$$

and

$$\mu^{\mathcal{X}} := \mu_{\omega_2(e_j),e_j} : T_{N_{\mathbf{s}}} \dashrightarrow T_{N_{\mu_j(\mathbf{s})}} \tag{5.2}$$

. One can then mutate each $\mu_j(\mathbf{s})$ again with respect to each $j \in I \setminus F$, and repeat the birational maps and gluing and so forth indefinitely, constructing two varieties $\mathcal{A}$ and $\mathcal{X}$ associated to $\mathbf{s}$. By [GHK15a], the space $\mathcal{X}$ agrees up to codimension-two[2] with the space obtained by just mutating with respect to each $j \in I \setminus F$ once, and if ω is non-degenerate, the analogous statement is true for $\mathcal{A}$.

To help us better understand these spaces, let $\lambda : M \to K_1$ denote the dual to the inclusion $\kappa_1 : K_1 \hookrightarrow N$, i.e., $\lambda(m) = m|_{K_1} \in K_1^*$ and note that we have a sequence

$$0 \to K_2 \overset{\kappa_2}{\to} N \overset{p_2}{\to} M \overset{\lambda}{\to} K_1^* \to 0.$$

This sequence is exact except possibly at M, where it is exact only up to torsion ($\lambda(m) = 0$ iff $km \in p_2(N)$ for some nonzero $k \in \mathbb{Z}$). Let $G = \ker(\lambda)/p_2(N) = (M/p_2(N))^{\mathrm{tor}}$. Tensoring this sequence with $\Bbbk^*$ and noting that the resulting sequence commutes with mutation, we obtain another sequence

$$1 \to T_{K_2} \overset{\kappa_2}{\to} \mathcal{A} \overset{p_2}{\to} \mathcal{X} \overset{\lambda}{\to} T_{K_1^*} \to 1. \tag{5.3}$$

One may view p_2 as the quotient by the action of the torus T_{K_2}. The space $\mathcal{X}$ fibers over the torus $T_{K_1^*}$. In fact, $\mathcal{X}$ is a Poisson manifold with Poisson structure determined by $\{z^{n_1}, z^{n_2}\} = \omega(n_1, n_2)z^{n_1+n_2}$, and the fibers of λ are precisely the leaves of the associated foliation of $\mathcal{X}$. The map p_2 maps $\mathcal{A}$ to $\mathcal{X}_e = \lambda^{-1}(e)$ (for e the identity in $T_{K_1^*}$) and this in fact identifies $\mathcal{A}$ as a $(T_{K_2} \times G)$-torsor[3] over $\mathcal{X}_e$.

The map $\mathcal{A} \to \mathcal{X}_e$ should be viewed as analogous to the quotient construction of toric varieties as in §3.6 (in [Man19] I even include boundary divisors associated to the rays generated by frozen

[2]Actually, making sense of codimension for a space like this is confusing, so to be safe, I should define $\mathcal{X}^{ft}$ to be the space obtained by some arbitrary finite tree of mutations which includes each $j \in I \setminus F$ at least once, and similarly with $\mathcal{A}^{ft}$, and then I should compare these to the spaces obtained by mutating each j just once. But the statement we actually care about is that the spaces of global regular functions agree, and for that we're fine).

[3]In [Man19] I missed the need for G here. This doesn't present serious problems for the results there, but it may be worth revisiting and seeing what effect this has.

vectors—extending this to higher-codimension strata would make this into a proper generalization of the toric setting, but codimension-two issues get annoying here). One can modify the coefficients in the construction of $\mathcal{A}$ to obtain spaces $\mathcal{A}_t$ which are torsors over other fibers $\mathcal{X}_t$ of λ. The main result[4] of [Man19] is that $\mathcal{A}_t$ is the universal torsor over $\mathcal{X}_t$, which very roughly means that $\Gamma(\mathcal{A}, \mathcal{O}_{\mathcal{A}}) = \bigoplus_{\mathcal{L} \in \mathrm{Pic}(\mathcal{X})} \Gamma(\mathcal{X}, \mathcal{L})$. (Actually this is the statement that the global sections on $\mathcal{A}$ forms the Cox ring on $\mathcal{X}$—being the universal torsor is a sheaf-theoretic version of this).

In the case where ω has rank-two (i.e., $p_2(N)$ is a two-dimensional sublattice of M), the fibers of $\mathcal{X}$ are Looijenga interiors (and can naturally be compactified to Looijenga pairs). Moving around over $T_{K_1^*}$ corresponds to deforming the Looijenga interior by sliding around the blowup points from the toric model. There is a precise sense in which this is actually a universal family [LZ] (generalizing this to higher-rank might be an interesting project).

[4] In the absence of boundary divisors/frozen vectors this is actually in [GHK15a]; I just extended this to allow boundary components.

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