Tropical curve counting and canonical bases

Travis Mandel

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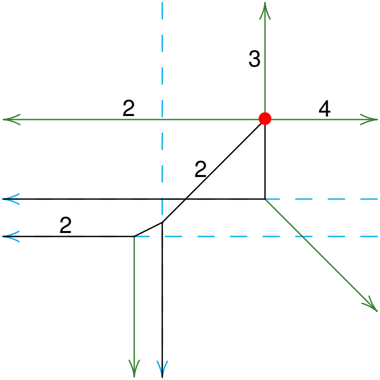

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  - Mirror symmetry: theta functions $\leftrightsquigarrow$ Gromov-Witten data of the mirror.
- Today:
  - GHKK theta functions $\leftrightsquigarrow$ tropical curves (closer to Gromov-Witten theory).
  - Quantized theta functions $\leftrightsquigarrow$ refined counts of tropical curves (Block-Göttsche [BG14]).

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- Cluster variety U roughly constructed by performing the gluings $T_N \dashrightarrow T_N$ corresponding to $\mu_i := \mu_{e_i, W(e_i, \cdot)}$ for all $i \in I$.
- For mirror, $N \leftrightsquigarrow M$ and use $\mu_{W(\cdot, e_i), e_i}$.

The geometric meaning of $\mu_{e,u}$

Recall $\mu_{e,u}^{*} : z^m \mapsto z^m(1+z^u)^{-\langle e,m\rangle}$.

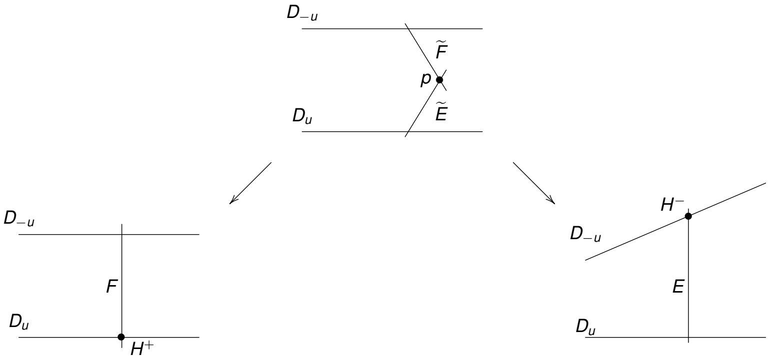

Figure: $\mu_{e,u}$ = blow up $H^+ := \{1+z^u=0\} \cap D_e$ (left arrow), then contract proper transform $\widetilde{F}$ of the fibers F which hit H_+ (right arrow).

[GHKK14]'s Main result

- Let $\mathcal{A}$ be cluster variety constructed from N, W, $\{e_i \in N\}_{i\in I}$ as above, $\mathcal{X}$ the mirror.
- (Formal version of) $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ admits canonical additive basis of "theta functions" $\{\vartheta_p | p \in N\}$.

Quantizing mutations

$$\mathbb{C}_q[M] := \mathbb{C}[q^{\pm 1}][z^m | m \in M] / \langle z^{m_1} z^{m_2} = q^{W(m_1,m_2)} z^{m_1+m_2} \rangle.$$

Quantum mutations: $\mu_{e,u}^* : \mathbb{C}_q[M] \dashrightarrow \mathbb{C}_q[M]$,

$$z^m \mapsto \Psi_q(z^u) z^m \Psi_q(z^u)^{-1}$$

where

$$\Psi_q(x) := \exp(-\operatorname{Li}_2(-x; q)) = \prod_{a=1}^{\infty} \frac{1}{1 + q^{2a-1} x}.$$

Due to [BZ05] for $\mathcal{A}$, [FG09] for $\mathcal{X}$.

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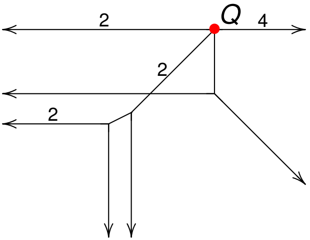
The [GHKK14] theta function construction works here too.

Tropical disks



- Tropical curve: Weighted graph Γ immersed in $N_{\mathbb{R}}$ with rational slopes and satisfying the balancing condition.
- Assume genus 0 (Γ is simply connected).

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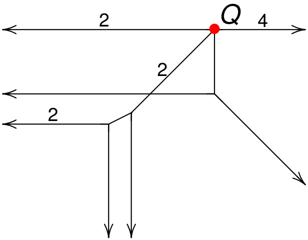

- Tropical curve: Weighted graph Γ immersed in $N_{\mathbb{R}}$ with rational slopes and satisfying the balancing condition.
- Assume genus 0 (Γ is simply connected).
- Tropical *disks*: One special vertex Q might not be balanced.

Multiplicity

- Vertex V in edge E. Let $v_{V,E}$:=primitive tangent vector from V into E. Let w_E denote the weight of E.
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- Balancing condition: $\sum_{E \ni V} w_E v_{V,E} = 0$.
- For $V \neq Q$ trivalent and $E_1, E_2 \ni V$,

$$\text{Mult}(V) := |W(w_{E_1} v_{V,E_1}, w_{E_2} v_{V,E_2})|.$$

- $\text{Mult}(Q) := 1$.
-

$$\text{Mult}(\Gamma) := \prod_{V \in \Gamma^{[0]}} \text{Mult}(V).$$

Quantum refined multiplicity

- Due to [BG14] in dimension 2.
- For any $c \in \mathbb{Z}$,

$$[c]_q := \frac{q^c - q^{-c}}{q - q^{-1}} = q^{-c+1} + q^{-c+3} + \ldots + q^{c-3} + q^{c-1} \text{ (for } c > 0\text{)}$$

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- The edges containing Q will have an ordering. Then

$$\mathrm{Mult}_q(Q) := \prod_{i<j} q^{\sum_{i<j} W(w_{E_i} v_{Q,E_i}, w_{E_j} v_{Q,E_j})}.$$

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Main Theorem

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Let $p_1, \ldots, p_s, n \in N$, $Q \in N_{\mathbb{R}}$ generic and very close to $\mathbb{R}_{\gg 0}n$. The ϑ_n-coefficient of

$$\vartheta_{p_1}\vartheta_{p_2}\cdots\vartheta_{p_s}$$

is:

$$\sum_C \mathrm{Mult}(C)\frac{R_{\mathbf{w}_C}}{|\operatorname{Aut}(\mathbf{w}_C)|},$$

where $R_{\mathbf{w}_C}$ and $|\operatorname{Aut}(\mathbf{w}_C)|$ will be defined below, and the sum is over tropical curves C in $N_{\mathbb{R}}$ with:

1. $\sum_C \text{Mult}(C) \frac{R_{\mathbf{w}_C}}{|\text{Aut}(\mathbf{w}_C)|}$

- One unbounded direction parallel to p_i for all $i = 1, \ldots, s$, with weights index(p_i).

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- One unbounded direction parallel to p_i for all $i = 1, \ldots, s$, with weights index(p_i).
- Geometrically: Corresponding holomorphic disks in $\overline{T_N}$ intersect D_{p_i} with multiplicity index(p_i).

2. $\sum_C \mathrm{Mult}(C) \frac{R_{\mathbf{w}_C}}{|\mathrm{Aut}(\mathbf{w}_C)|}$

- Other unbounded directions parallel to the e_i's, contained in generically specified hyperplanes parallel to $e_i^{W\perp}$. Call the weights w_{ij}.

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$$R_{\mathbf{w_c}} := \prod_{ij} \frac{(-1)^{w_{ij}-1}}{w_{ij}^2}.$$

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- Geometrically: Intersects D_{e_i} in set of form $\{1 + z^{u_i} = a_{ij}\}$ for generically specified a_{ij}'s, with multiplicity w_{ij}.

3. $\sum_C \text{Mult}(C)\frac{R_{\mathbf{w}_C}}{|\text{Aut}(\mathbf{w}_C)|}$

- Q is s-valent and $\mapsto$ generically specified point in $N_{\mathbb{R}}$.
- Geometrically: Marked point $Q \mapsto$ generically specified point in T_N. s-valence corresponds (conjecturally) to ψ_Q^{s-2}-condition.

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- Another Theorem: the $n = 0$ (i.e., constant) terms determine everything.

Sample tropical curve

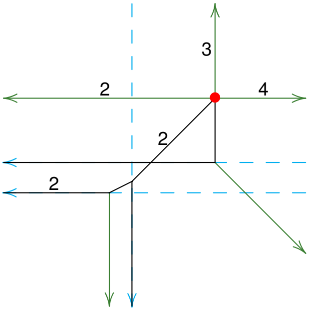

Figure: A tropical curve making a contribution of (-1) (or $-q^{20}[2]_q/2$ for the quantum version) to the constant term in the product $\vartheta_{(4,0)} \cdot \vartheta_{(0,3)} \cdot \vartheta_{(-2,0)} \cdot \vartheta_{(0,-1)} \cdot \vartheta_{(1,-1)}$ in a cluster algebra with $(N = \mathbb{Z}^2, \{-e_1, -e_2\}, W(e_1, e_2) = 1)$.

The End

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