Valuative independence of theta functions

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(based on joint work with M.W. Cheung, T. Magee, and G. Muller)

Introduction:
The Valuative Independence Theorem

Discrete valuations

A (discrete) valuation val on a field K is a function

$$\mathrm{val} : K \to \mathbb{Z} \cup \{\infty\}$$

such that

- $\mathrm{val}(xy) = \mathrm{val}(x) + \mathrm{val}(y)$
- $\mathrm{val}(x + y) \geq \min(\mathrm{val}(x), \mathrm{val}(y))$
- $\mathrm{val}(x) = \infty \iff x = 0.$

for all $x, y \in K$.

Example: $D \subset Y$ a prime divisor, $K = K(Y)$, have val_D mapping f to the order of vanishing/pole of f along D.

Boundary divisors for toric varieties

- Let
  - $N \cong \mathbb{Z}^r$,
  - $M = \mathrm{Hom}(N, \mathbb{Z})$,
  - and $T_N = N \otimes \mathbb{C}^* = \operatorname{Spec} \mathbb{C}[M]$ where

$$\mathbb{C}[M] = \mathbb{C}[z^m | m \in M] / \langle z^{m_1} z^{m_2} = z^{m_1+m_2} | m_1, m_2 \in M \rangle.$$

- $n \in N \setminus \{0\} \rightsquigarrow \rho_n = \mathbb{R}_{\geq 0} n \rightsquigarrow$ boundary divisor $D_{\rho_n} =: D_n$.
- Let $\mathrm{val}_n := |n| \, \mathrm{val}_{D_n}$.
- **Fact**: $\mathrm{val}_n(z^m) = m \cdot n$.
- Furthermore:

$$\mathrm{val}_n \left(\sum_m a_m z^m \right) = \min_{a_m \neq 0} (m \cdot n).$$

Valuative Independence Theorem (VIT)

- For Y a cluster variety, we again have a lattice N parametrizing divisorial valuations $\mathrm{val}_n := |n|\,\mathrm{val}_{D_n}$.
- Dual lattice M parameterizing theta functions ϑ_m.

Theorem (Cheung-Magee-M-Muller)

$$\mathrm{val}_n\left(\sum_m a_m \vartheta_m\right) = \min_{a_m \neq 0} \mathrm{val}_n(\vartheta_m).$$

- In fact, our argument works in the more general context of theta functions constructed from a positive consistent scattering diagram.
- Application: Get theta function bases for line bundles on compactifications.

Review of scattering diagrams and theta functions

Scattering diagrams

- A scattering diagram in $M_{\mathbb{R}}$ is a set of walls with attached functions. More precisely:
- A **wall** in $M_{\mathbb{R}}$ is a pair $(\mathfrak{d}, f)$ where
  - $f \in 1 + z^{m_{\mathfrak{d}}}\Bbbk[\![z^{m_{\mathfrak{d}}}]\!]$ for some $m_{\mathfrak{d}} \in M$;
  - $\mathfrak{d}$ is a cone in $n_{\mathfrak{d}}^{\perp}$ for some $n_{\mathfrak{d}} \in m_{\mathfrak{d}}^{\perp}$.
- A **scattering diagram** $\mathfrak{D}$ is a set of walls (finite up to any finite order).
- $(\mathfrak{d}, f)$ is called **incoming** if $m_{\mathfrak{d}} \in \mathfrak{d}$.
- **Fact** [GS, KS, GHKK]: Given

$$\mathfrak{D}^{\text{in}} = \{(n_i^{\perp}, f_i)\}_i,$$

$\exists!$ *consistent* scattering diagram $\mathfrak{D} = \text{Scat}(\mathfrak{D}^{\text{in}})$ whose incoming walls are $\mathfrak{D}^{\text{in}}$.

Interpretations of the initial scattering diagram

- If $\mathfrak{D}^{\text{in}} = \{(n_i, f_i)\}_i$, you should think of the theta functions we will construct as, roughly, being functions on Y constructed as follows:
  - Start with $\mathrm{TV}(\Sigma)$ for Σ a fan in $N_{\mathbb{R}}$ including rays generated by the n_i's.
  - For each i, blow up $D_{n_i} \cap \{f_i = 0\}$.
- **Cluster varieties:** Special cases where

$$f_i = (1 + z^{m_i})^{|n_i|}$$

  and the matrix

$$(m_i \cdot n_j)_{i,j \text{ indices for } \mathfrak{D}^{\text{in}}}$$

  is skew-symmetric (or skew-symmetrizable).

Theta functions

- Let $p \in M, x \in M_{\mathbb{R}}$.
- A broken line with ends (p, x) is a piecewise-straight path $\Gamma : (-\infty, 0] \to M_{\mathbb{R}}$ such that
  - $\Gamma(0) = x$
  - each straight segment Γ_i has an attached monomial $c_i z^{m_i}$ such that $\Gamma_i' = -m_i$
  - the initial attached monomial is z^p
  - Γ can only bend when crossing a wall $(\mathfrak{d}, f)$. The attached monomials satisfy

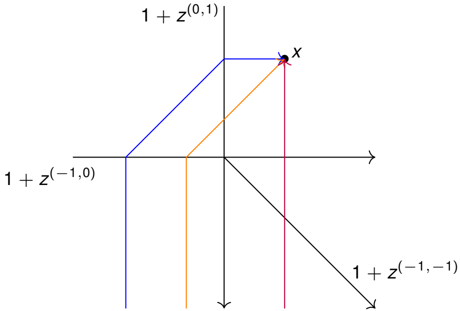
    $$c_{i+1} z^{m_{i+1}} \quad \text{is a term in} \quad c_i z^{m_i} f^{m_i \cdot n_{\mathfrak{d}}}$$

    where $n_{\mathfrak{d}} \in \mathfrak{d}^{\perp}$ is primitive in N and positive on m_i.
- Now define

  $$\vartheta_{p,x} = \sum_{\mathrm{Ends}(\Gamma)=(p,x)} c_{\Gamma} z^{m_{\Gamma}}$$

  where $c_{\Gamma} z^{m_{\Gamma}}$ denotes the final attached monomial of Γ.
- $\vartheta_{p,x}$ for different x are related by transition functions called path-ordered products.

Theta function example



$$\vartheta_{(0,-1),x} = z^{(-1,0)} + z^{(-1,-1)} + z^{(0,-1)}$$

Valuations

- For each generic $x \in M_{\mathbb{R}}$, $\vartheta_{p,x}$ form a (topological) basis for the formal Laurent series ring $A = \Bbbk[\![M]\!] = \sum_{m \in M} c_m z^m$.
- Given $n \in N_{\mathbb{R}}$, define val_n on A by

$$\mathrm{val}_n : \sum_m c_m z^m = \inf_{c_m \neq 0} m \cdot n.$$

- **Valuative Independence Theorem**:

$$\mathrm{val}_n(\sum_m c_m \vartheta_{m,x}) = \min_{c_m \neq 0} \mathrm{val}_n(\vartheta_{m,x})$$

whenever the right-hand side is finite.

Proof sketch for the Valuative Independence Theorem

Tautness

- Assuming positivity,

$$\mathrm{val}_n(\vartheta_{m,x}) = \mathrm{val}_n\left(\sum_{\mathrm{Ends}(\Gamma)=(m,x)} c_\Gamma z^{m_\Gamma}\right) = \min_{\mathrm{Ends}(\Gamma)=(m,x)} m_\Gamma \cdot n.$$

- Show this is true even if we allow "rational broken lines" whose bends are c times bends of ordinary broken lines for $c \in \mathbb{Q} \cap [0,1]$.
- **Theorem:** The val_n-minimizing rational broken lines are "*n*-**taut**" (greedy approach to minimizing). Otherwise they wouldn't even give local minima.

Tropical theta functions

- Let $\vartheta_{m,x}^{\mathrm{trop}}$ be the piecewise-linear function on $N_{\mathbb{R}}$ given by

$$\vartheta_{m,x}^{\mathrm{trop}}(n) = \mathrm{val}_n(\vartheta_{m,x}).$$

- **Theorem:** The slope of $\vartheta_{m,x}^{\mathrm{trop}}$ at a generic n uniquely determines m. So if $\vartheta_{m_1,x}^{\mathrm{trop}} = \vartheta_{m_2,x}^{\mathrm{trop}}$ at a generic n, then $m_1 = m_2$.
- **Proof:** The slope determines m_Γ for the minimizing Γ, and then we can reconstruct Γ by tracing backwards from x using the n-tautness condition until we recover m. □
- **VIT is a corollary:** In general,

$$\mathrm{val}_n\left(\sum_i f_i\right) \geq \min_i\left(\mathrm{val}_n(f_i)\right)$$

and strict inequality can only happen if $\mathrm{val}_n(f_i) = \mathrm{val}_n(f_j)$ for some $i \neq j$. □

Applications of VIT and tautness

Linear independence of theta functions

- Theta functions are linearly independent (easy).
- But what about after specializing coefficients?
- **Theorem**: Theta functions remain linearly independent after specializing coefficients.
- **Proof idea:**
  - Let $f = \sum c_m \vartheta_m$.
  - For generic n, $\mathrm{val}_n(f)$ is determined by a taut broken line, and no other broken line contributing to $\mathrm{val}_n(f)$ gives the same valuation.
  - So coefficient-specialization cannot affect $\mathrm{val}_n(f)$.
  - In fact, f^{trop} is unchanged.
  - In particular, f does not become 0 after specializing coefficients.

Bases for line bundles on compactifications

- Suppose $\{\vartheta_m : m \in M\}$ is a theta function basis for regular functions on a cluster variety U.
- Primitive $n \in N \leftrightsquigarrow$ boundary divisors D_n.
- Let $F \subset N$ be a finite set of primitive vectors and $Y = U \cup \bigcup_{n\in F} D_n$.
- Let $W = \sum a_i D_{n_i}$,

$$\Gamma(Y, \mathcal{O}(W)) = \{f \in K(Y) | \operatorname{val}_{n_i}(f) \geq a_i \text{ for all } i\}.$$

- **Corollary of VIT:** We have a theta function basis for $\Gamma(Y, \mathcal{O}(W))$:

$$\{\vartheta_m | \operatorname{val}_{n_i}(\vartheta_m) \geq a_i \text{ for all } i\}$$

- Every line bundle in Y is isomorphic to $\mathcal{O}(\widetilde{W})$ for some boundary divisor $\widetilde{W}$ on Y^{prin} [M, '19, Cluster algebras are Cox rings]
- So we get theta function bases for all line bundles! (Assuming U^{prin} has a theta function basis).

Theta functions and unfreezing

- Let **s** be a seed (i.e., an initial scattering diagram)
- **s**$'$ obtained from **s** by unfreezing (i.e., creating new initial scattering walls).
- Let ϑ_m be a theta function for **s**.
- **Theorem:** If ϑ_m extends to a global regular function $\widetilde{\vartheta}_m$ for **s**$'$, then $\widetilde{\vartheta}_m$ is a theta function for **s**$'$.
- **Proof sketch** (for principal coefficients):
  - Note $\mathrm{val}_n(\widetilde{\vartheta}_m) = \mathrm{val}_n(\vartheta_m)$ for each n.
  - Also, $\widetilde{\vartheta}_m = \vartheta'_m + \sum_{v \in m+M^+} a_v \vartheta'_v$.
  - So VIT $\implies \mathrm{val}_n(\vartheta_m) \leq \mathrm{val}_n(\vartheta'_m)$ for all n.
  - But if $\vartheta'_m \neq \widetilde{\vartheta}_m$, then $\mathrm{val}_n(\vartheta'_m) < \mathrm{val}_n(\vartheta_m)$ for some n.

Moduli of local systems

For Σ a marked triangulable surface, and let $\mathcal{Y}_\Sigma$ be one of Fock-Goncharov's moduli of local systems on Σ.

Corollary

Let Σ' be obtained from Σ via gluing boundary arcs. Then theta functions on $\mathcal{Y}_\Sigma$ give theta functions on $\mathcal{Y}_{\Sigma'}$.

Proof idea: Gluing boundary arcs can be understood as identifying and then unfreezing certain frozen indices.

Notation: valuations and tropical theta functions

- Let ϑ_m denote $\vartheta_{m,x}$ for x in the positive chamber.
- $n \in N$ determines $\mathrm{val}_n : M \to \mathbb{Z}$, $m \mapsto \mathrm{val}_n(\vartheta_m)$.
- $m \in M$ determines $\vartheta_m^{\mathrm{trop}} : N \to \mathbb{Z}$, $n \mapsto \mathrm{val}_n(\vartheta_m)$.

Theta reciprocity

- Let $\mathfrak{D}^{\text{in}} = \{n_i^\perp, (1+z^{m_i})^{|n_i|}\}$, $\mathfrak{D} = \text{Scat}(\mathfrak{D}^{\text{in}})$
- Define $\mathfrak{D}^\vee$ similarly switching the roles of N and M, so

$$(\mathfrak{D}^\vee)^{\text{in}} = \{m_i^\perp, (1+z^{n_i})^{|m_i|}\}.$$

- **Theorem:** For all $n \in N, m \in M$, we have

$$\text{val}_n(\vartheta_m) = \text{val}_m(\vartheta_n).$$

- Application [Keel]:
  - Let $Y = U \cup \bigcup_{n \in F} D_n$ like we saw before.
  - Let $W = \sum_{n \in F} \vartheta_n$ (potential on mirror $U^\vee$)
  - Let $\Xi = \{W^{\text{trop}} \geq 0\} \subset M_{\mathbb{R}}$.
  - Then $\{\vartheta_m | m \in \Xi\}$ form a basis for $\Gamma(Y, \mathcal{O}_Y)$.

Applications in representation theory

- Let G be a reductive group, B a maximal Borel, N the unipotent radical of B.
- Irreducible representations of G correspond to weight-spaces of functions on G/N
- G/N is a partial compactification of a cluster variety $\mathcal{A}$ which has a theta function basis.
- **Corollary:** We get a theta function basis for every irreducible representation of G, cut out by a slice of some $\{W^{\mathrm{trop}} \geq 0\}$.
- $G = \mathrm{SL}_n$ case was in T. Magee's thesis. VIT lets us generalize (without finding an "optimized seed").