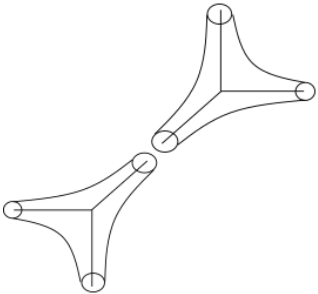
Tropical multiplicities from polyvector fields and QFT

Travis Mandel
(based on joint work with Helge Ruddat)



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- **Goal:** Develop localized formulas for tropical multiplicities.

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For curves in toric surfaces passing through specified points, Mikhalkin defined the multiplicity of a tropical curve Γ as

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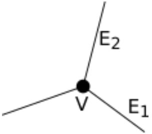

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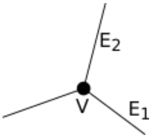

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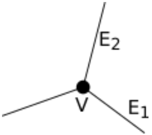

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Answer: Not quite.

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  - **Balancing condition:** For each $V \in \Gamma^{[0]}$ and $E \ni V$, let $u_{(V,E)}$ denote the primitive vector from $h(V)$ into $h(E)$. Then

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  - A **tropical curve** is a parametrized tropical curve up to isomorphism.

Genus, type, and degree

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- Type: (Γ, w, g, ϵ) plus $u_{(V,E)}$ for each flag $E \ni V$.
- Degree: $\Delta : I \to N$, $\Delta(i) = w(E_i) u_{(V,E_i)}$.

Incidence conditions

- Let $\mathbf{A} = (A_i)_{i \in I}$, where each A_i is an affine hyperplane in $N_{\mathbb{R}}$ with rational slope.
- We say Γ matches the constraints $\mathbf{A}$ if $h(E_i) \subset A_i$ for each i.
- Could also consider ψ-classes, i.e., higher-valency requirements.

Tropical tangent space

- **Notation:** For $S = A_i \in \mathbf{A}$ or $S = E \in \Gamma^{[1]}$, let $\mathbb{L}(S)$ be the linear space parallel to S. Let $\mathbb{L}_N(S) = \mathbb{L}(S) \cap N$.

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- Given a tropical curve Γ, define

$$\Phi := \prod_{V \in \Gamma^{[0]}} N \to \left(\prod_{E \in \Gamma_c^{[1]}} N/\mathbb{L}_N(E) \right) \times \left(\prod_{i \in I} N/\mathbb{L}_N(A_i) \right)$$

$$H \mapsto ((H_{\partial^+ E} - H_{\partial^- E})_{E \in \Gamma_c^{[1]}}, (H_{\partial E_j})_{j \in I}).$$

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- Note: $\ker(\Phi \otimes \mathbb{R})$ is the tangent space to Γ in $\mathfrak{T}_{g,\Delta}(\mathbf{A})$, the moduli of tropical curves of genus g and degree Δ matching $\mathbf{A}$.

Multiplicity definition

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- $\mathfrak{D}_\Gamma := \operatorname{index}(\Phi)$.
- $$\operatorname{Mult}(\Gamma) := \mathfrak{D}_\Gamma \prod_{E\in\Gamma_c^{[1]}} w(E).$$

Correspondence theorem

- Suppose that $\sum_i \operatorname{codim}(A_i) = \#I + (r-3)(1-g)$, and that $\mathfrak{T}_{g,\Delta}(\mathbf{A})$ is finite.
- Let
$$\mathrm{GW}^{\mathrm{trop}}_{g,\Delta}(\mathbf{A}) := \sum_{\Gamma \in \mathfrak{T}_{g,\Delta}(\mathbf{A})} \mathrm{Mult}(\Gamma).$$
Let $\mathrm{GW}^{\log}_{g,\Delta}(\mathbf{A})$ denote the correspond count of holomorphic curves.

Theorem (Mikhalkin, Nishinous-Siebert, A. Gross, M-Ruddat)

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Polyvector fields

- Let $M = \text{Hom}(N, \mathbb{Z})$.
- Can view $m \in M$ as a derivation ∂_m on $\mathbb{G}_m(M) = \text{Spec}\,\mathbb{Z}[N]$.

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- Polyvector fields on $\mathbb{G}_m(M)$ given by

$$\mathcal{P} := \mathbb{Z}[N] \otimes \Lambda^* M.$$

Polyvector fields and incidence conditions

- Consider $\Gamma \in \mathfrak{T}_{g,\Delta}(\mathbf{A})$.
- Let α_i be the unique-up-to-sign primitive element of $\Lambda^{\mathrm{codim}(A_i)} M$ which vanishes on $\mathbb{L}(A_i)$.

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Brackets of polyvector fields

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- Define $\ell_k : \mathcal{P}^{\otimes k} \to \mathcal{P}$ by

$$\ell_k(z^{n_1}\alpha_1, \ldots, z^{n_k}\alpha_k) := \ell_1\left(\prod_{j=1}^{k} z^{n_j}\alpha_j\right) = z^{n_1+\ldots+n_k}\iota_{n_1+\ldots+n_k}(\alpha_1 \wedge \cdots \wedge \alpha_k).$$

Multiplicities from polyvector fields

Given a rigid $\Gamma \in \mathfrak{T}_{0,\Delta}(\mathbf{A})$ with a flow towards a specified $V_\infty \in \Gamma^{[0]}$, inductively associate $\zeta_E \in \mathcal{P}_0$ (up to sign) to each edge E as follows:

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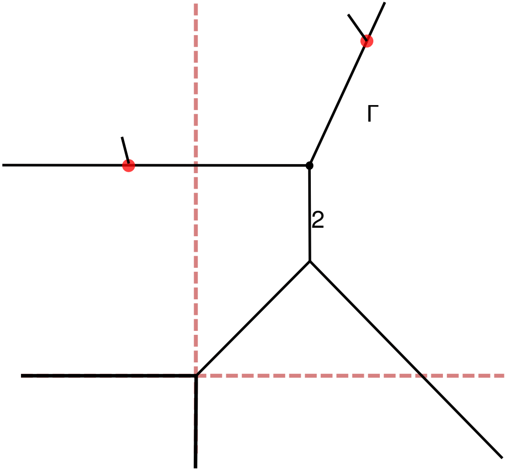
Theorem (M-Ruddat)

$$\mathrm{Mult}(\Gamma) = \left| \prod_{E \ni V_\infty} \zeta_E \right|,$$

where on the right-hand side we take the index in $\Lambda^{\mathrm{top}} M$.

Example



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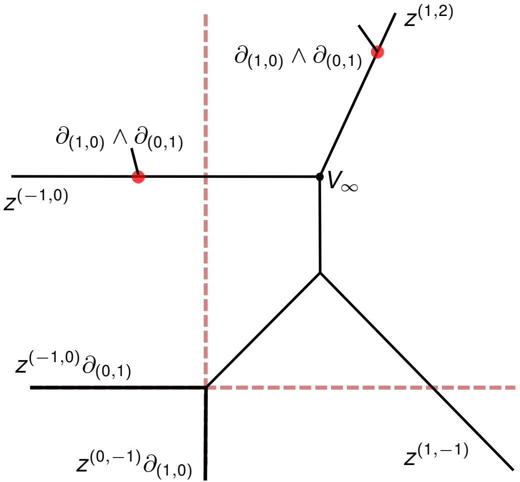

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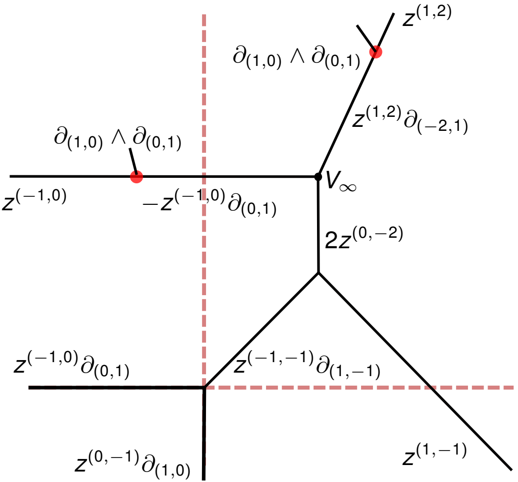

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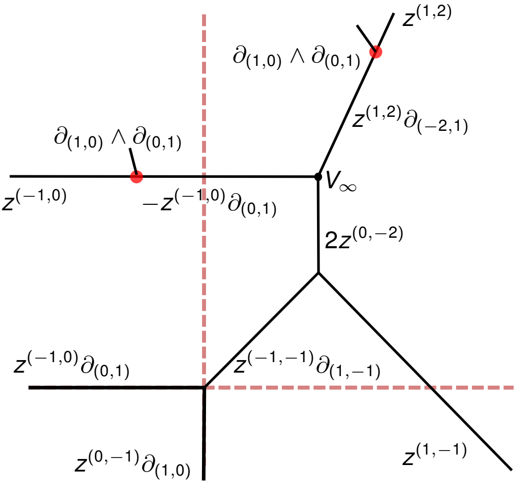

$$\mathrm{Mult}(\Gamma) = |2\partial_{(0,1)} \wedge \partial_{(-2,1)}| = 2\left|\det\begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix}\right| = 4$$

Cool structure

Up to some sign twistings in the definitions of the ℓ_k's:

- The ℓ_k make $\mathcal{P}_0$ into an L-infinity algebra.
- $\ell_2|_{\mathcal{P}_0}$ extends to the Schouten-Nijenhuis Lie bracket defined on all of $\mathcal{P}$.

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- Mirror to string topology/symplectic cohomology on $\mathbb{G}_m(N)$.

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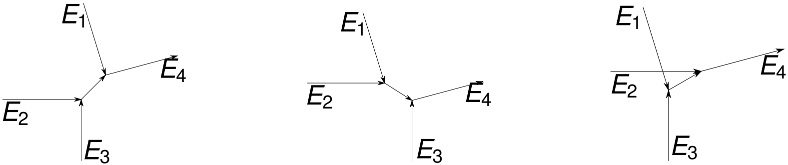
Tropical invariance and the Jacobi identity



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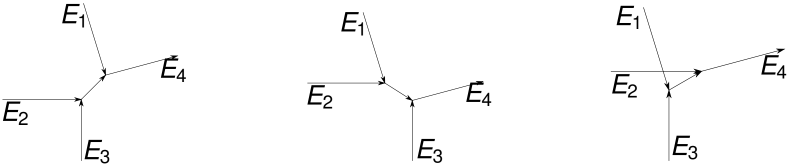

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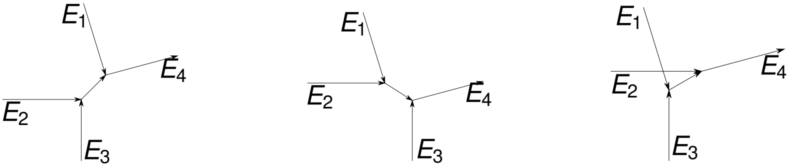

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- Things go wrong when applying this to Hall algebra theta functions [Cheung-M].

Proof of the polyvector field formula

Corollary of our more general **tropical quantum field theory** description of multiplicities.

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  - Objects: closed $(n-1)$-dimensional manifolds.

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- An n-dimensional topological quantum field theory (TQFT) is a symmetric monoidal functor from $\mathtt{nCob}$ to another symmetric monoidal category, e.g., $(\mathrm{Vect}, \otimes)$.

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- A Frobenius algebra over R is an associative, finite-dimensional R-algebra A together with a co-unit $\eta : A \to R$ such that the pairing $\mathrm{Tr} : A \otimes_R A \to R$, $\mathrm{Tr}(a \otimes b) := \eta(ab)$ is non-degenerate.

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- **Theorem:** The data of a 2-dimensional TQFT is equivalent to that of a commutative Frobenius algebra.
  - Take a handle-body decomposition of M:
  - Unit/counit $\leftrightsquigarrow$ cup/cap.
  - Product/coproduct $\leftrightsquigarrow$ pair of pants.

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  - Composition: concatenation.
  - Monoidal operator: $\sqcup$.
- 2-dimensional tropical quantum field theory (TrQFT): A symmetric monoidal functor from $\mathrm{Trop2Cob}$ to another symmetric monoidal category $\mathfrak{C}$, e.g., super $\mathbb{Z}$-modules $s\mathbb{Z}\,\mathrm{Mod}$.

Characterization of TrQFT's

Theorem (M-Ruddat)

The following data is equivalent to the data of a TrQFT valued in $\mathfrak{C}$:

- *A commutative Frobenius object $\mathcal{C}_0$ in $\mathfrak{C}$.*
- *For each $n \in \overline{\mathbb{N}} \setminus \{0\}$, an object $\mathcal{C}_n$ of $\mathfrak{C}$ equipped with morphisms $\mathrm{Tr}_n : \mathcal{C}_n \otimes \mathcal{C}_n \to 1_{\mathfrak{C}}$ and $\mathrm{Tr}_n^\vee : 1_{\mathfrak{C}} \to \mathcal{C}_n \otimes \mathcal{C}_n$.*
- *Morphisms $\kappa_n : \mathcal{C}_n \to \mathcal{C}_0$ and $\kappa_n^\vee : \mathcal{C}_0 \to \mathcal{C}_n$ for each $n \in \overline{\mathbb{N}} \setminus \{0\}$ such that*

$$\mathrm{Tr}_n \circ (\mathrm{Id}_n \otimes \kappa_n^\vee) = \mathrm{Tr}_0 \circ (\kappa_n \otimes \mathrm{Id}_0) \qquad \text{as morphisms } \mathcal{C}_n \otimes \mathcal{C}_0 \to 1_{\mathfrak{C}}, \text{ and}$$

$$(\mathrm{Id}_n \otimes \kappa_n) \circ \mathrm{Tr}_n^\vee = (\kappa_n^\vee \otimes \mathrm{Id}_0) \circ \mathrm{Tr}_0^\vee \qquad \text{as morphisms } 1_{\mathfrak{C}} \to \mathcal{C}_n \otimes \mathcal{C}_0.$$

The multiplicity TrQFT

- Consider $F_{\mathrm{Mult}} : \mathrm{Trop2Cob} \to s\mathbb{Z}\,\mathrm{Mod}$, defined as follows:
  - On objects: for $u \in N$, let $M_u := u^{\perp} \subset M$. Then

$$F_{\mathrm{Mult}}(\overline{\Delta} : I \to \overline{N}) := \bigotimes_{i \in I} \Lambda^*(M_{\Delta(i)} \oplus M_{\Delta(i)})$$

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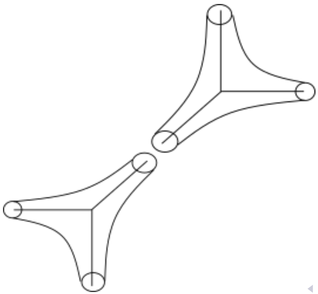
$$F_{\mathrm{Mult}}(\overline{\Delta} : I \to \overline{N}) := \bigotimes_{i \in I} \Lambda^*(M_{\Delta(i)} \oplus M_{\Delta(i)})$$
  - On morphisms: Choose a flow on Γ. Take products/coproducts when flowing through vertices. Contract by $\iota_{(n,0)\wedge(0,n)}$ when flowing across an edge of weighted direction n.



The TrQFT multiplicity theorem

- For any lattice L, $m_1, \ldots, m_k \in L$, and $\alpha = m_1 \wedge \cdots \wedge m_k \in \Lambda^k L$, define

$$\alpha^{\square} := (m_1, 0) \wedge (0, m_1) \wedge \cdots \wedge (m_k, 0) \wedge (0, m_k) \in \Lambda^{2k}(L \oplus L).$$

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- Define $\alpha_{\mathbf{A}} := \bigotimes_{i \in I} \alpha_i^{\square} \in F_{\mathrm{Mult}}(\overline{\Delta})$.

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$\mathrm{Mult}(\Gamma)^2 = F_{\mathrm{Mult}}(\Gamma)(\alpha_{\mathbf{A}})$.

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Theorem (M-Ruddat)

$\mathrm{Mult}(\Gamma)^2 = F_{\mathrm{Mult}}(\Gamma)(\alpha_{\mathbf{A}})$.

- Proof idea: Make all vertices sinks and edges sources.
  - Coproducts of the units $\leftrightsquigarrow$ Künneth decompositions of diagonal classes.
  - Counits of produts $\leftrightsquigarrow$ tropical intersections.

Simplification in genus 0

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- The "squaring" operation $\square$ is needed to get coproducts, but coproducts can be avoided in genus 0.
  - Skip squaring and recover polyvector field formula from before.
- Can also use simpler TrQFT in genus 0 called $F^{\square}_{\mathrm{Mult}}$.
  - Use only those $\zeta \in \Lambda^*(M_u \oplus M_u)$ with $\zeta = \alpha^{\square}$ for some $\alpha \in \Lambda^* M_u$.
  - Yields a "tropical slitting formula" in genus 0, hence a formula of the form

$$\mathfrak{D}_\Gamma = \frac{\prod_{V \in \Gamma^{[0]}} \mathrm{Mult}(V)}{\prod_{E \in \Gamma_c^{[1]}} \mathrm{Mult}(E)}.$$